Computational Geometry
Chapter 4: Voronoi Diagrams

Prof. Dr. Sándor Fekete

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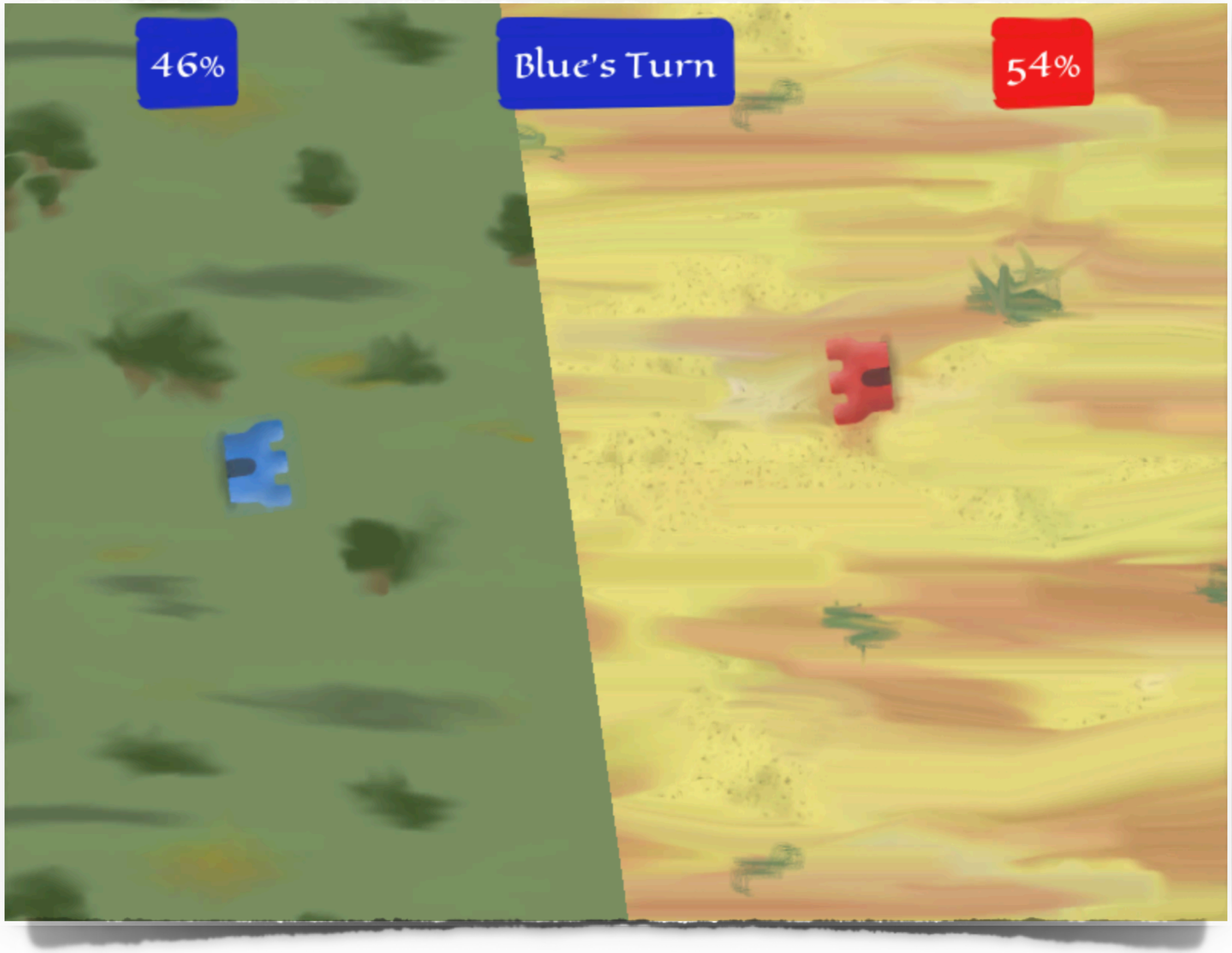
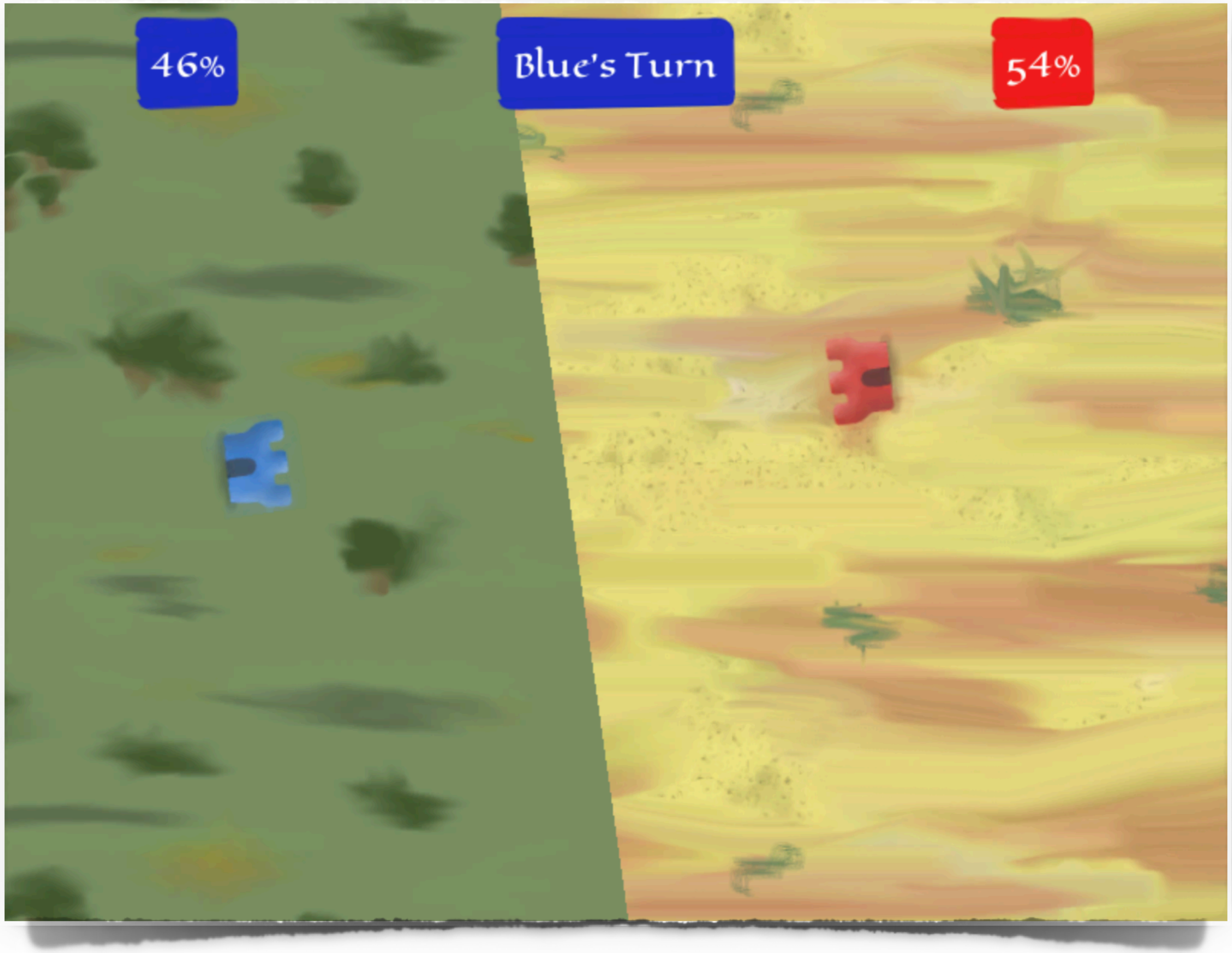
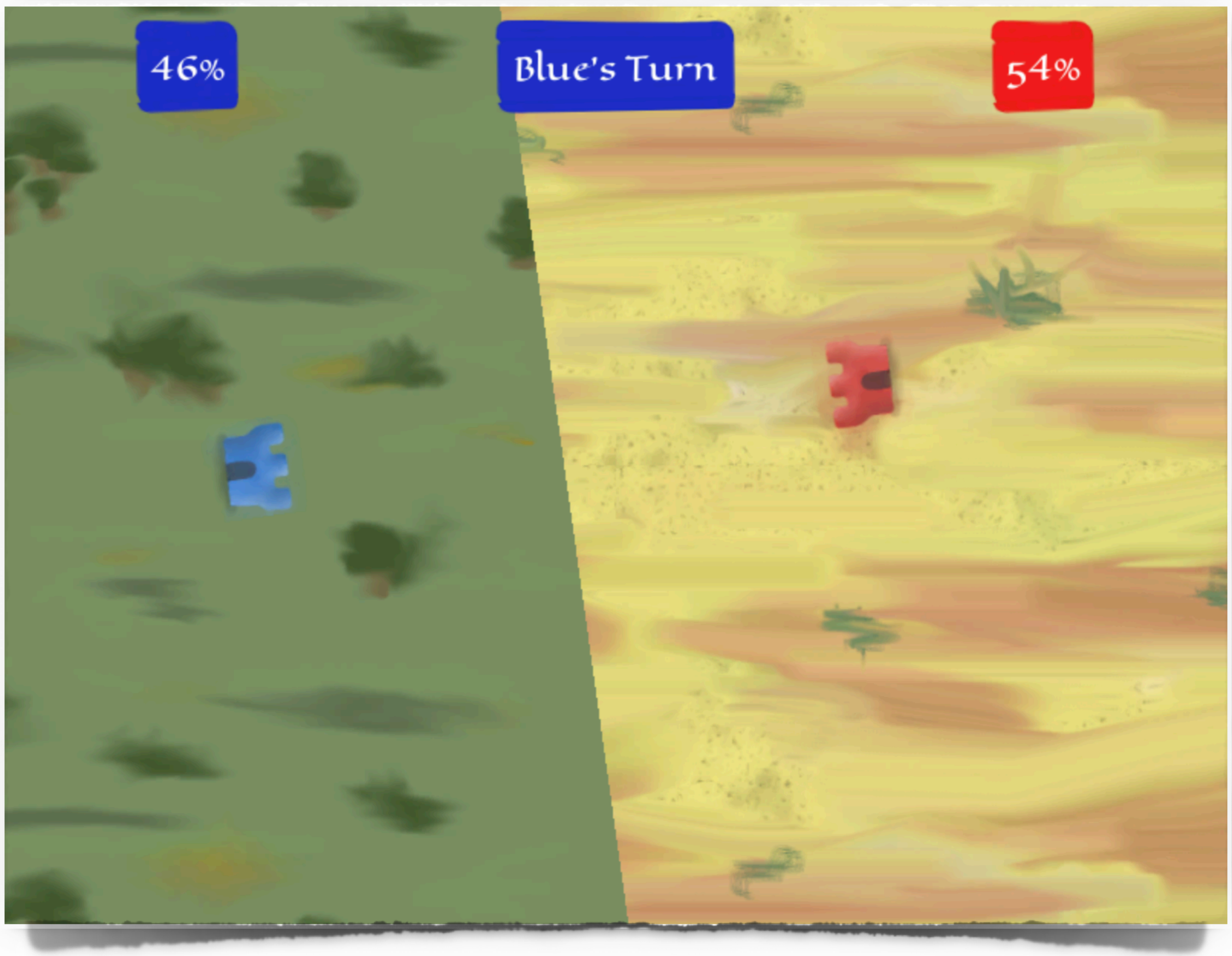
Ruler of the Plane

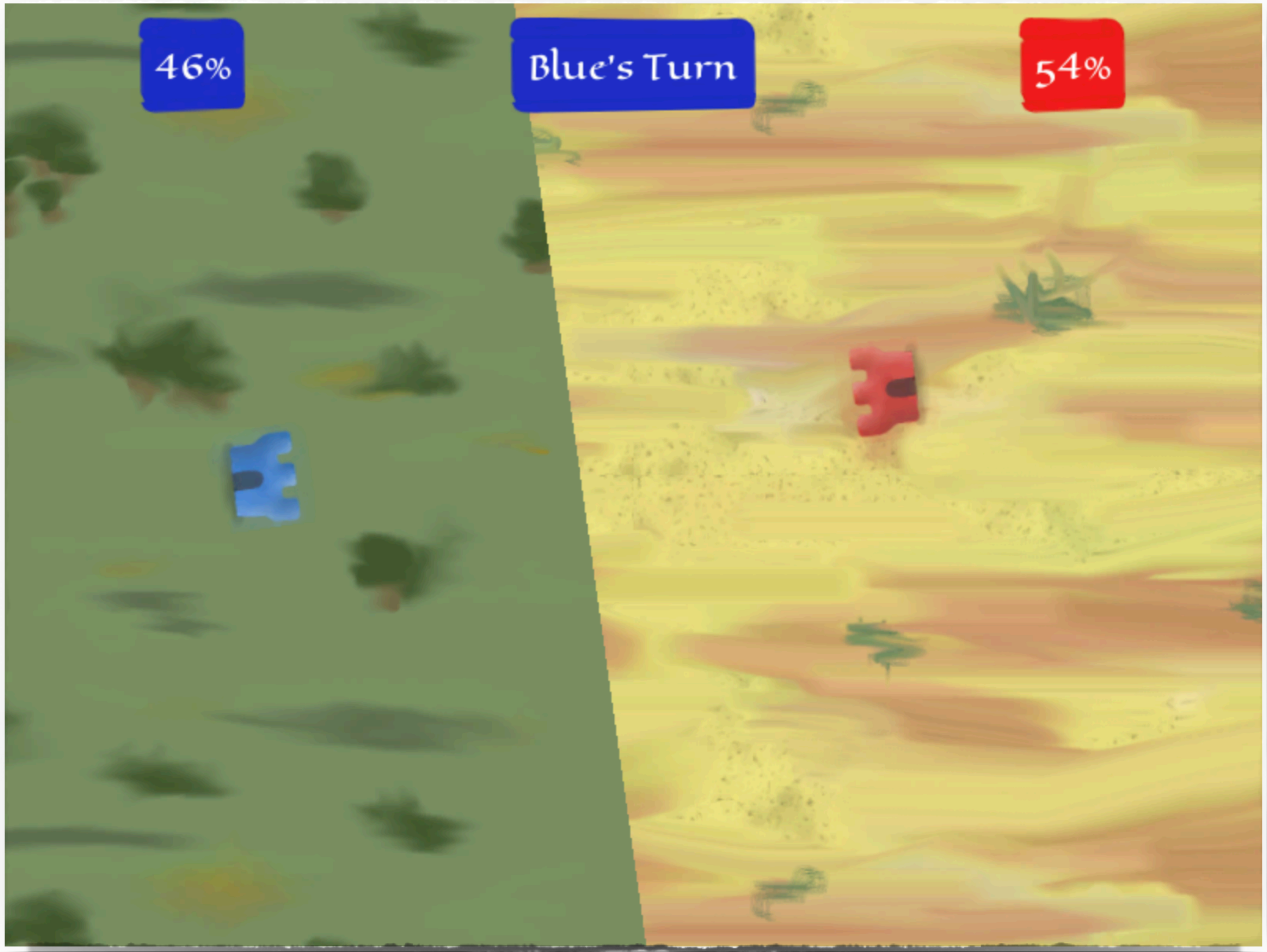

46%
Blue's Turn
54%



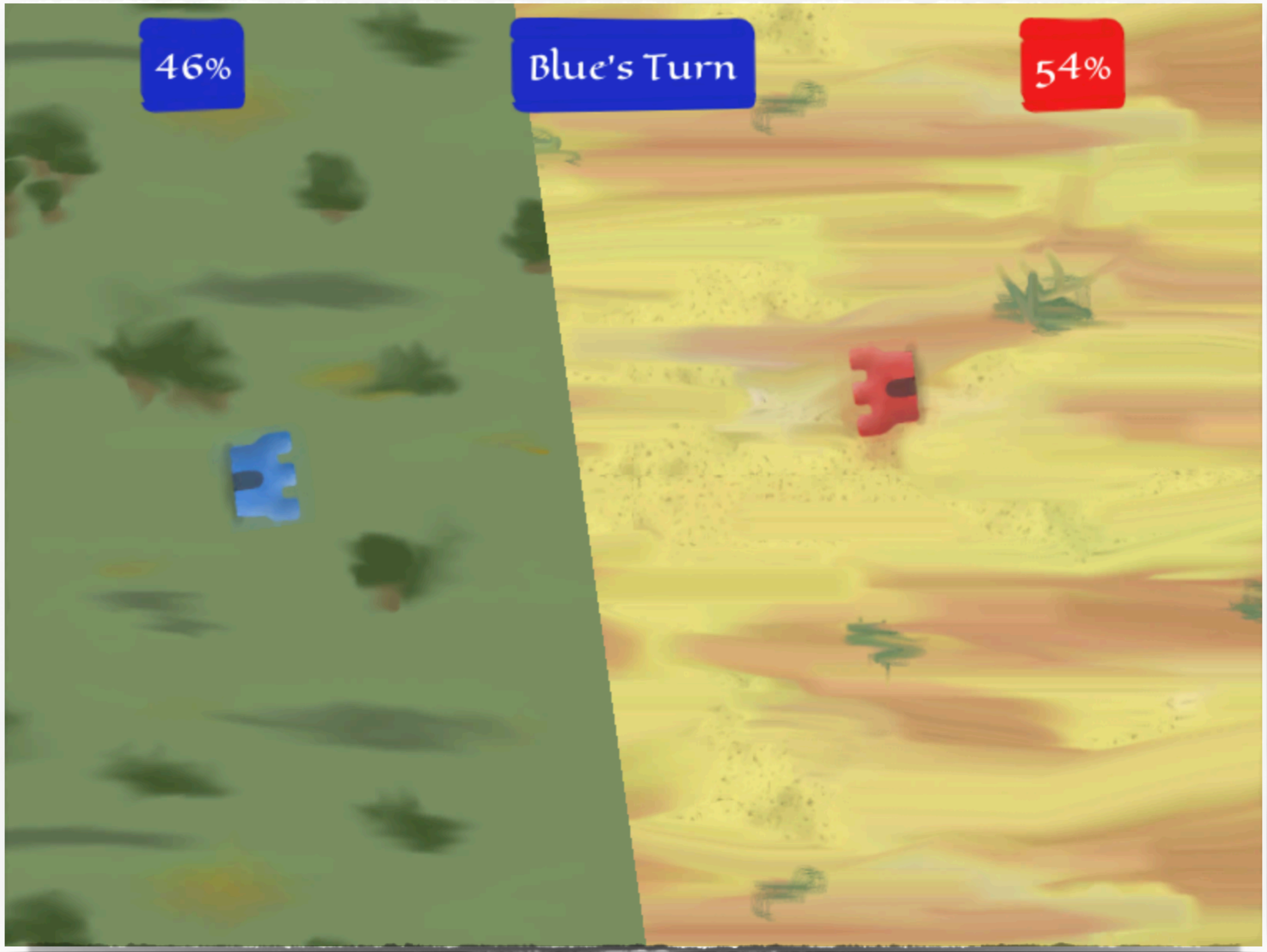
- **A domain**



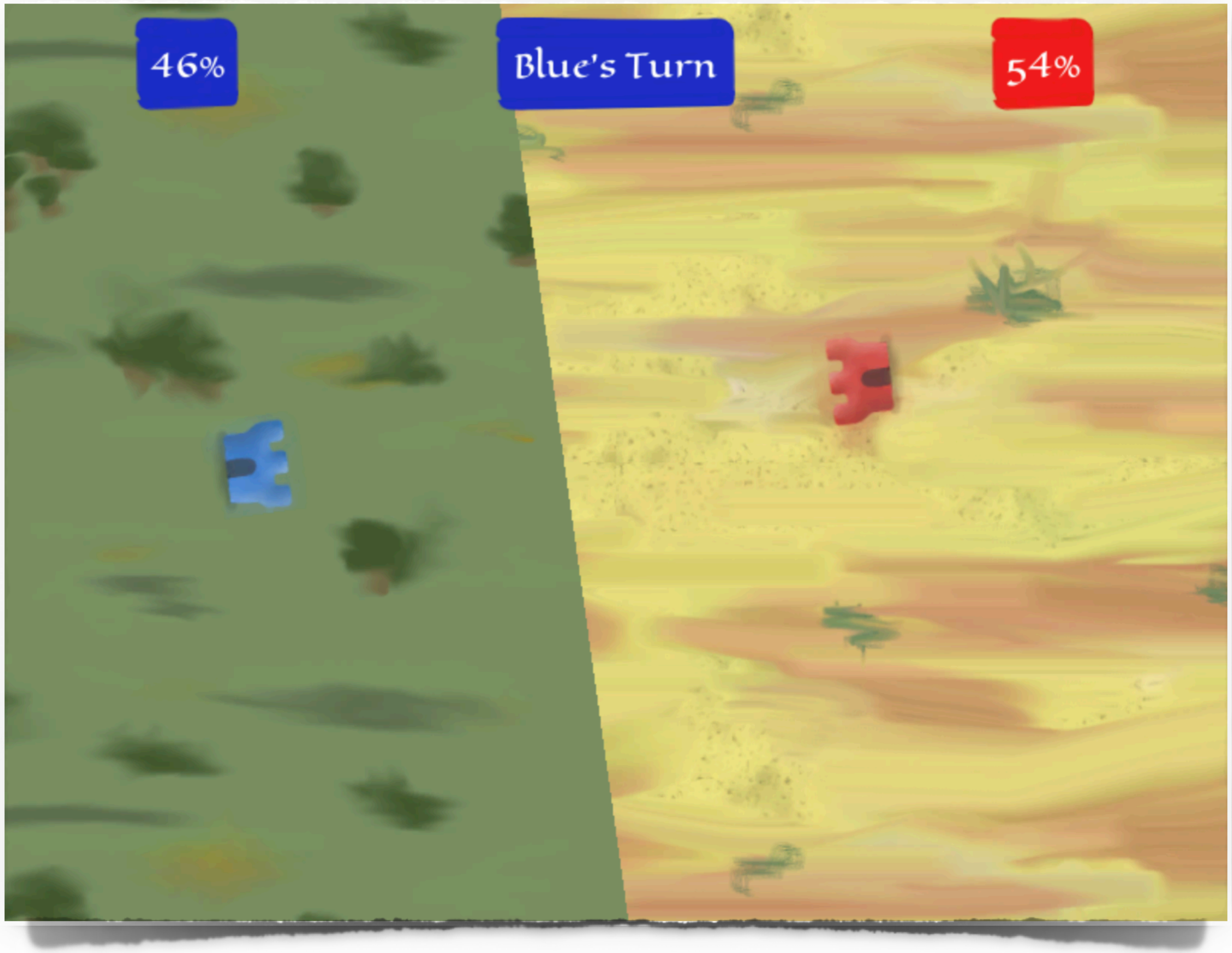
- **A domain**
- **Two players**



- A domain
- Two players
- Players take turns



- A domain
- Two players
- Players take turns
- Voronoi diagram is computed



- A domain
- Two players
- Players take turns
- Voronoi diagram is computed
- Player with larger area wins

1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]

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Competitive Facility Location along a Highway*

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- **Blue and Red**
- $n > 1$

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- **Blue and Red**
- $n > 1$
- **Circle or line segment**

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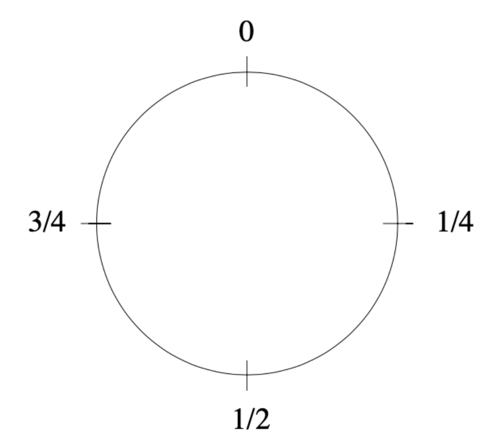

Figure 1: There are four keypoints when $n = 4$.

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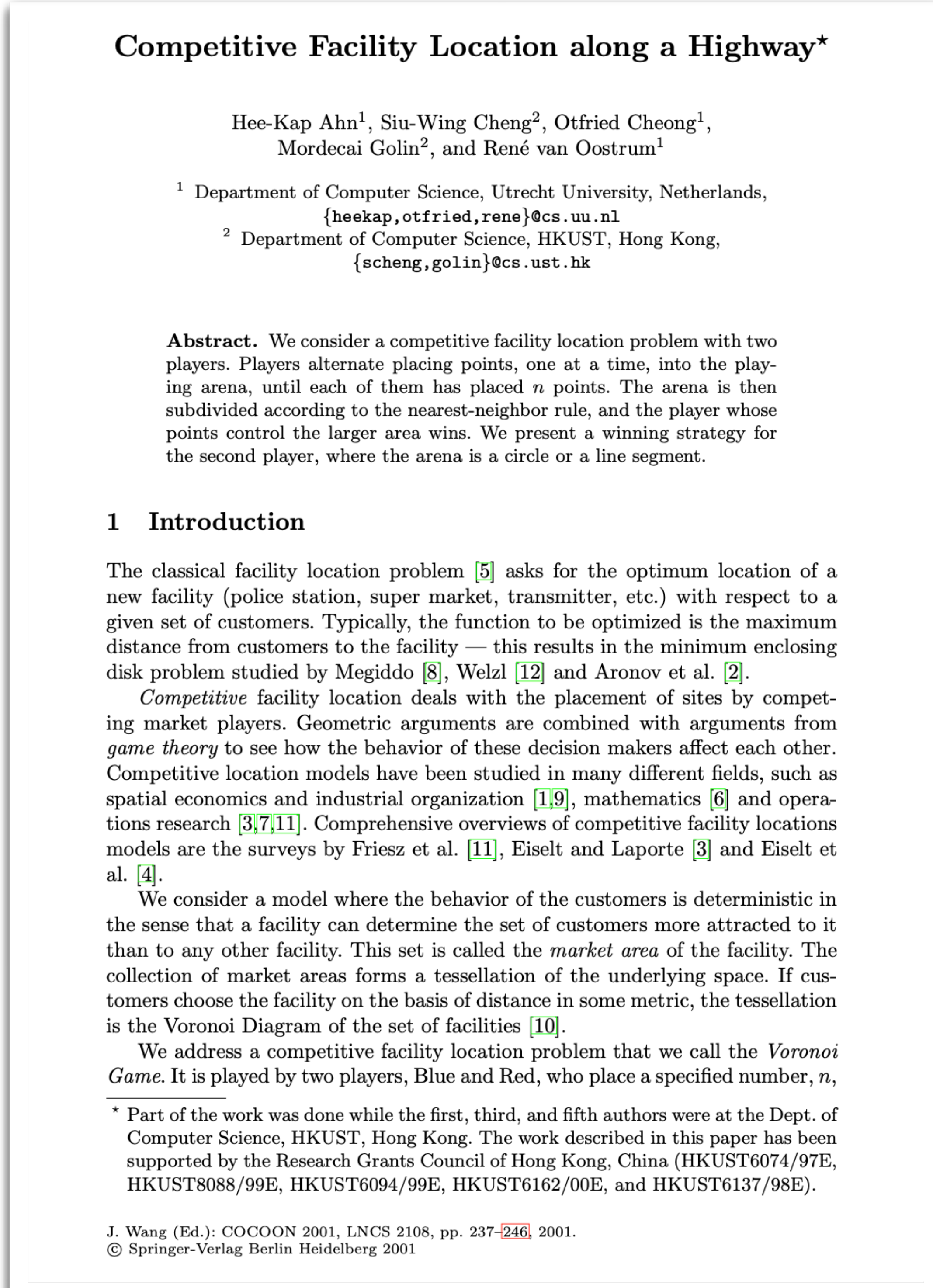

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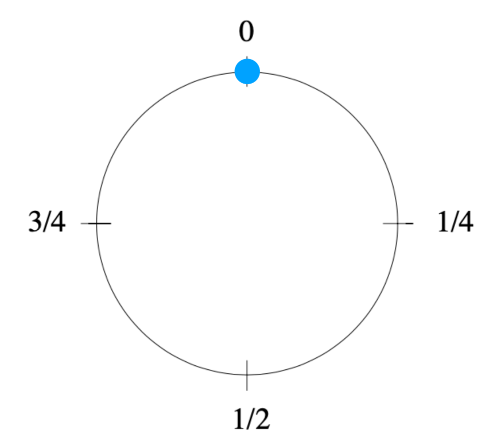

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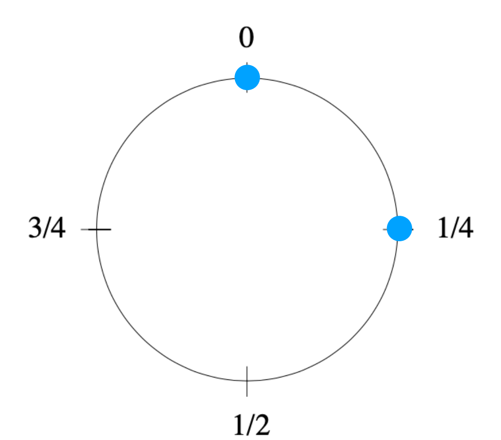

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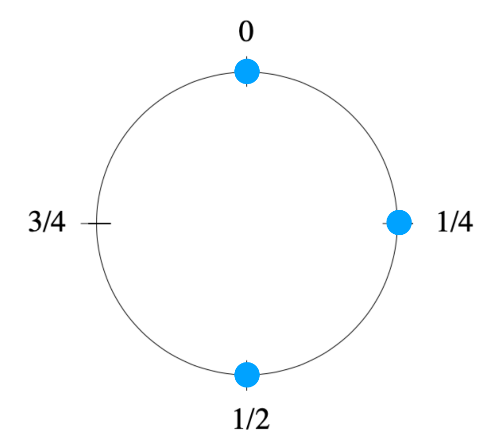

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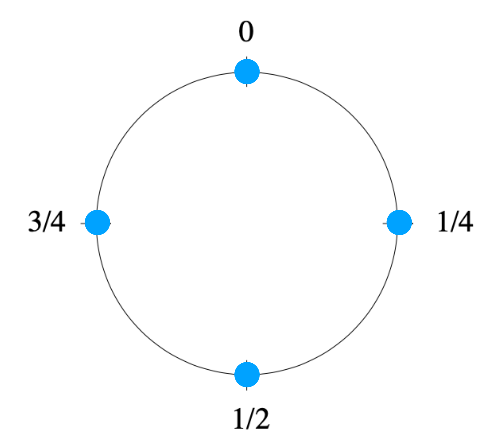

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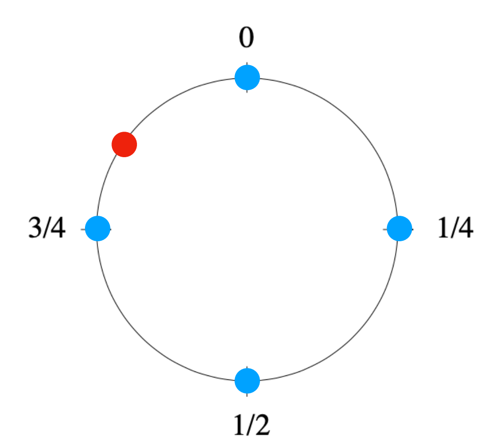

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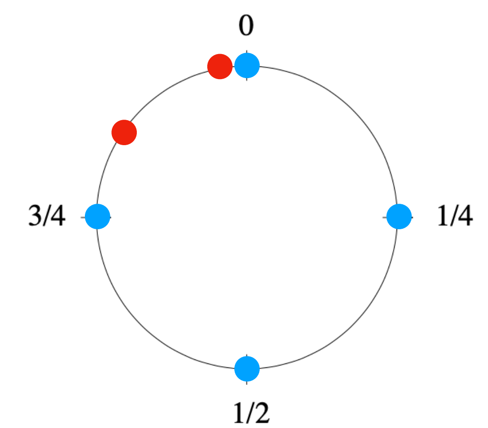

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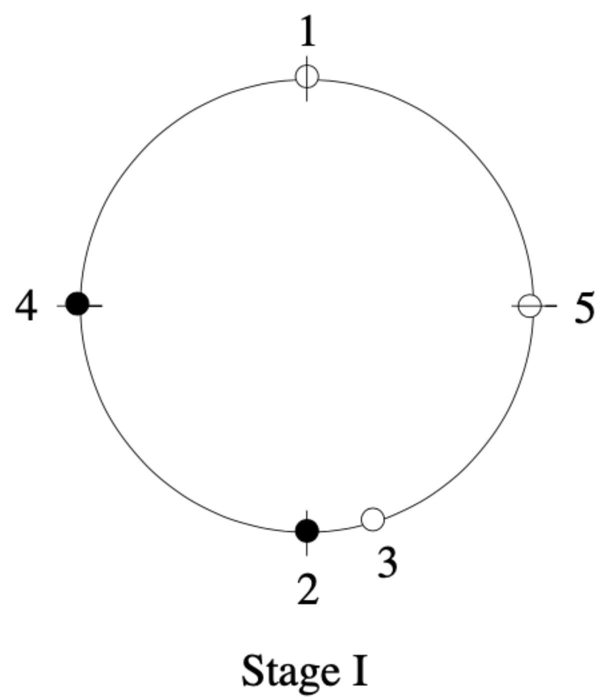

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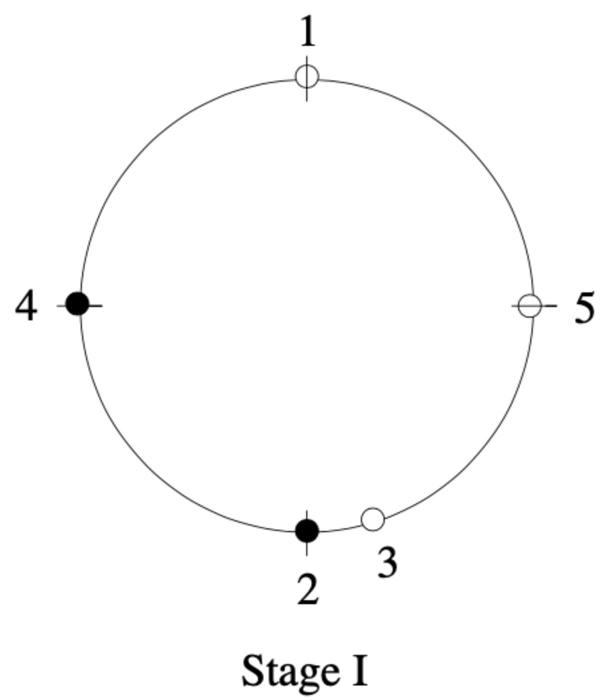

- Stage I: Play keypoint while available
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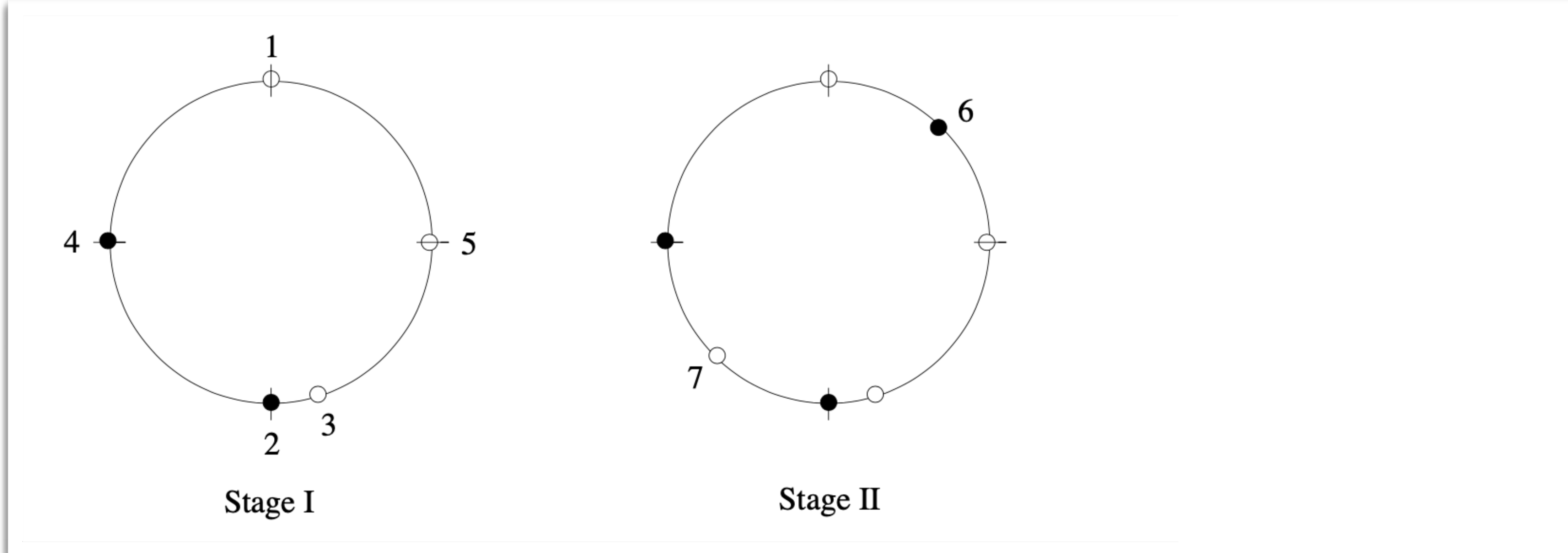

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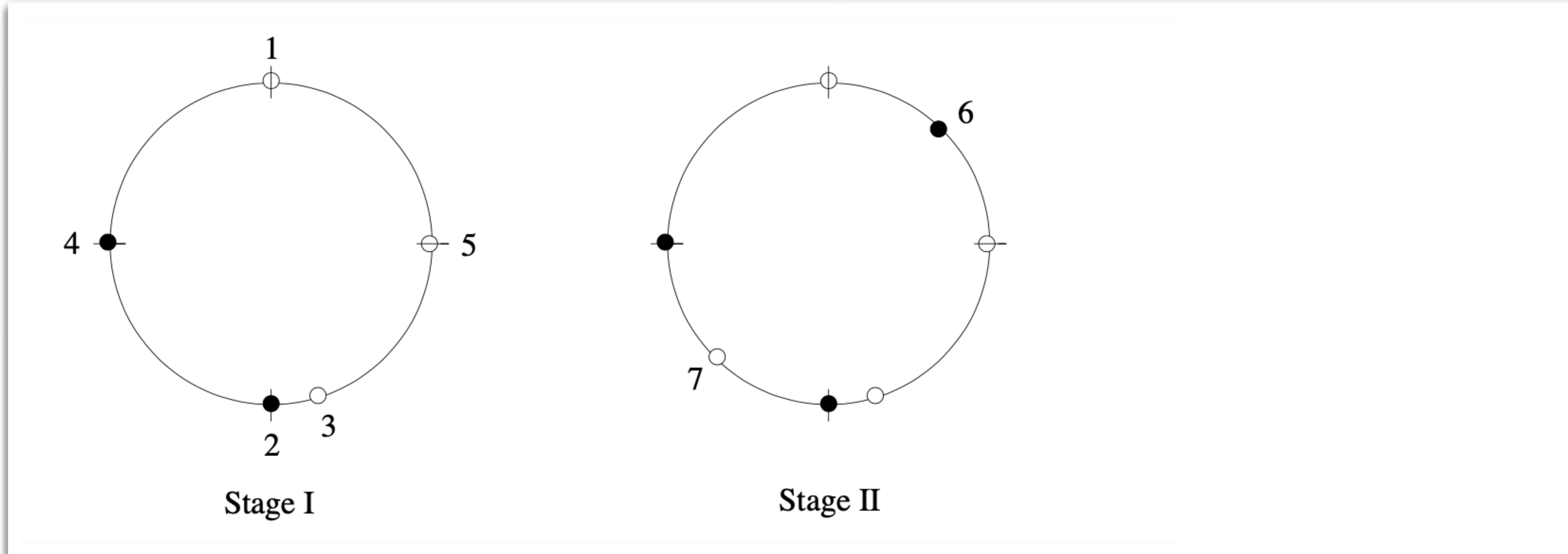

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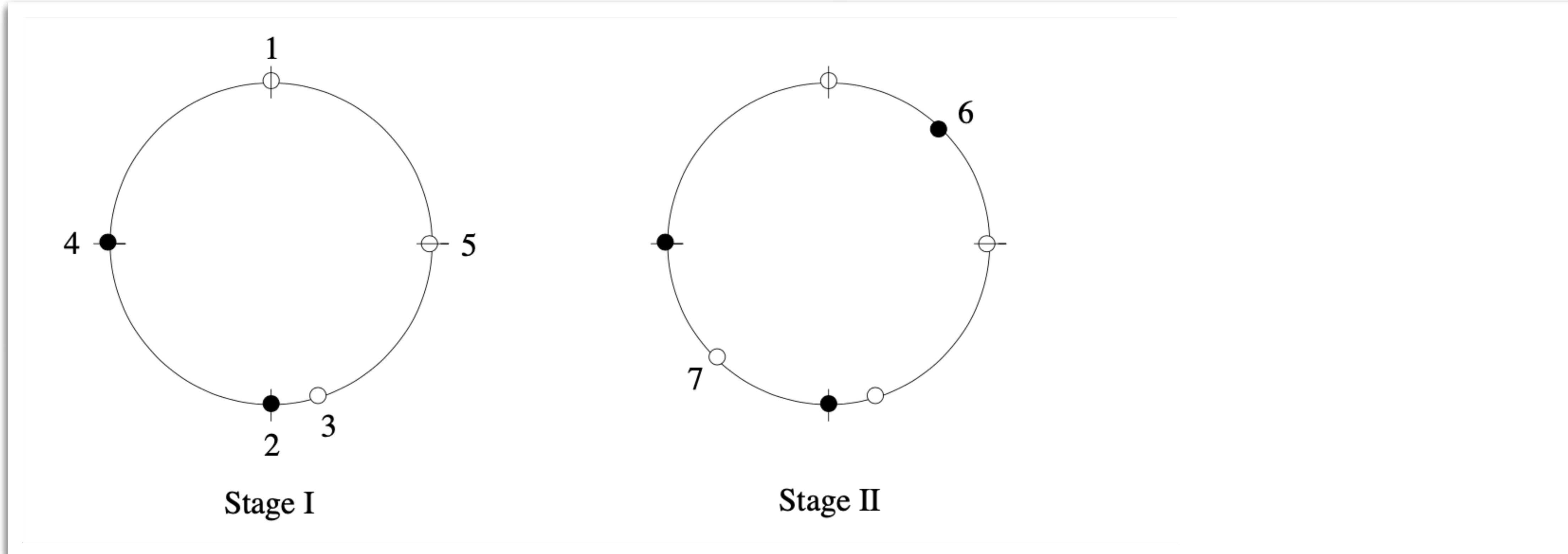

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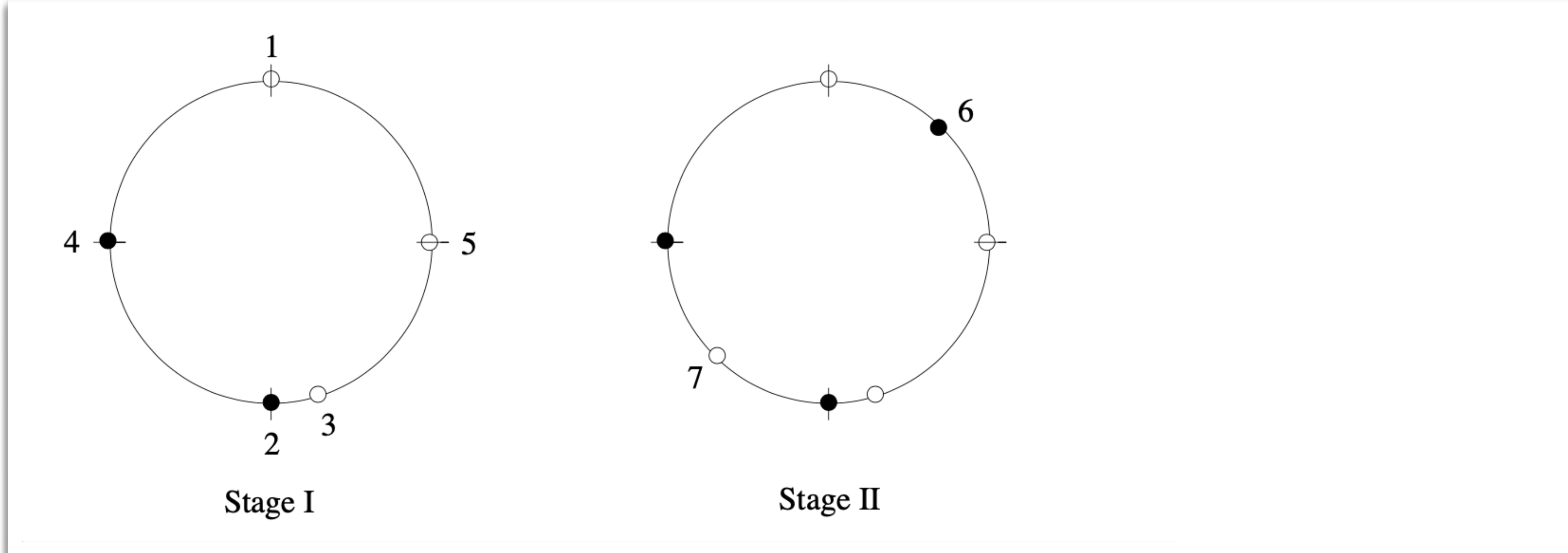

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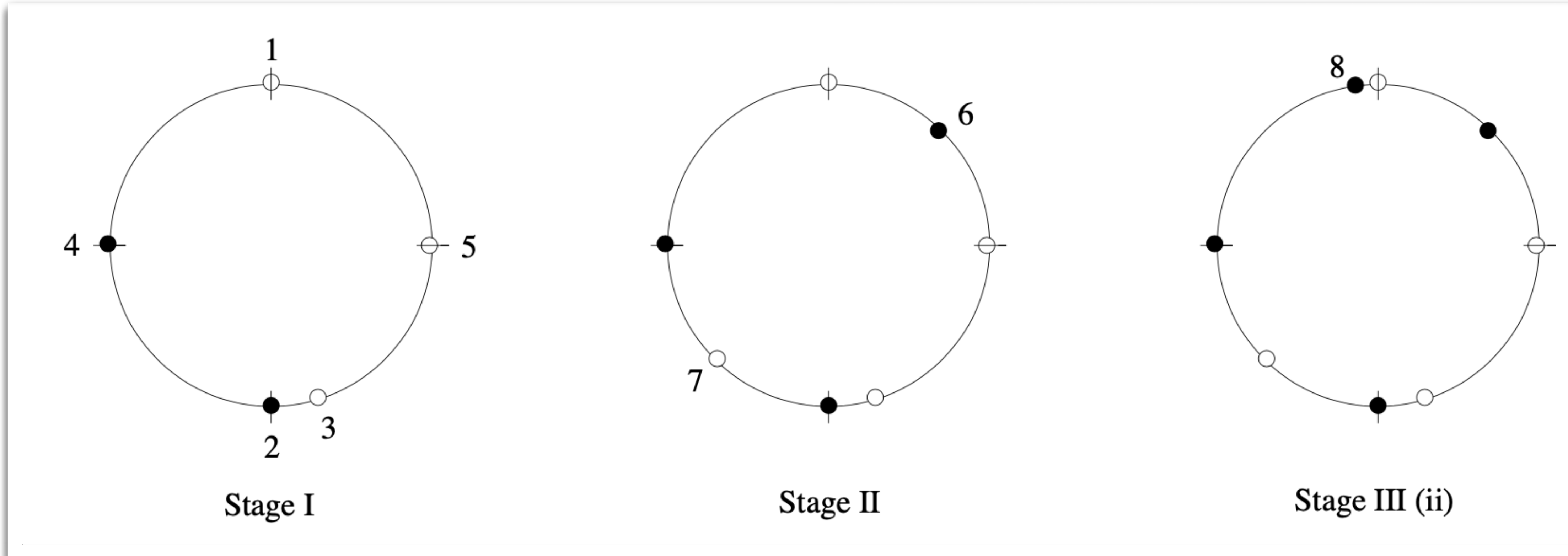

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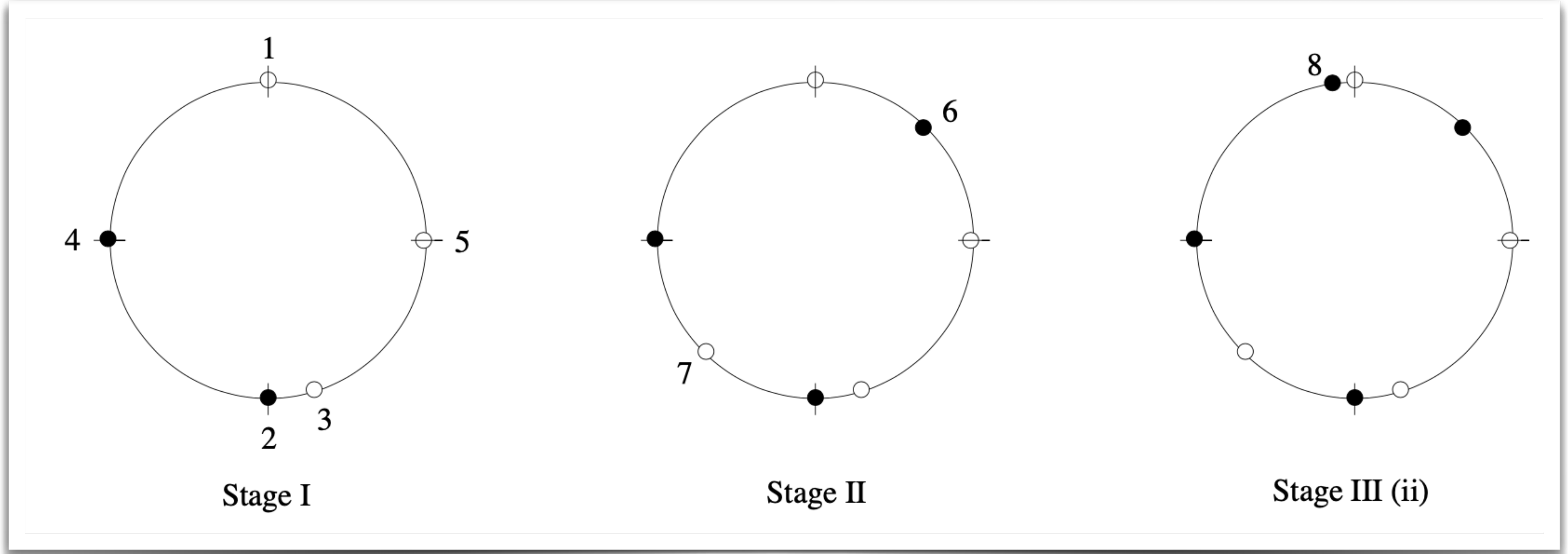

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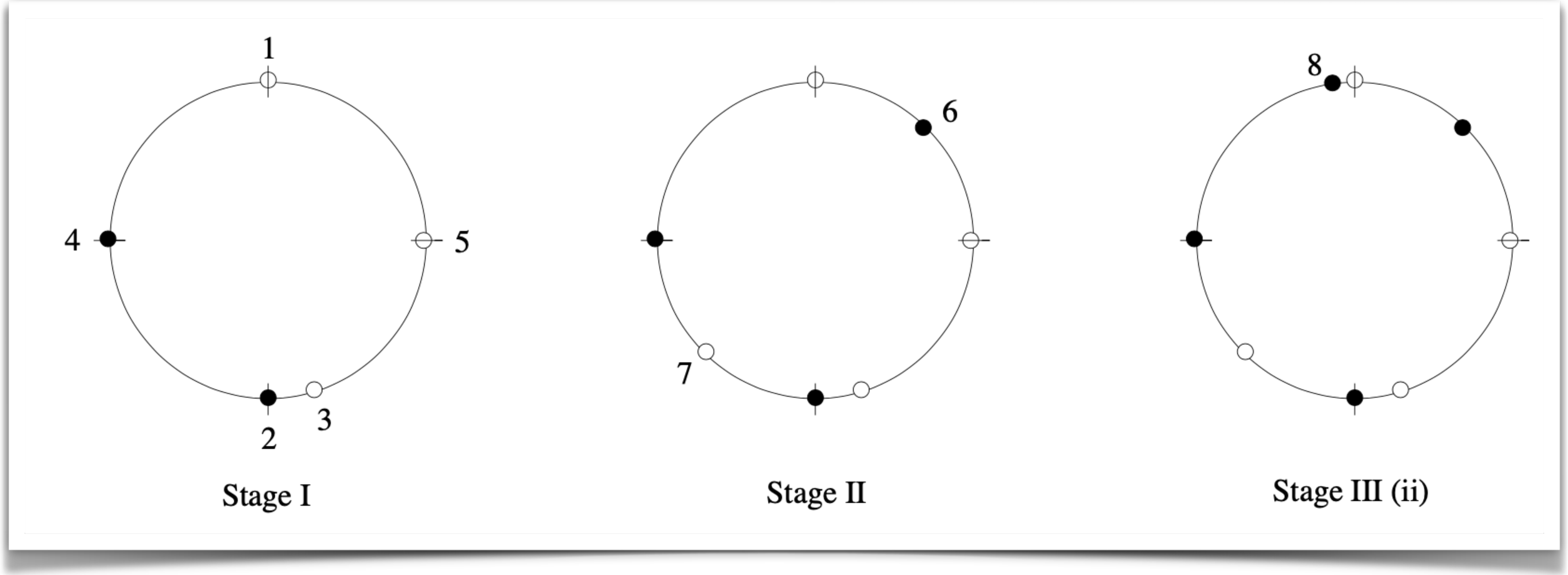

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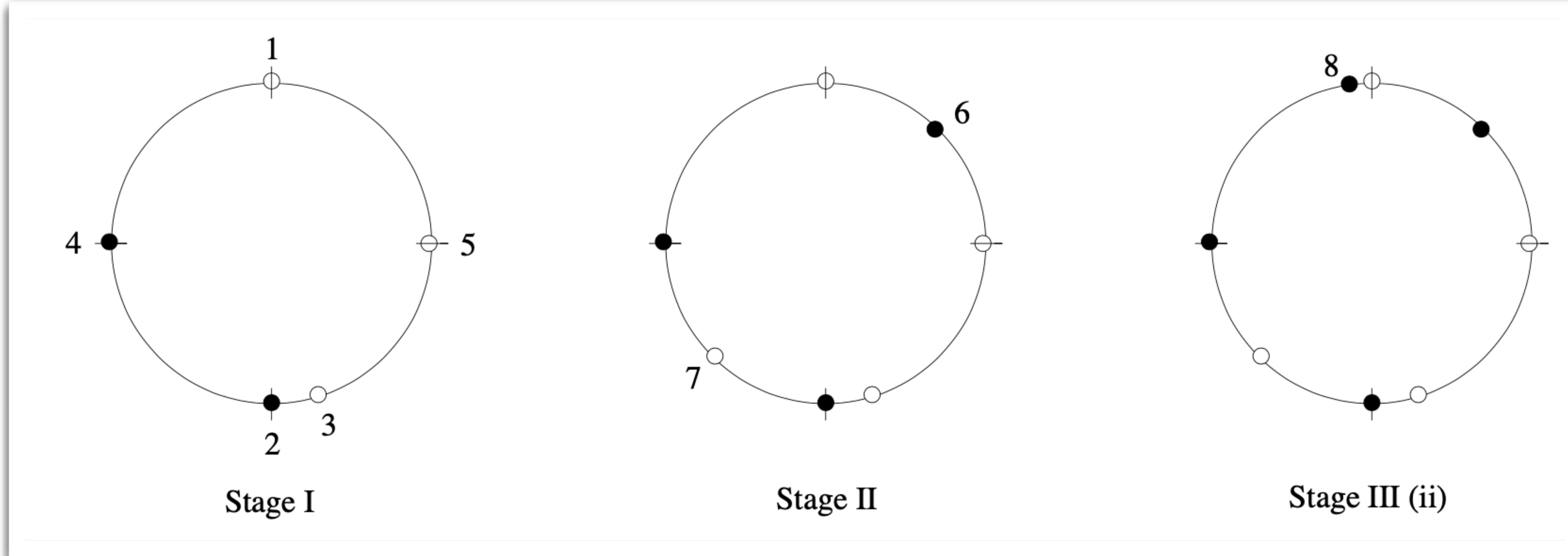
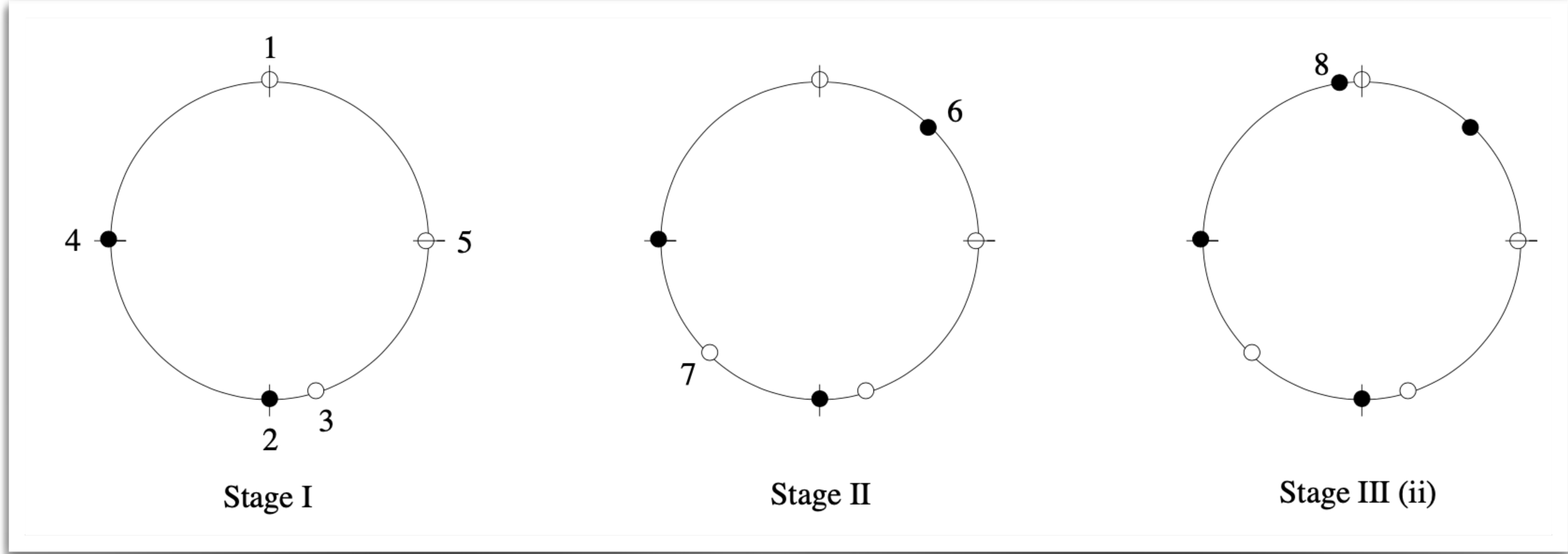


Observations:

- **Playing keypoints ensures that blue intervals are uneven.**



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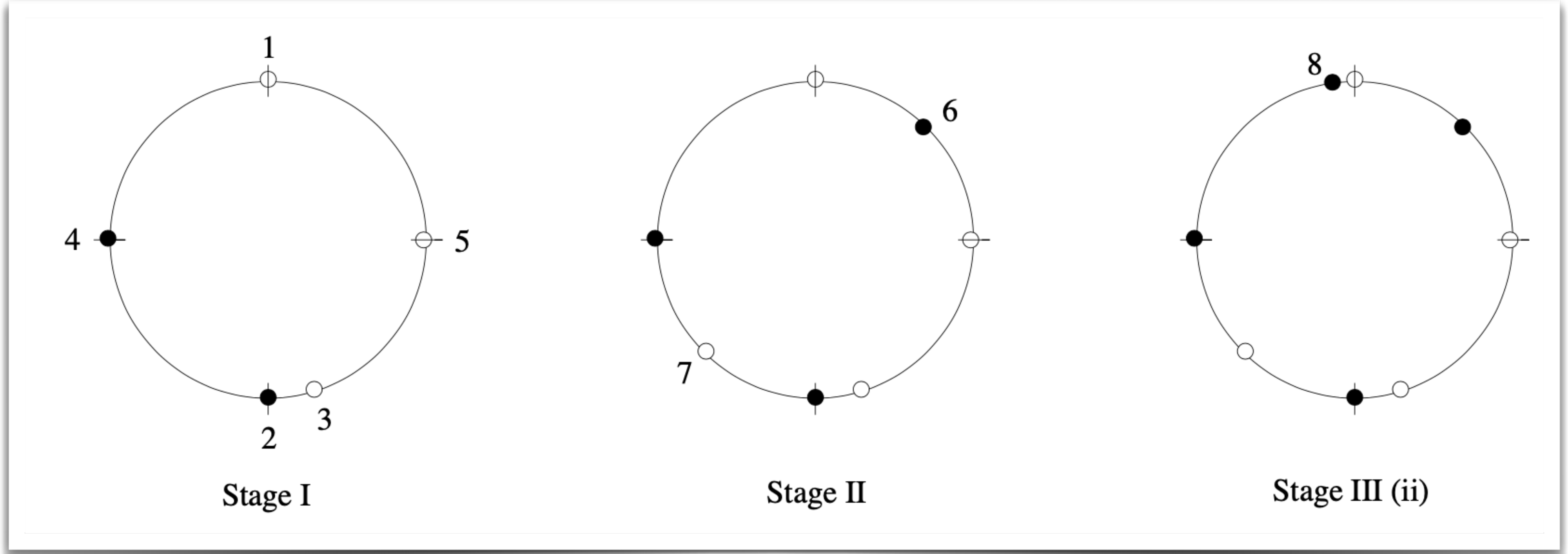

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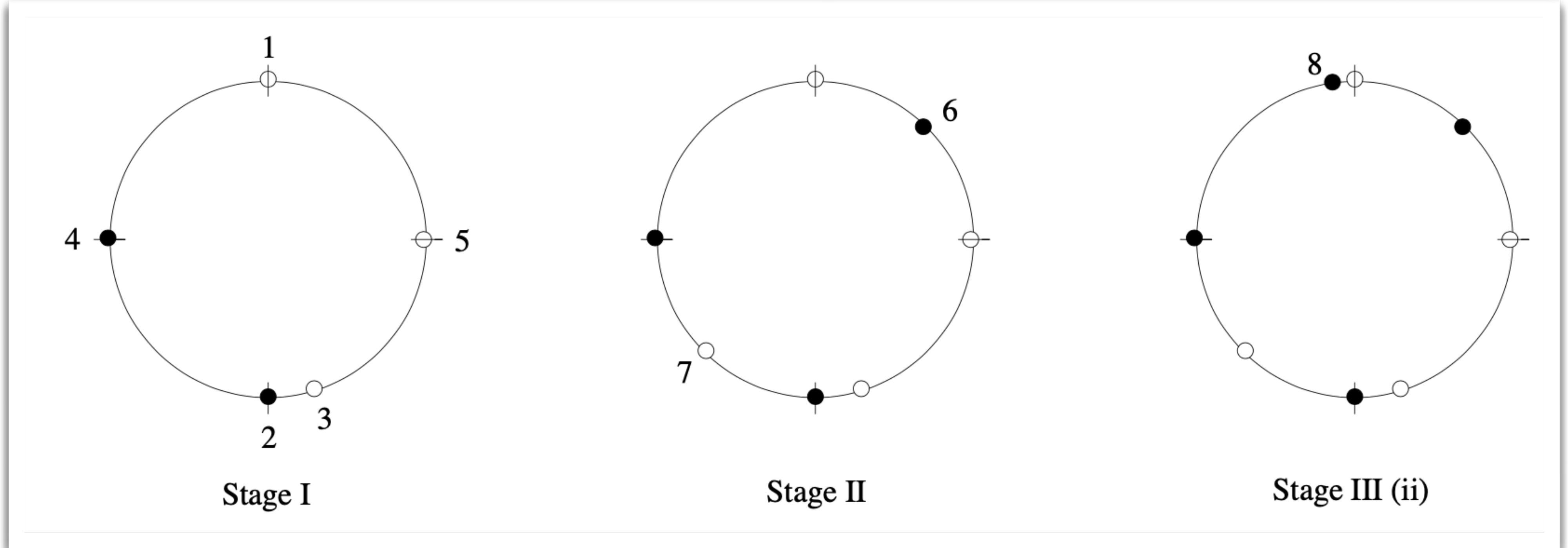

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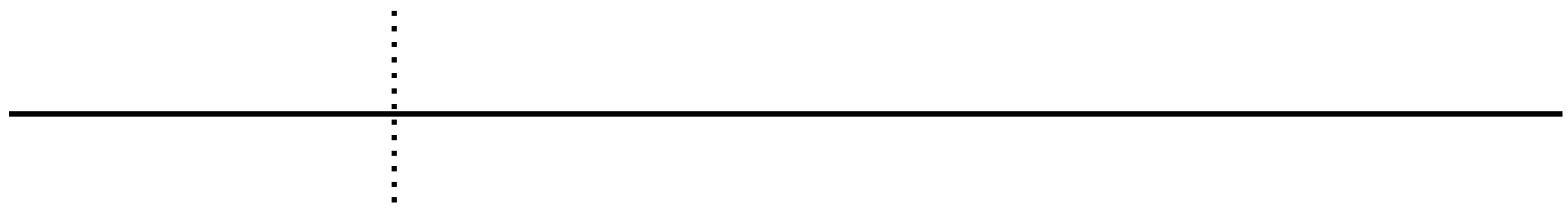

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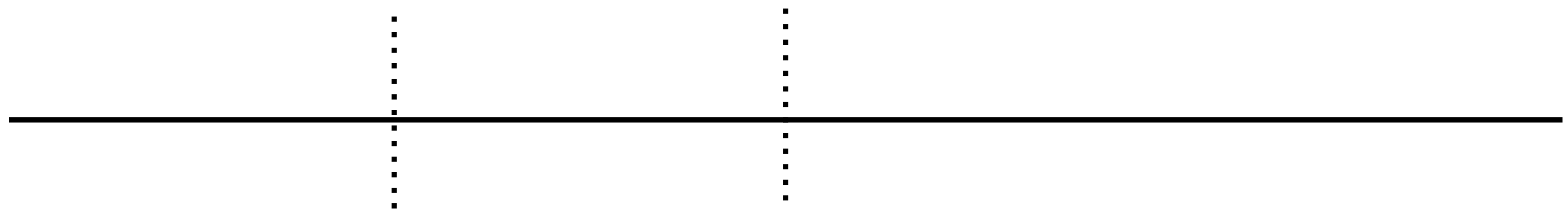

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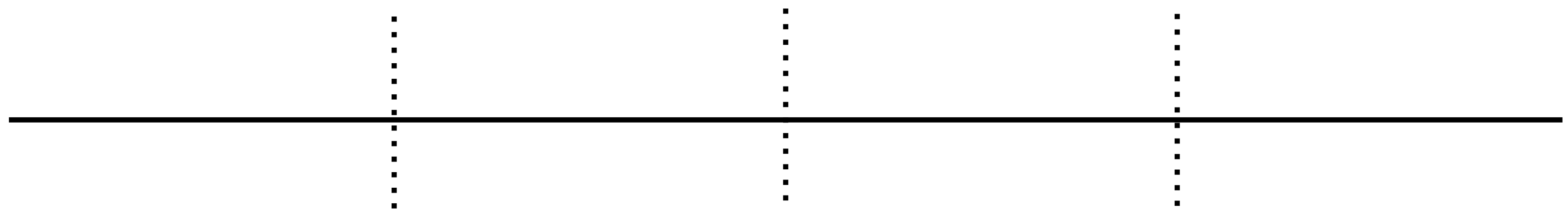

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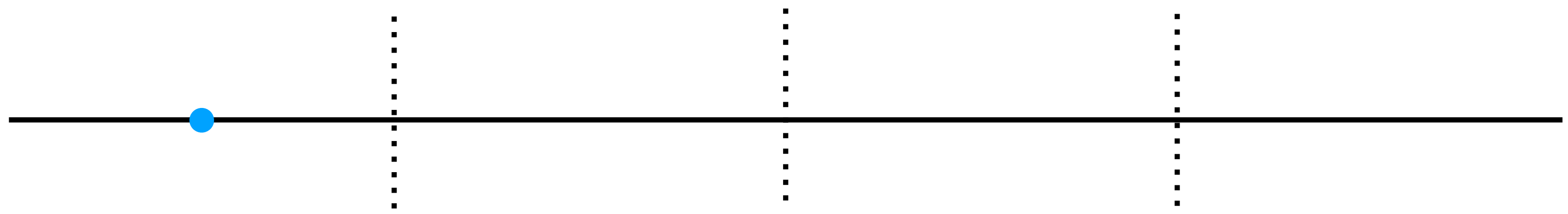

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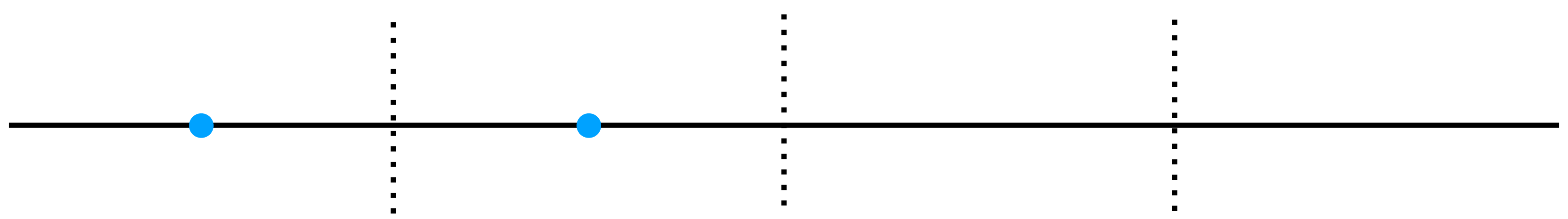

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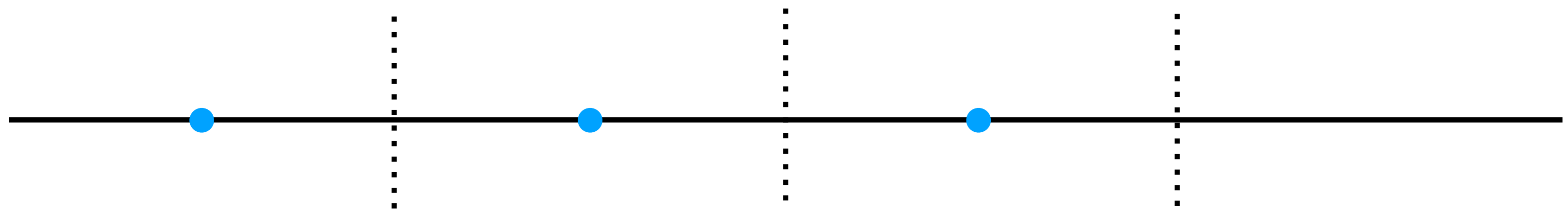

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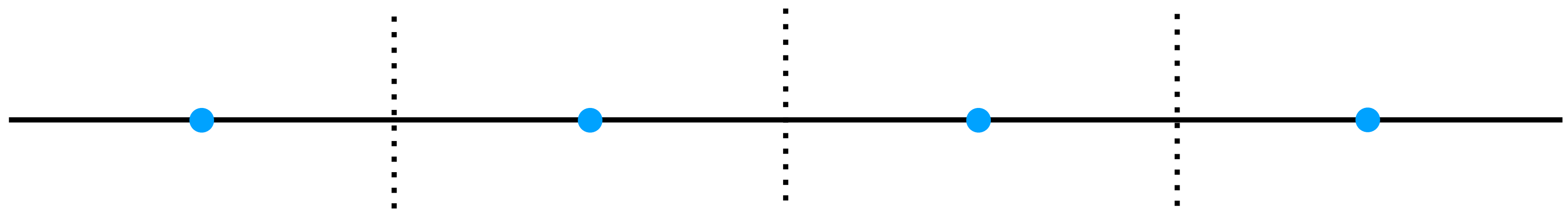

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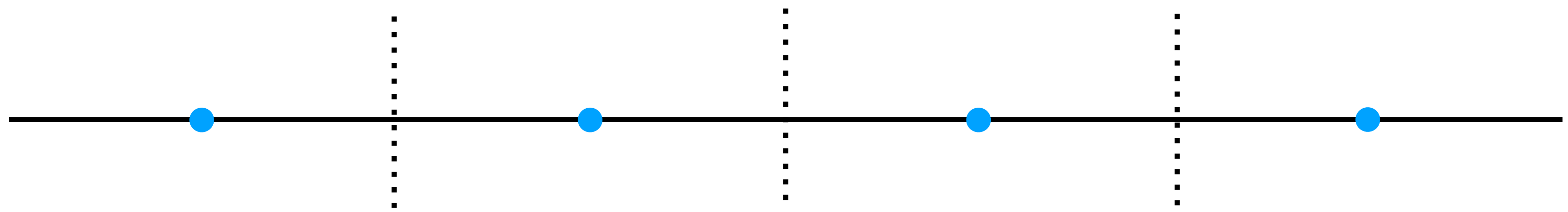

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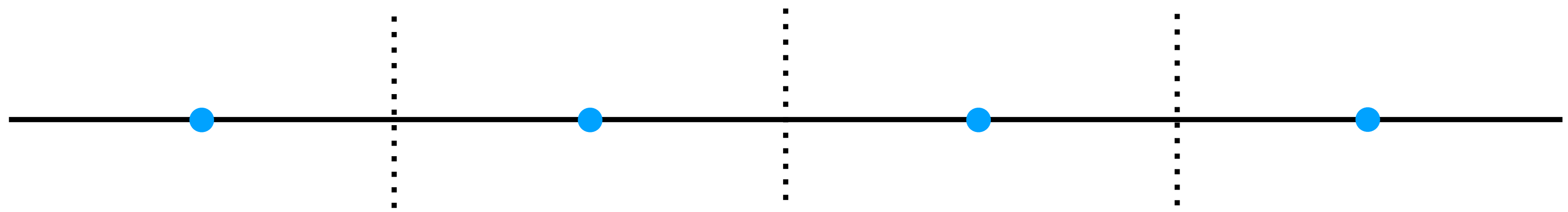

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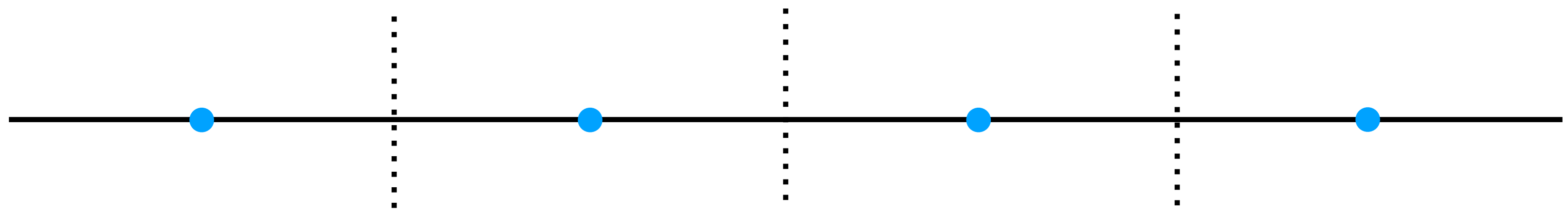

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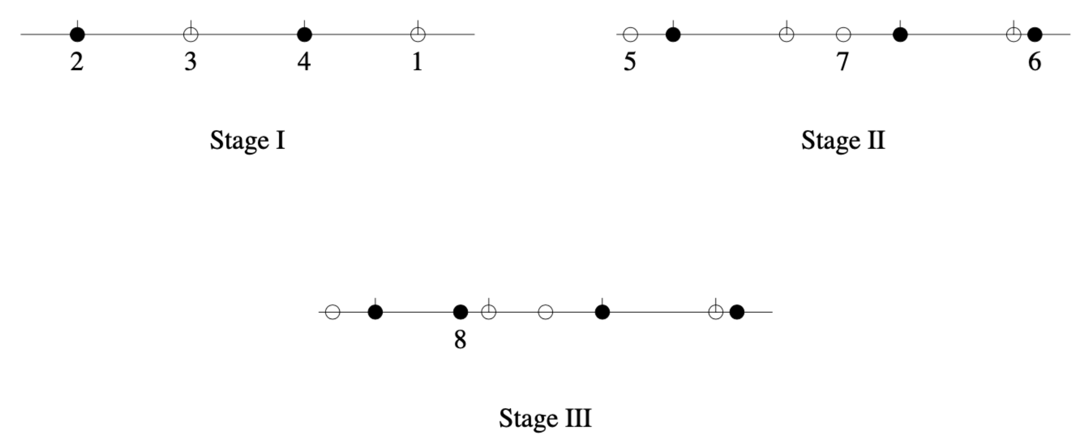

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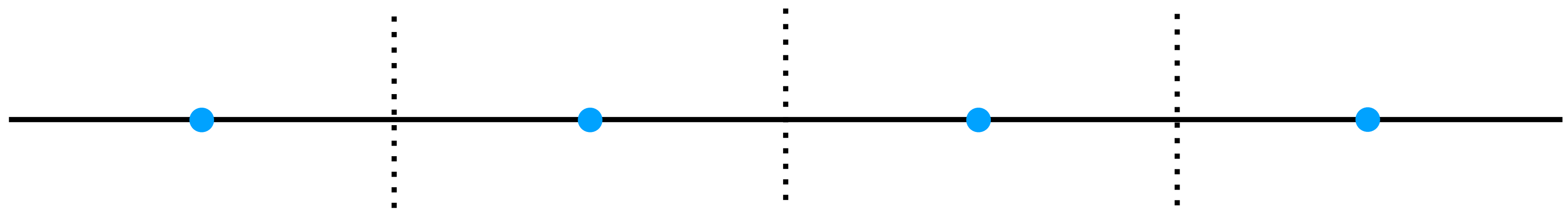
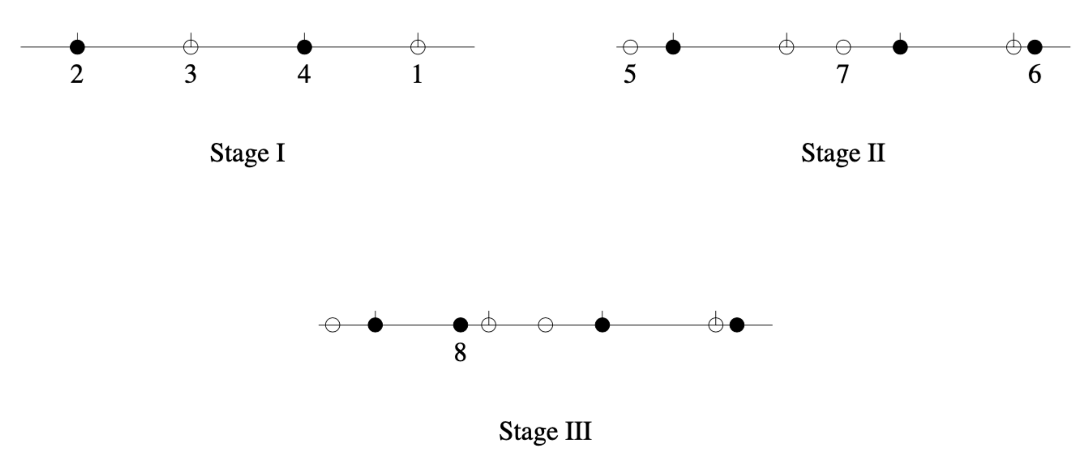
1D Voronoi Game [Ahn, Cheng, Cheong, Golin, van Oostrum 2001]



Definition 2 *The n points $u_i = \frac{1}{2n} + \frac{i}{n}$, $i = 0, 1, \ldots, n-1$ are* **keypoints**.

Modifications for game on a line segment:

- **Keypoints are predefined by breaking line segments into equal pieces.**
- **Distinguish interior and boundary intervals.**



Theorem 2 *The line strategy is a well-defined winning strategy for Red.*

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The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]

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Discrete Comput Geom 31:125–138 (2004)
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The One-Round Voronoi Game*

Otfried Cheong,[1] Sariel Har-Peled,[2] Nathan Linial,[3] and Jiří Matoušek[4]

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Abstract. In the one-round Voronoi game, the first player chooses an n-point set $\mathcal{W}$ in a square Q, and then the second player places another n-point set $\mathcal{B}$ into Q. The payoff for the second player is the fraction of the area of Q occupied by the regions of the points of $\mathcal{B}$ in the Voronoi diagram of $\mathcal{W} \cup \mathcal{B}$. We give a (randomized) strategy for the second player that always guarantees him a payoff of at least $\frac{1}{2} + \alpha$, for a constant $\alpha > 0$ and every large enough n. This contrasts with the one-dimensional situation, with $Q = [0, 1]$, where the first player can always win more than $\frac{1}{2}$.

1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

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- **A square**

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- A square
- Two players, White and Black

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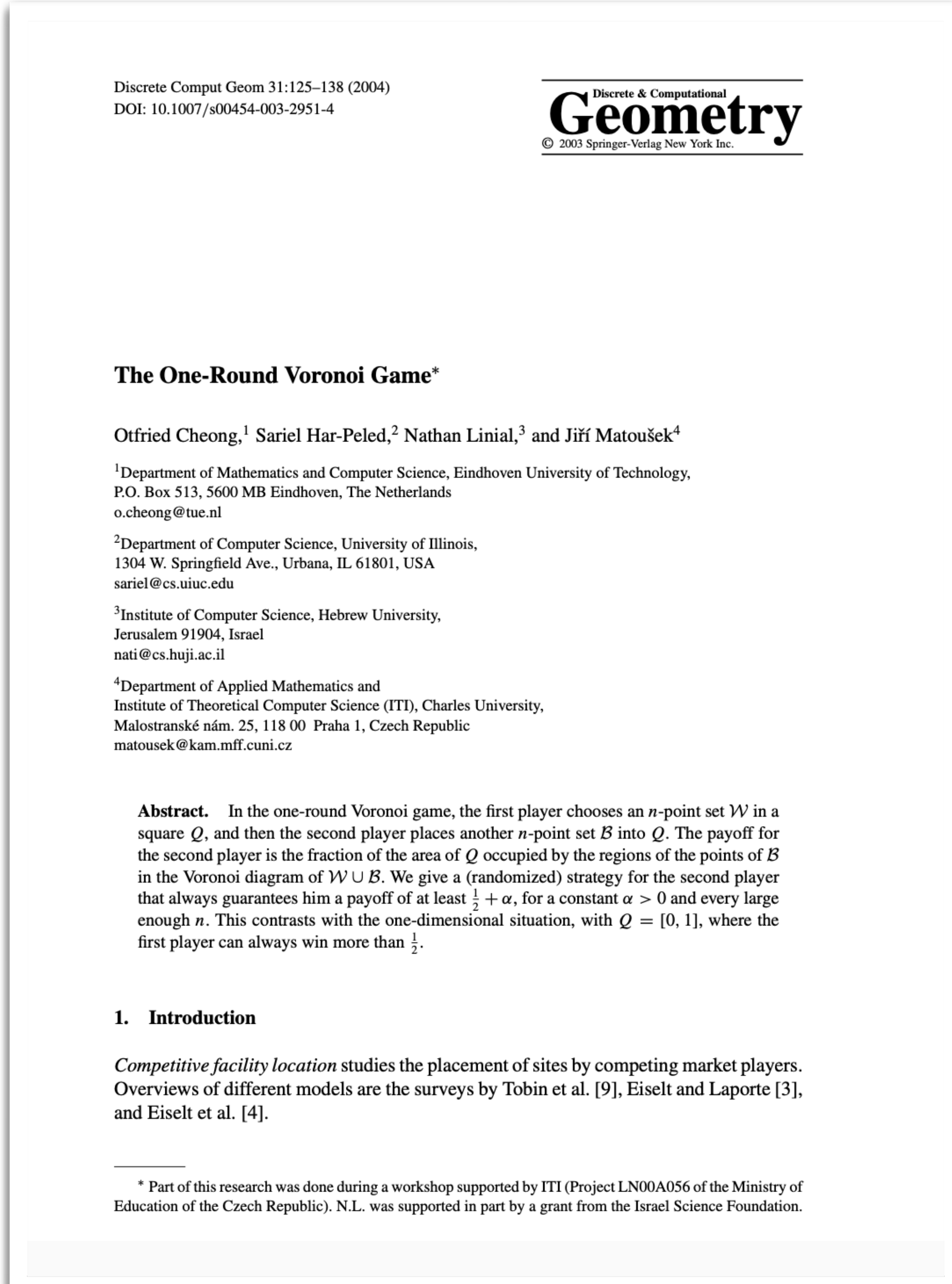
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- A square
- Two players, White and Black
- Players place all points at once

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- A square
- Two players, White and Black
- Players place all points at once
- Voronoi diagram is computed

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- A square
- Two players, White and Black
- Players place all points at once
- Voronoi diagram is computed
- Player with larger area wins

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]

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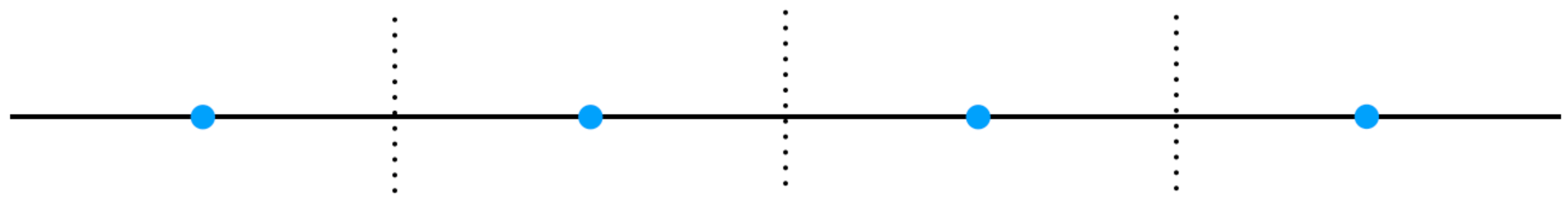
The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



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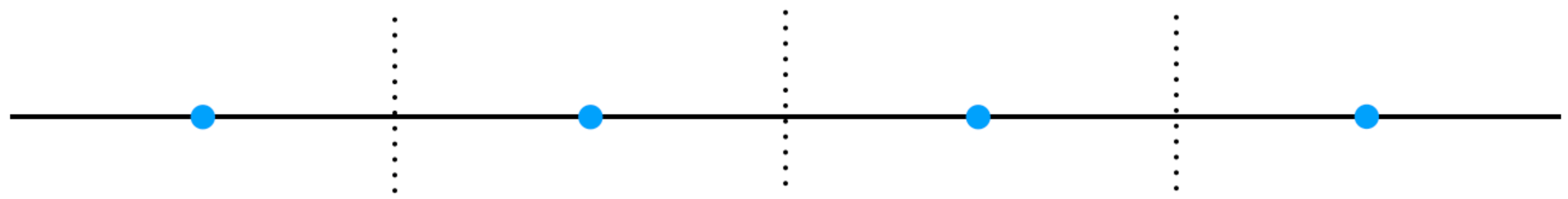

- **1D: First player wins**

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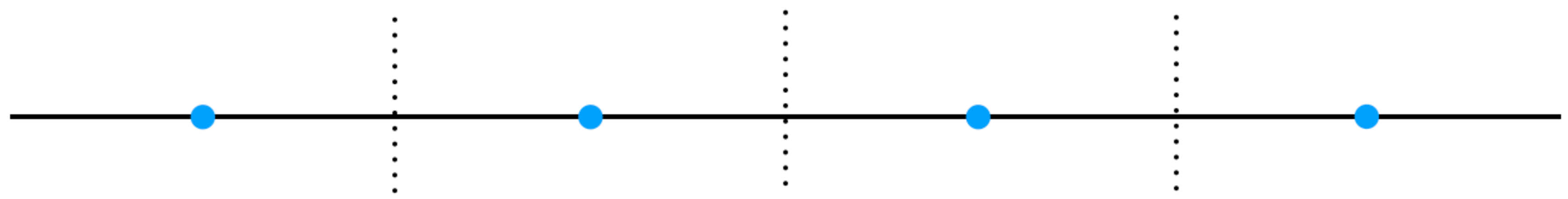

- 1D: First player wins
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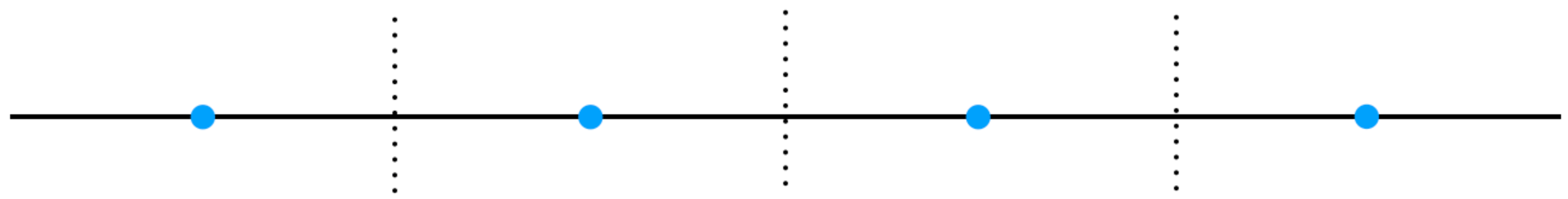

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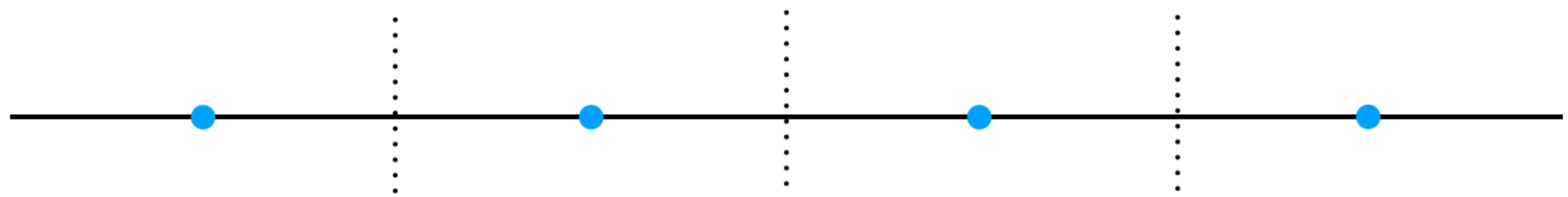
- **Complicated randomized arguments**



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$$\mathbf{E}[\mathrm{vol}(R(\mathcal{B}, \mathcal{W}))] \geq (\tfrac{1}{2} + \beta_1)\delta n.$$

If the total area A_ℓ of the long regions (of diameter at least D) exceeds $n/2D$, then

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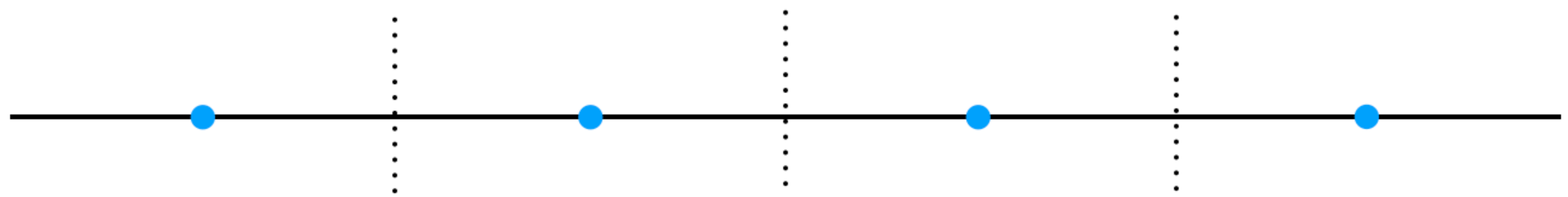

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$$\mathbf{E}[\mathrm{vol}(R(x, \mathcal{W}))] = \frac{1}{\mathrm{vol}(Q)} \int_Q \int_Q I_{R(x,\mathcal{W})}(y)\, \mathrm{d}y\, \mathrm{d}x = \frac{1}{n} \int_Q \mathrm{vol}(\{x \in Q : y \in R(x, \mathcal{W})\})\, \mathrm{d}y$$

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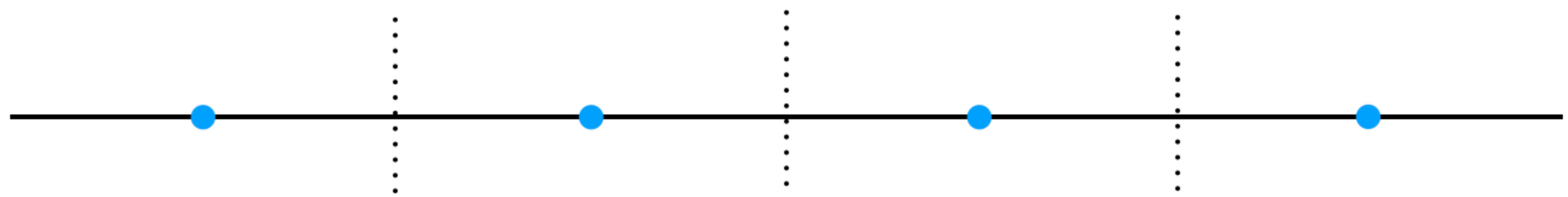

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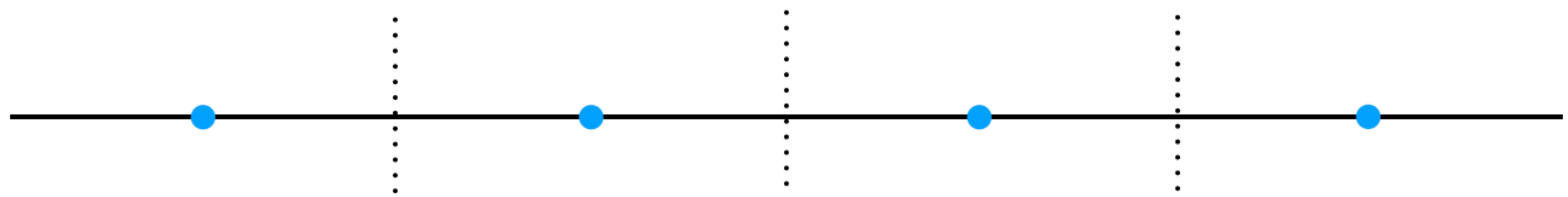

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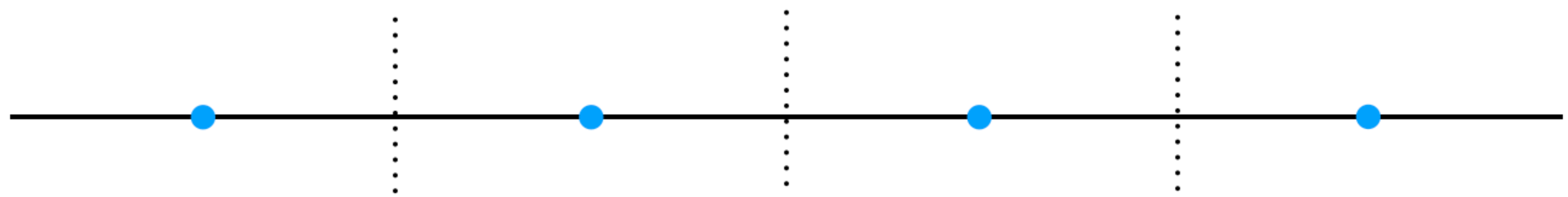

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$$\begin{aligned}P(y) &= \mathrm{Prob}\,[\mathcal{B} \cap B(y, \mathrm{dist}(y, \mathcal{W})) \neq \emptyset] \\ &= 1 - (\mathrm{Prob}\,[x \notin B(y, \mathrm{dist}(y, \mathcal{W}))])^{\delta n} \\ &= 1 - \left(1 - \frac{1}{n} \cdot \mathrm{vol}(B(y, \mathrm{dist}(y, \mathcal{W})) \cap Q)\right)^{\delta n}.\end{aligned}$$

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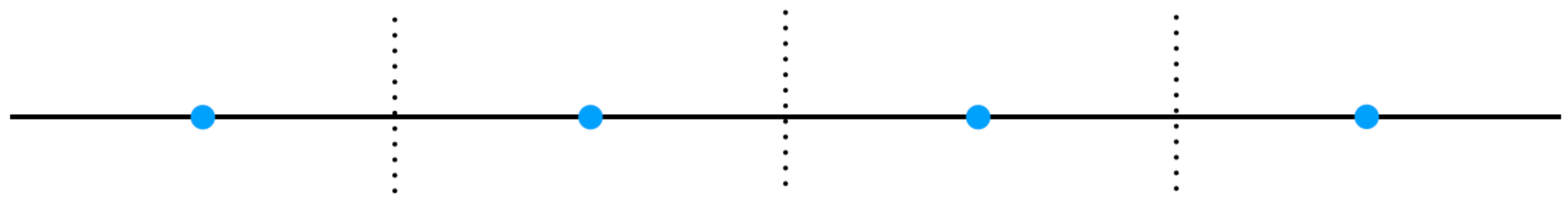

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Theorem 5. *There exist constants $\alpha > 0$ and n_0 such that for every $n \geq n_0$, Black can always win at least $\frac{1}{2} + \alpha$ in the Voronoi game. That is, for every n-point set $\mathcal{W} \subset Q$ there exists an n-point set $\mathcal{B} \subset Q \setminus \mathcal{W}$ with $\mathrm{vol}(R(\mathcal{B}, \mathcal{W})) \geq (\frac{1}{2} + \alpha)\,\mathrm{vol}(Q)$.*

$$\mathbf{E}[\mathrm{vol}(R(x, \mathcal{W}))] = \frac{1}{\mathrm{vol}(Q)} \int_Q \int_Q I_{R(x,\mathcal{W})}(y)\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{n} \int_Q \mathrm{vol}(\{x \in Q : y \in R(x, \mathcal{W})\})\,\mathrm{d}y$$

$$\int_{C_0} \mathrm{dist}(y, w)^2\,\mathrm{d}y = \int_0^{\sqrt{a/\pi}} r^2 \cdot 2\pi r\,\mathrm{d}r = \frac{a^2}{2\pi}.$$

$$F_0(\mathcal{W}) = \frac{\pi}{n} \sum_{w \in \mathcal{W}} \int_{\mathrm{cell}_{\mathcal{W}}(w)} \mathrm{dist}(y, w)^2\,\mathrm{d}y \geq \frac{1}{2n} \sum_{w \in \mathcal{W}} a_w^2 \geq \frac{1}{2n} \frac{\left(\sum_{w \in \mathcal{W}} a_w\right)^2}{n} \geq \frac{1}{2}$$

$$P(y) = \mathrm{Prob}\,[\mathcal{B} \cap B(y, \mathrm{dist}(y, \mathcal{W})) \neq \emptyset] = 1 - (\mathrm{Prob}\,[x \notin B(y, \mathrm{dist}(y, \mathcal{W}))])^{\delta n} = 1 - \left(1 - \frac{1}{n} \cdot \mathrm{vol}(B(y, \mathrm{dist}(y, \mathcal{W})) \cap Q)\right)^{\delta n}.$$

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]

Discrete Comput Geom 31:125–138 (2004)
DOI: 10.1007/s00454-003-2951-4

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The One-Round Voronoi Game*

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1. Introduction

Competitive facility location studies the placement of sites by competing market players. Overviews of different models are the surveys by Tobin et al. [9], Eiselt and Laporte [3], and Eiselt et al. [4].

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Open:

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Open:

- **White wins for $n=1$, Black for large n**
- **White wins for 1D, Black for 2D**

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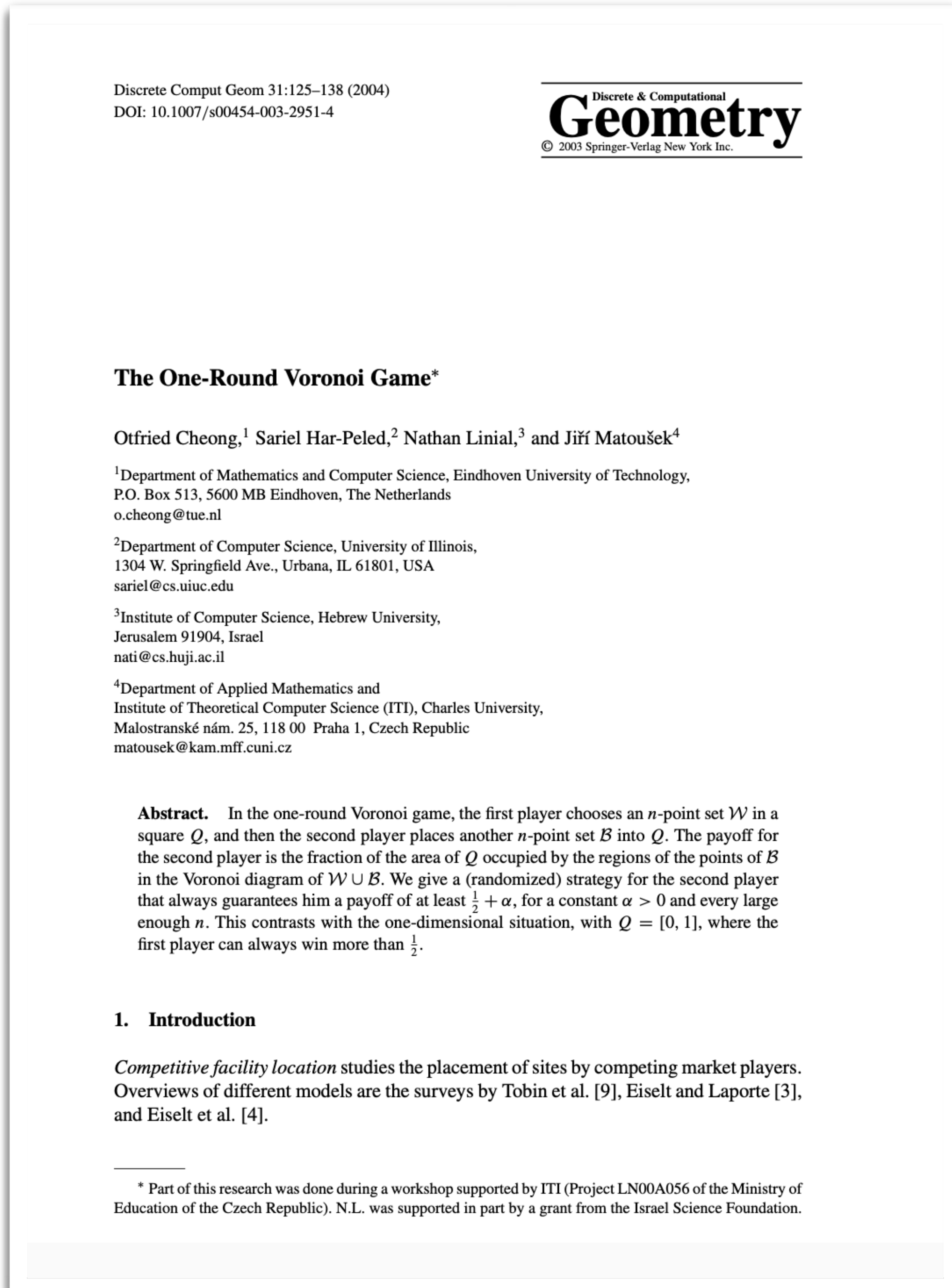

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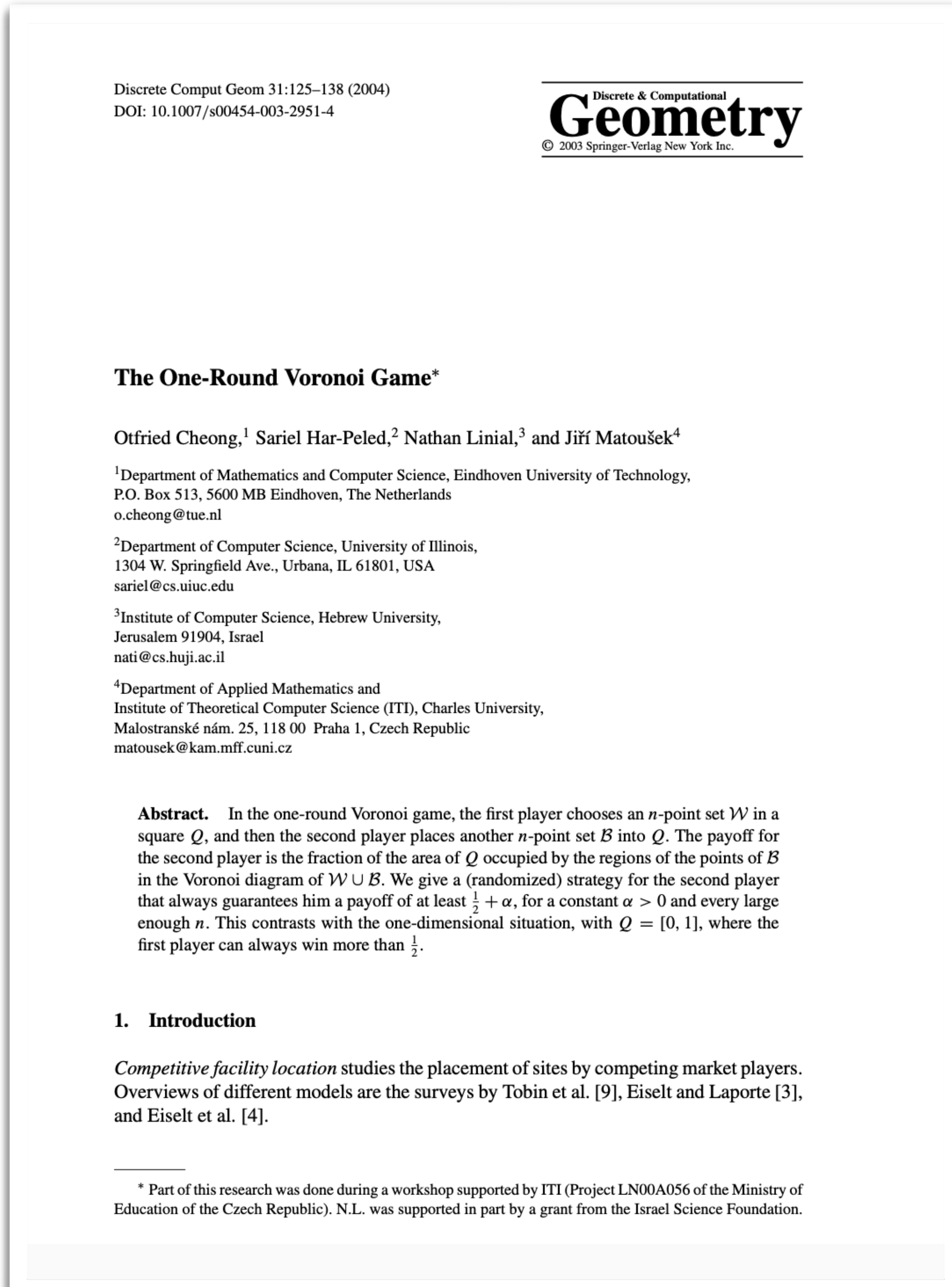
Open:

- **White wins for *n=1*, Black for large *n***
- **White wins for 1D, Black for 2D**
- **Strategy uses randomization**

The One-Round Voronoi Game [Cheong, Har-Peled, Linial, Matousek 2002/2004]



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Open:

- White wins for n=1, Black for large n
- White wins for 1D, Black for 2D
- Strategy uses randomization
- Explain and find simpler strategy!

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The one-round Voronoi game replayed ☆

Sándor P. Fekete [a,*], Henk Meijer [b,1]

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Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leqslant 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of the board closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

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Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

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The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

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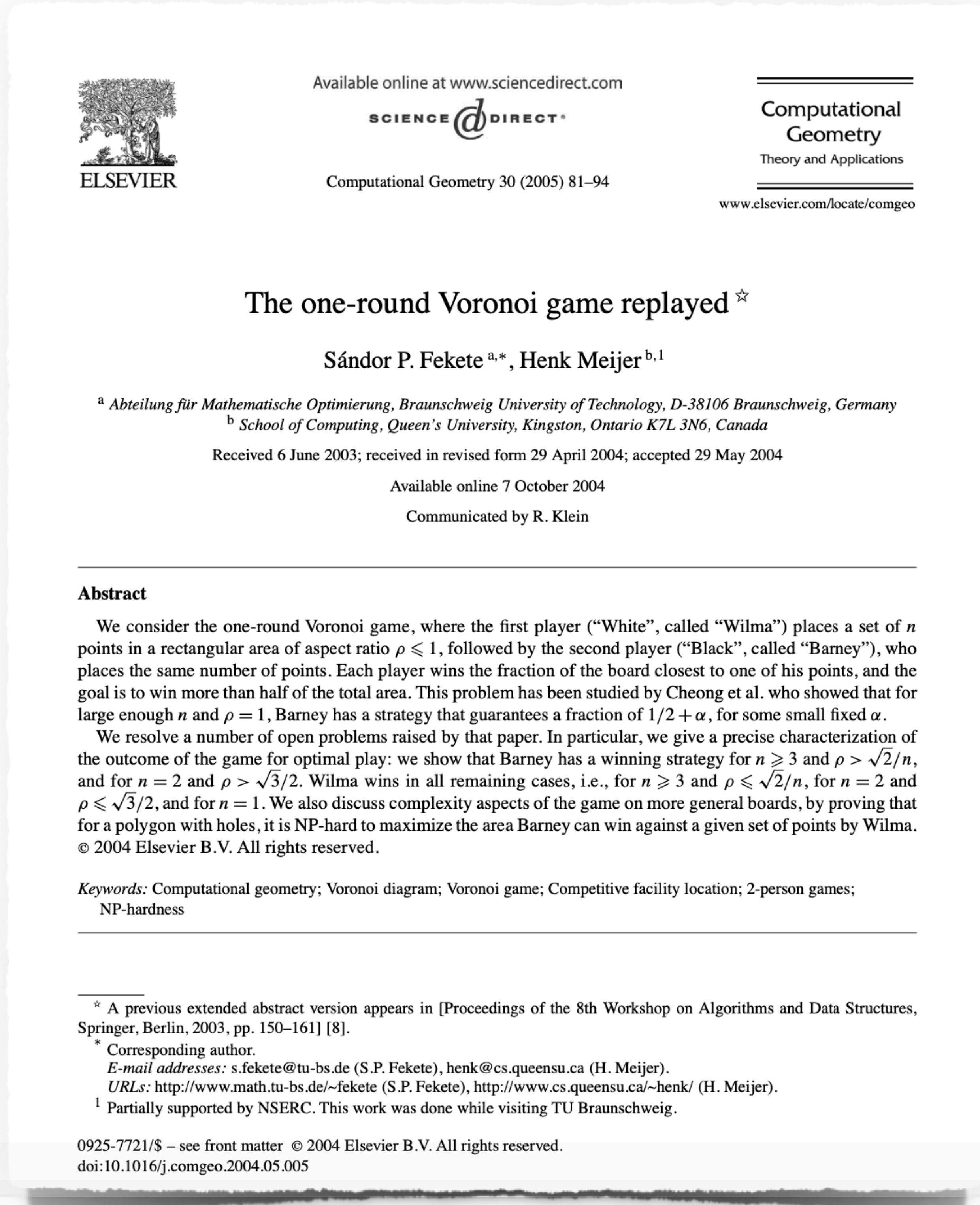
Insights:

- **Consider rectangle**

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]



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Insights:

- **Consider rectangle**
- **Outcome depends on aspect ratio**

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

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Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

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Insights:

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The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

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The one-round Voronoi game replayed ☆

Sándor P. Fekete [a,*], Henk Meijer [b,1]

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[b] *School of Computing, Queen's University, Kingston, Ontario K7L 3N6, Canada*

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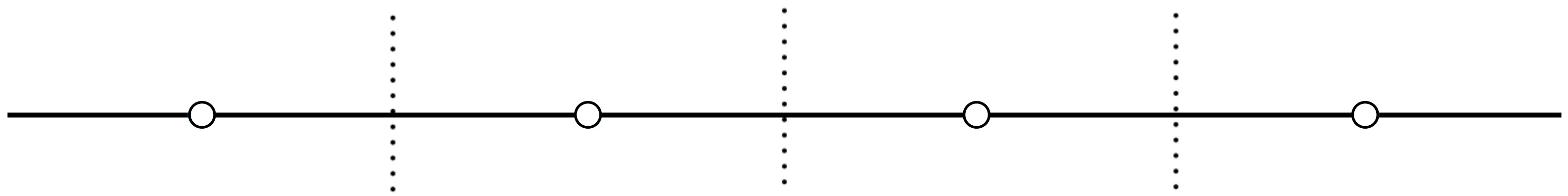

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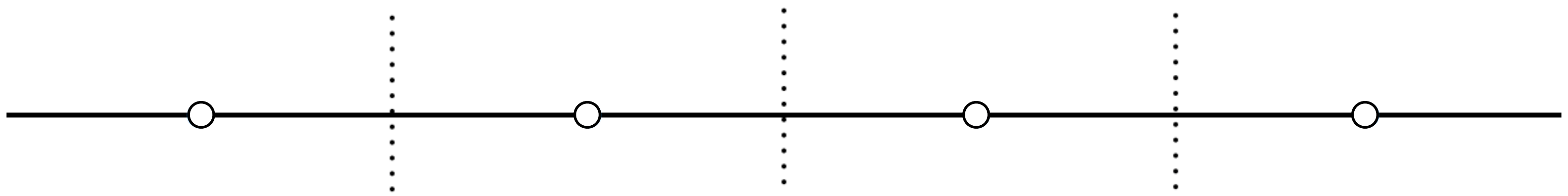

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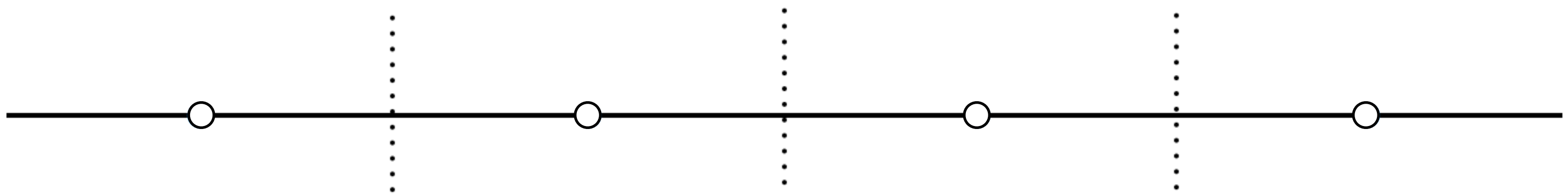

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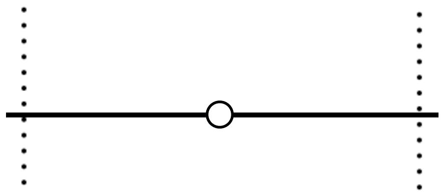

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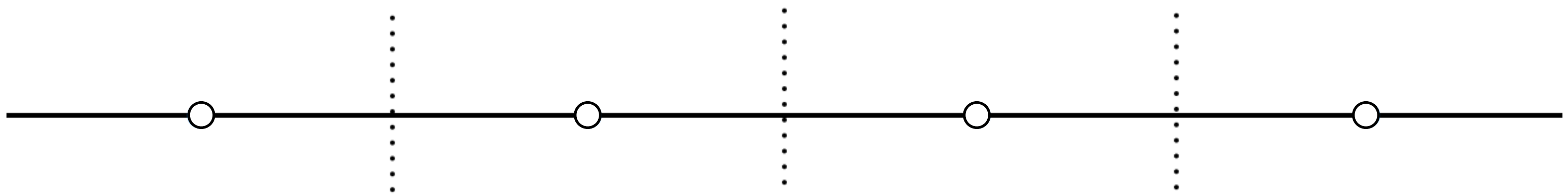

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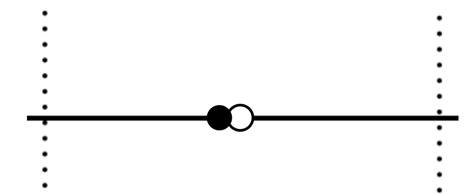

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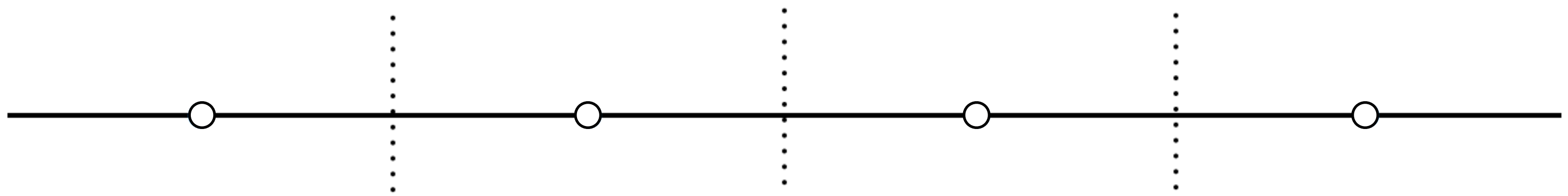

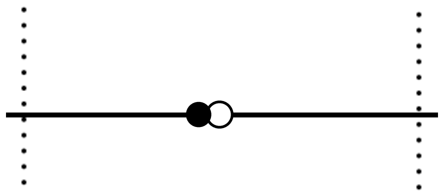

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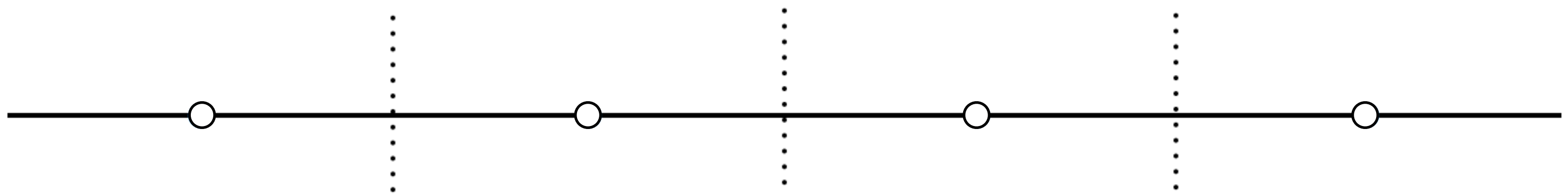

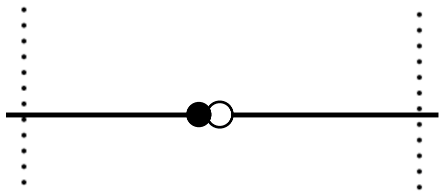

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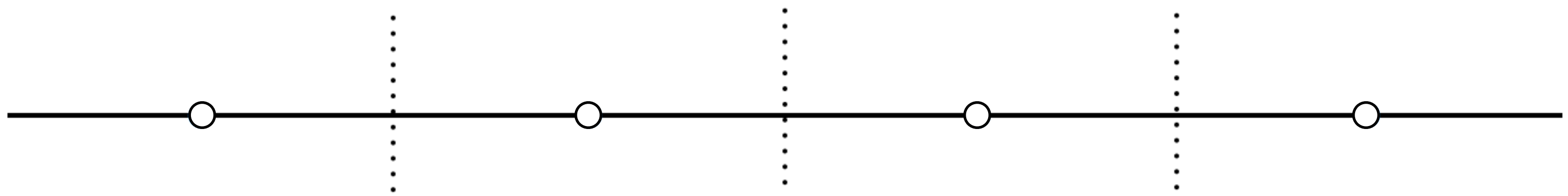

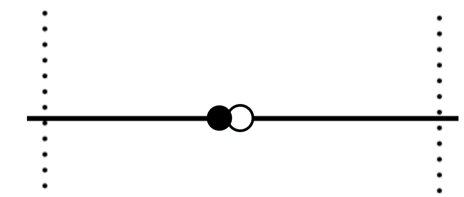

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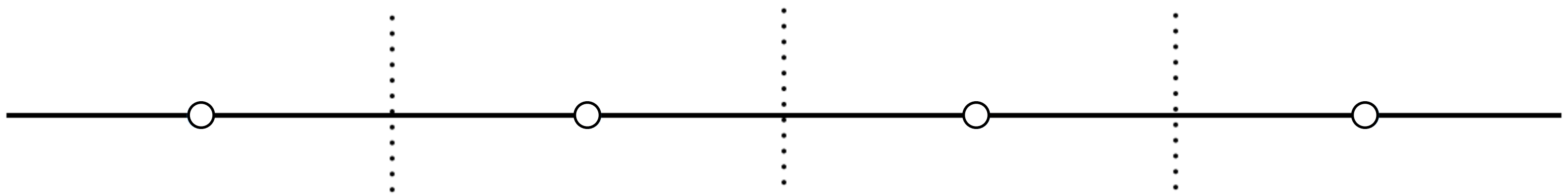

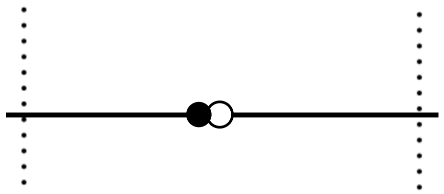

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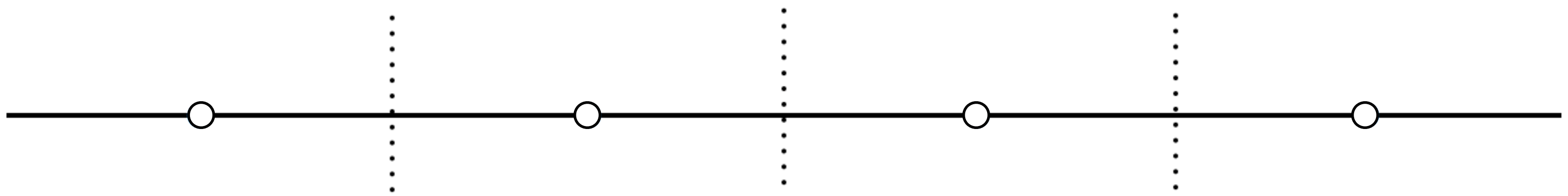

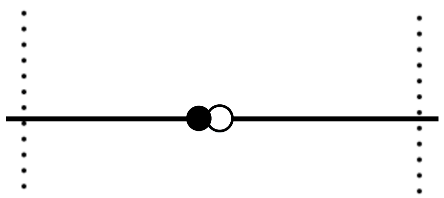

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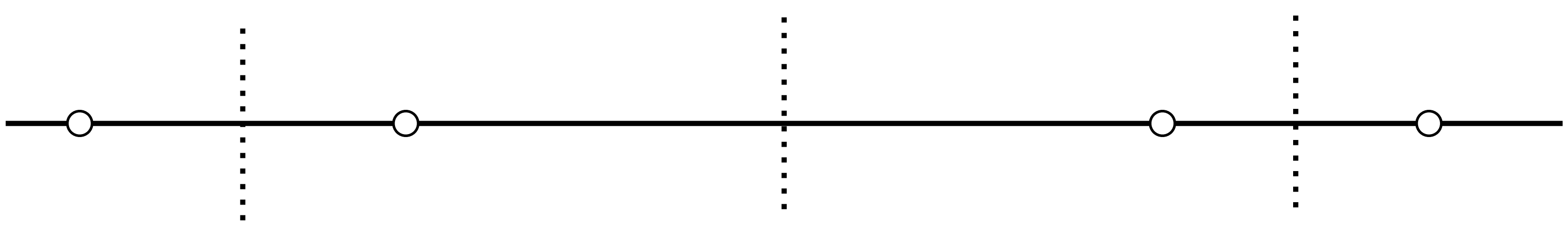

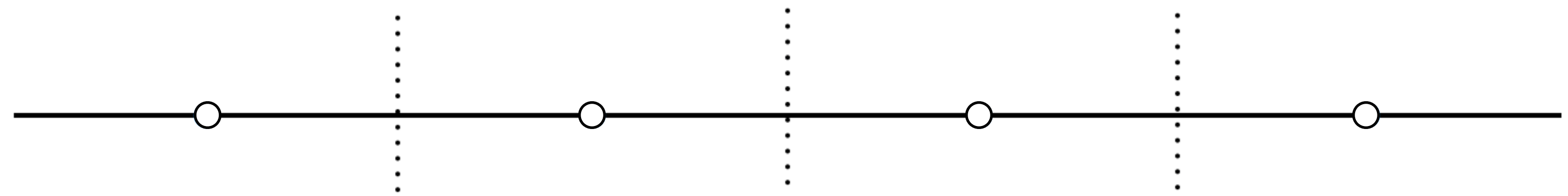

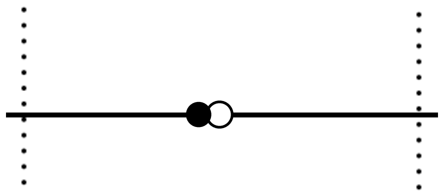

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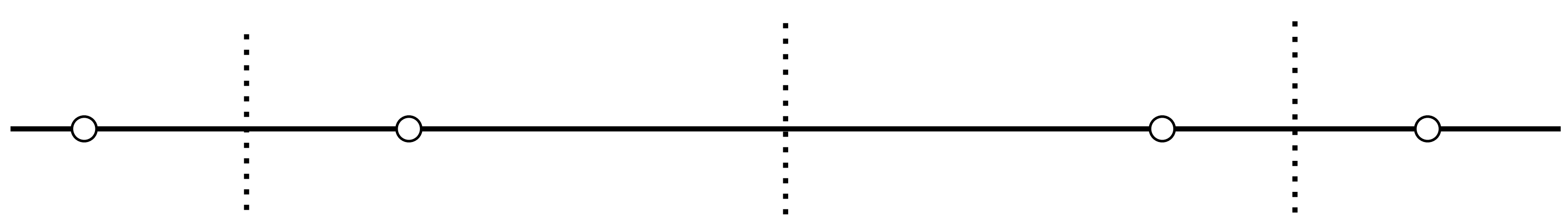

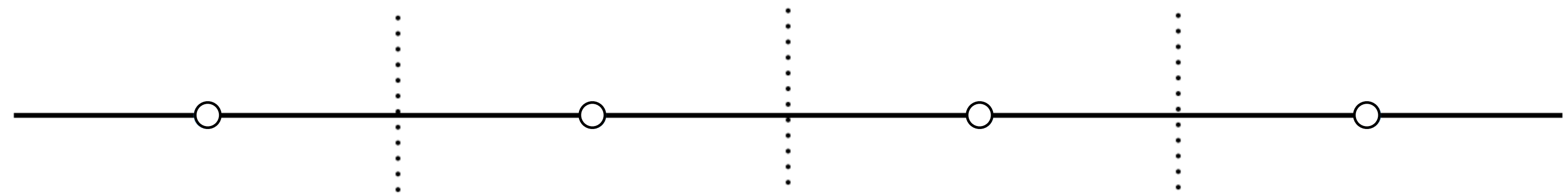

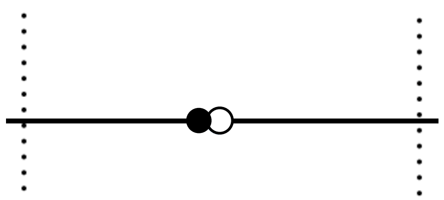

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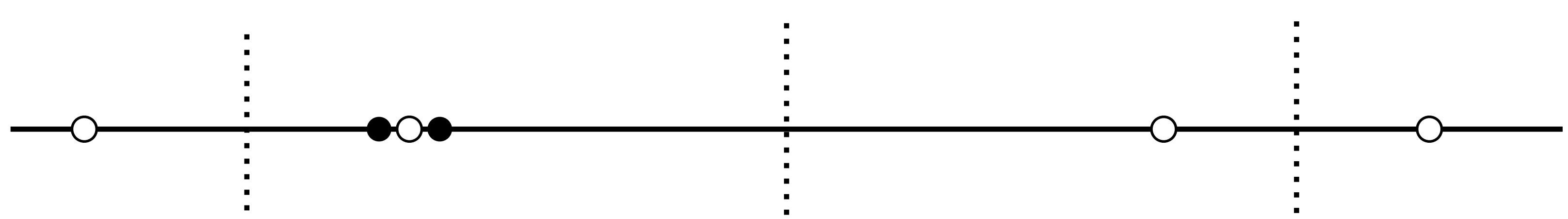

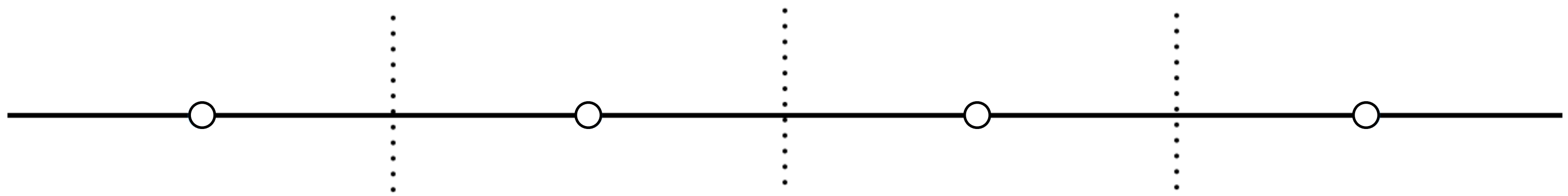

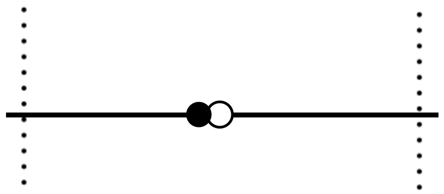

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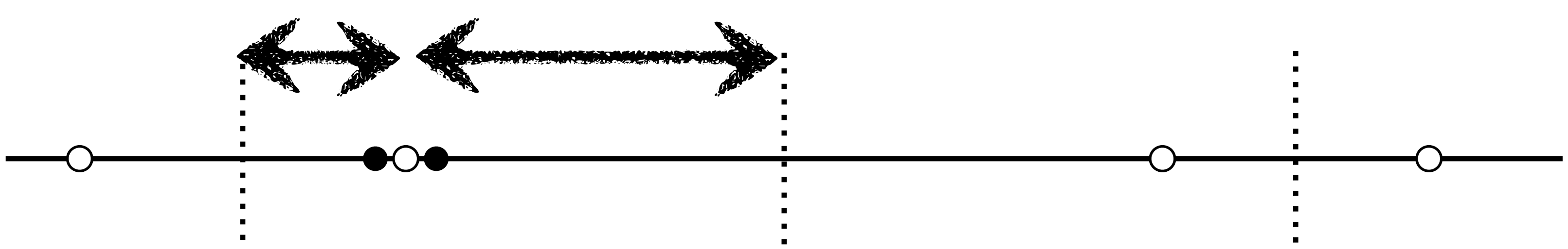

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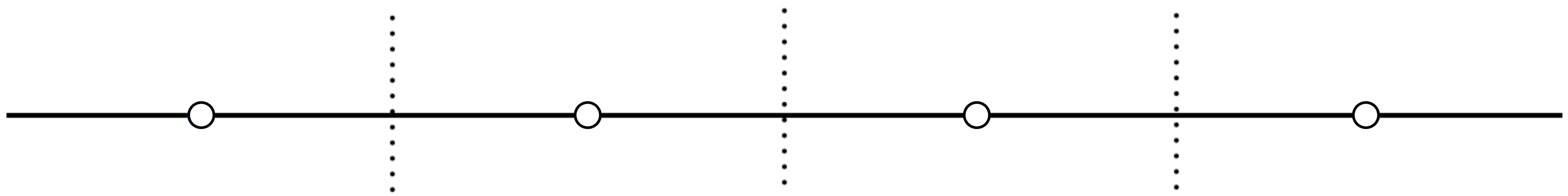

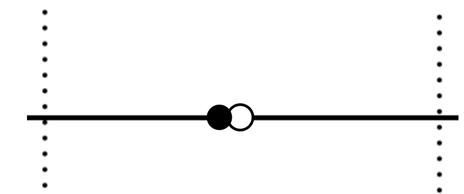

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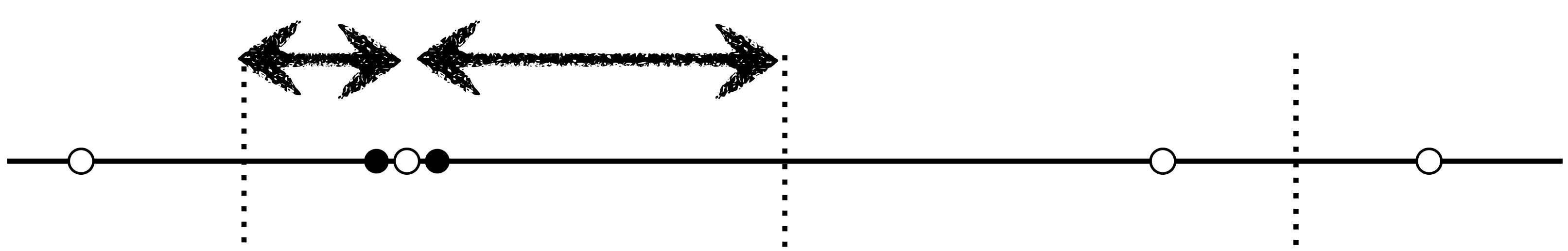

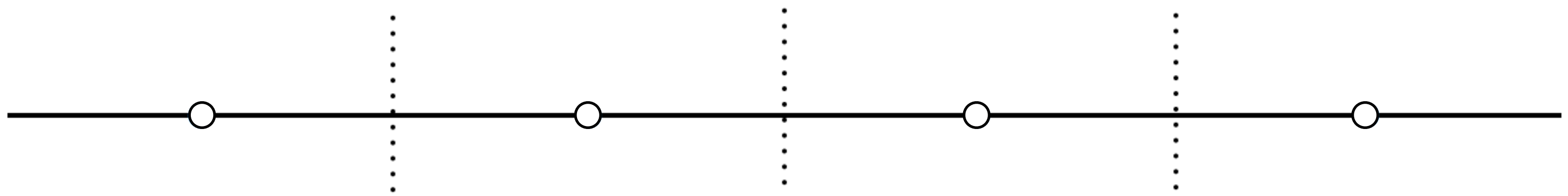

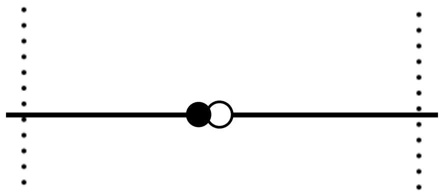

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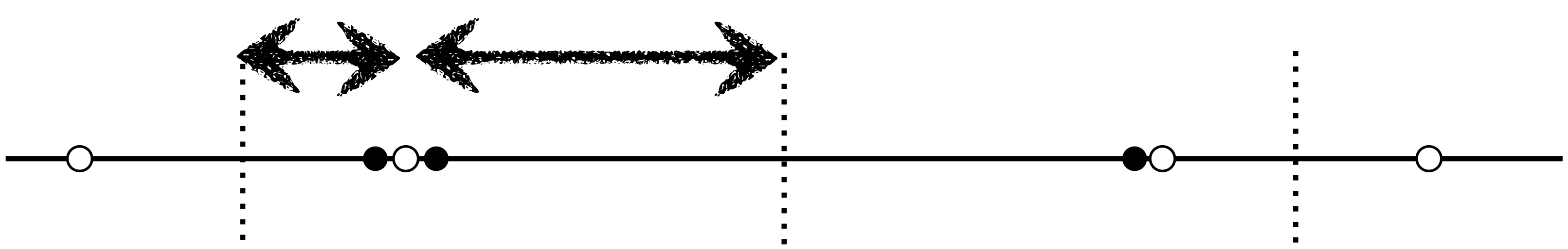

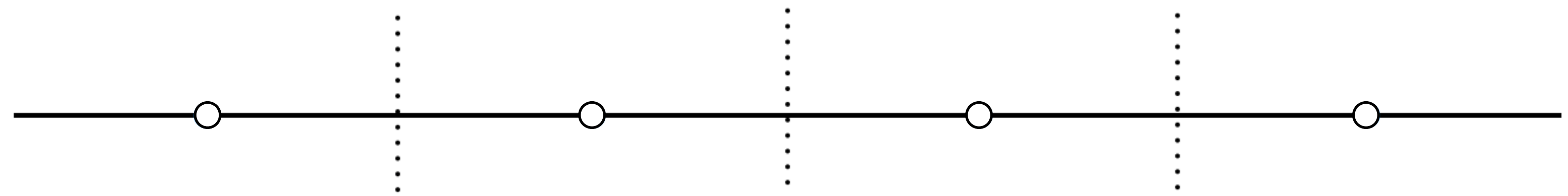

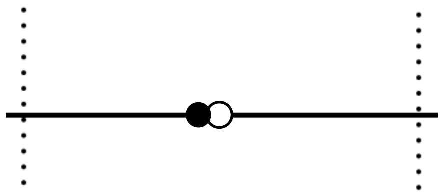

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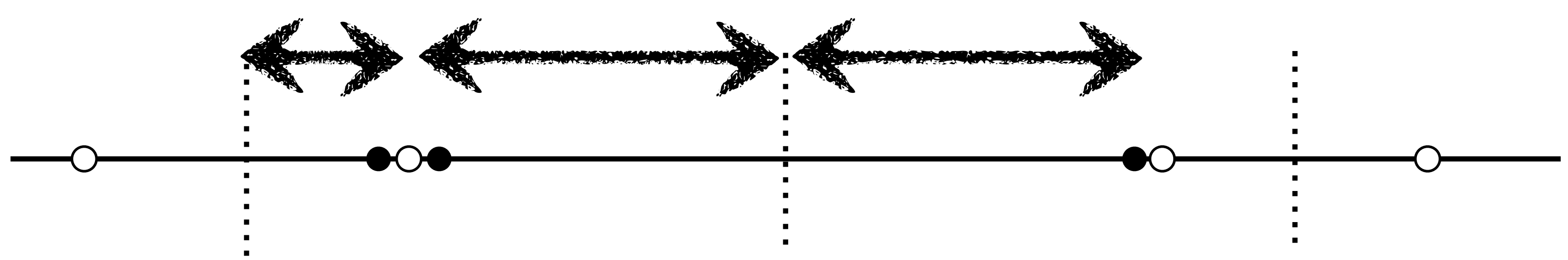

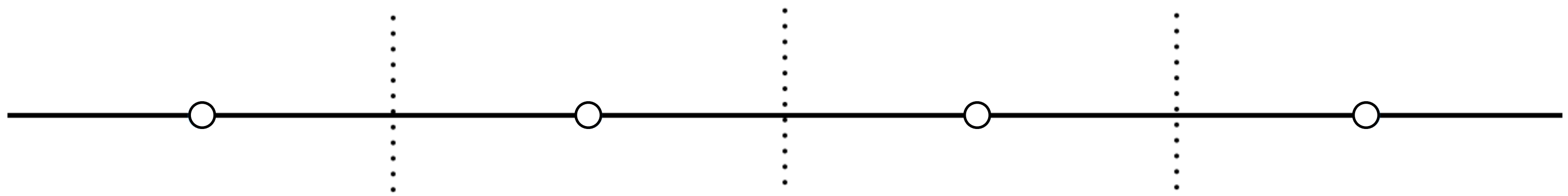

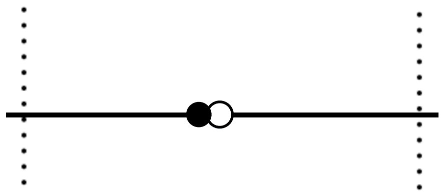

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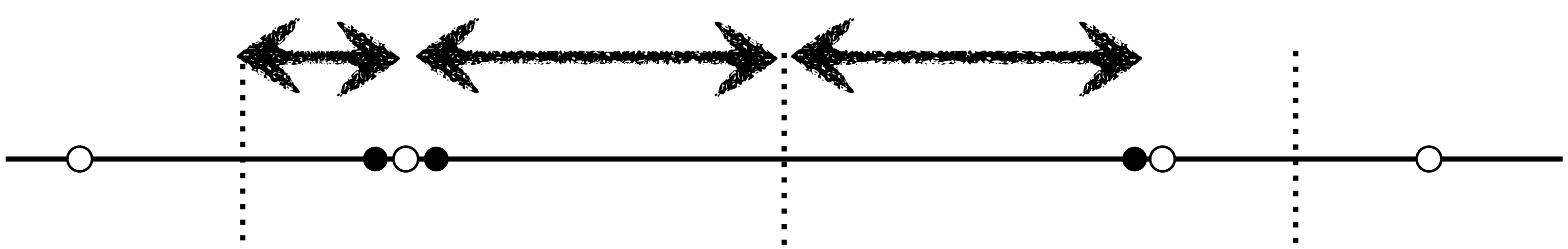

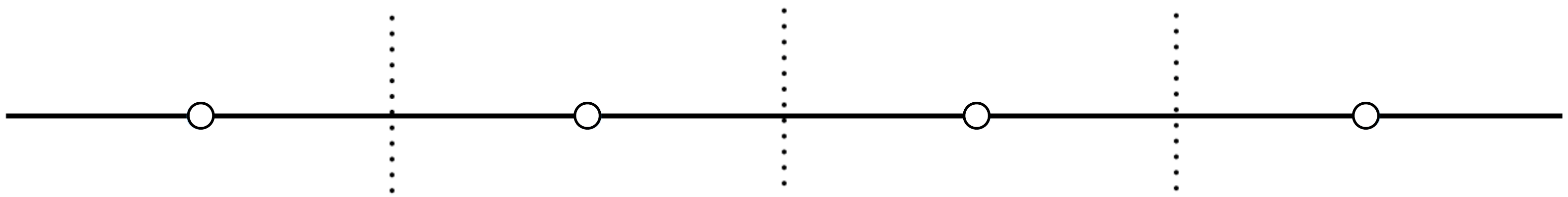

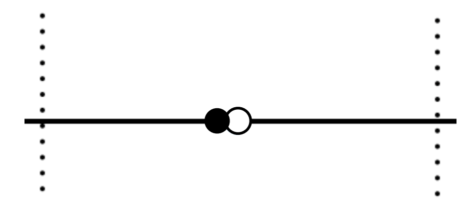

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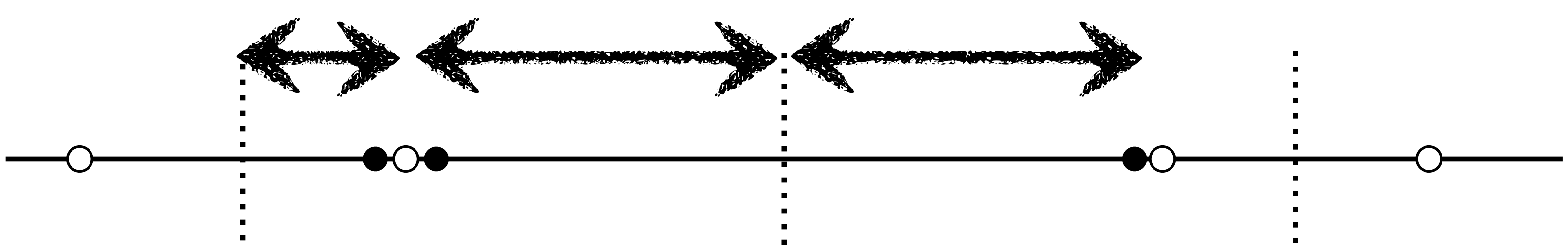

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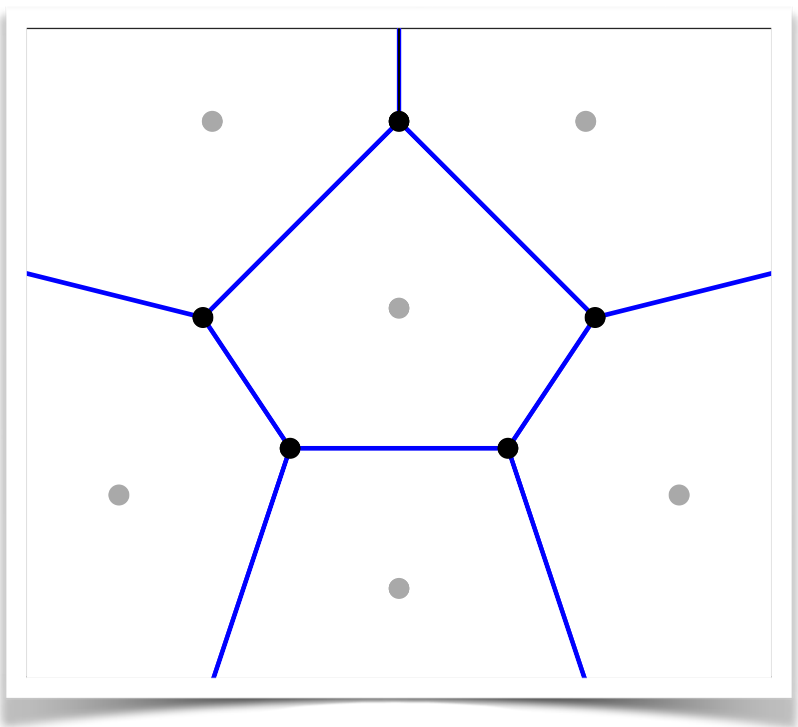

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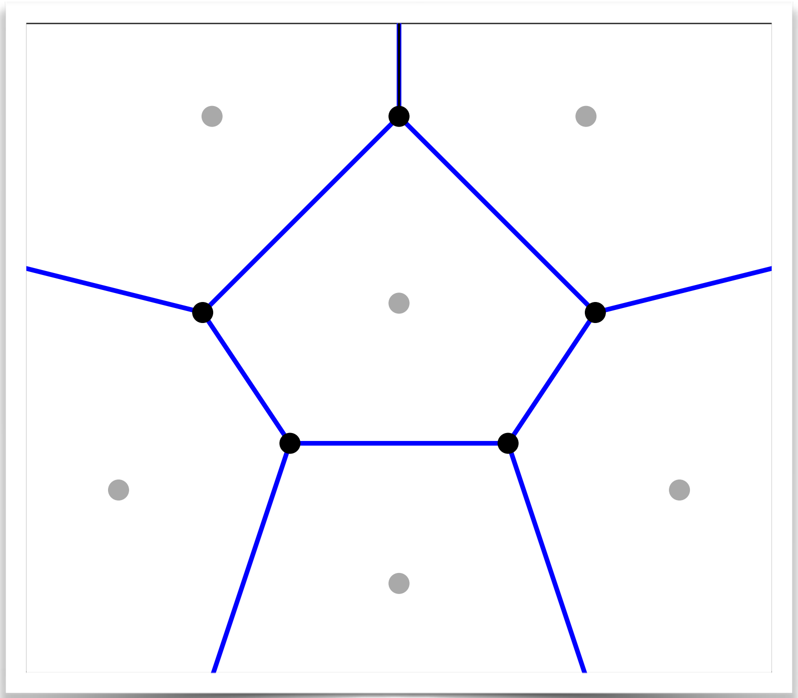

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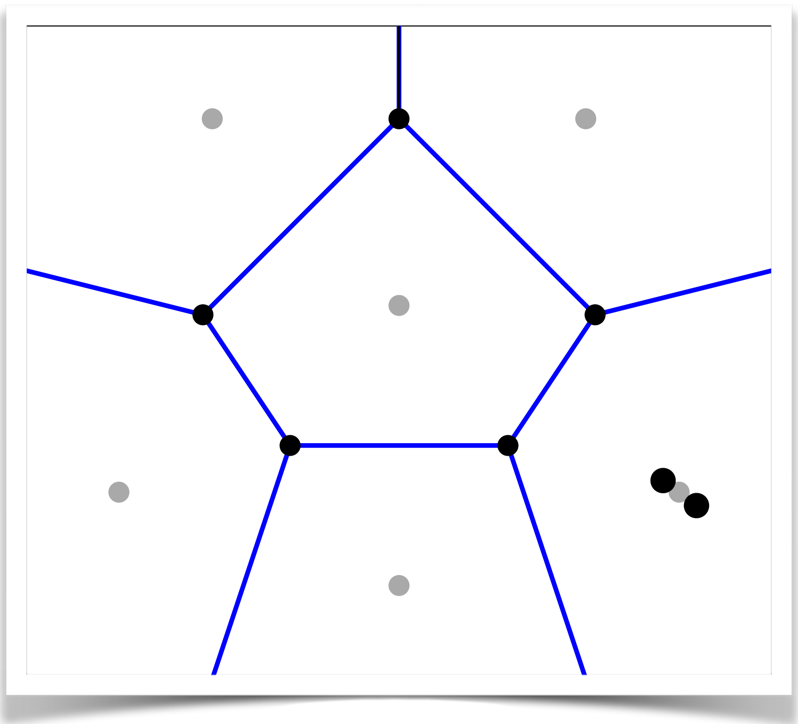

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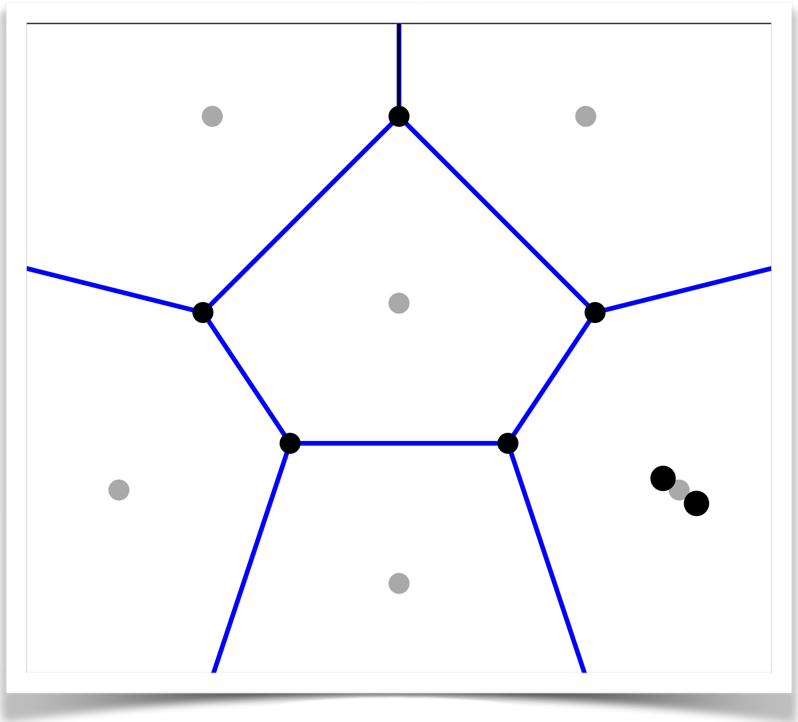

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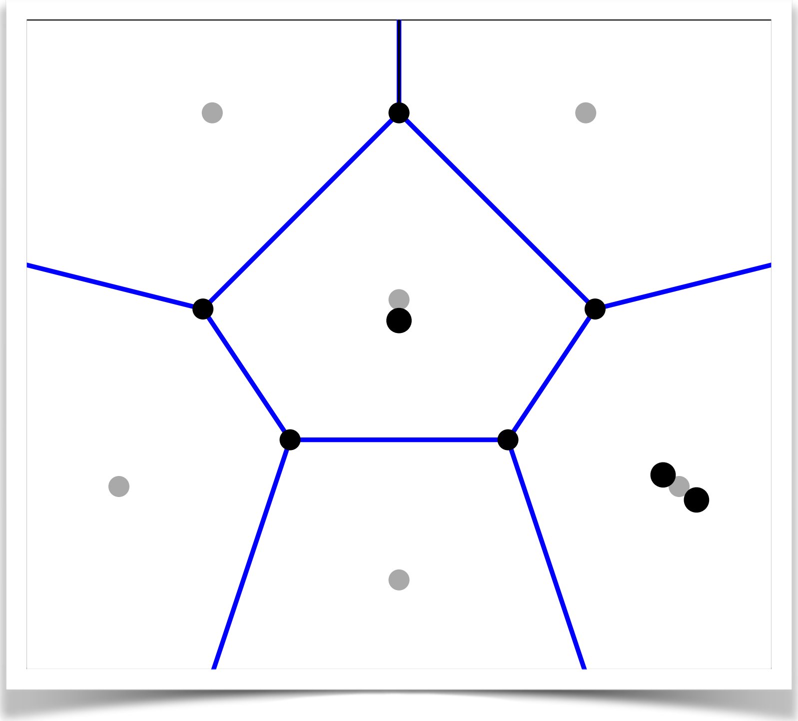

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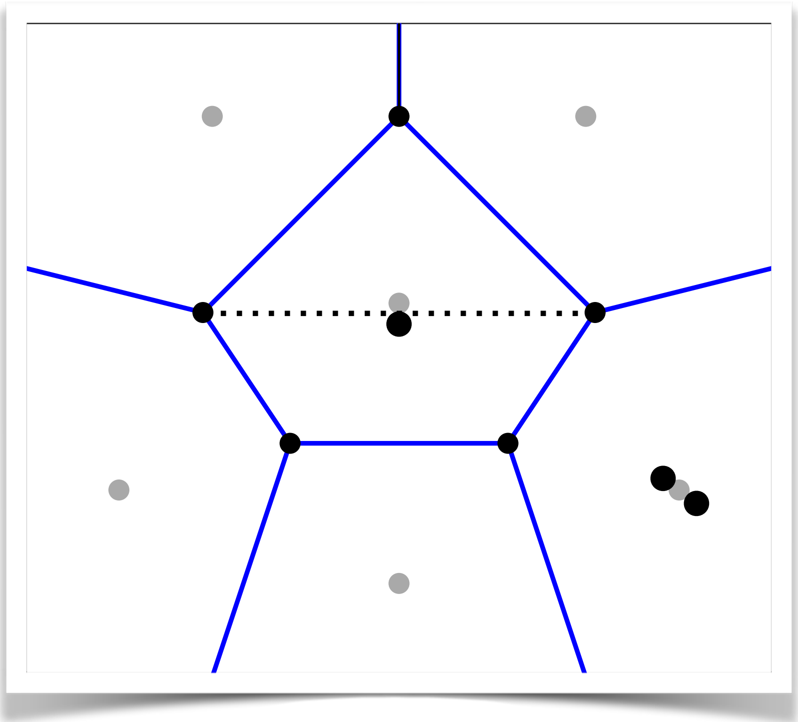

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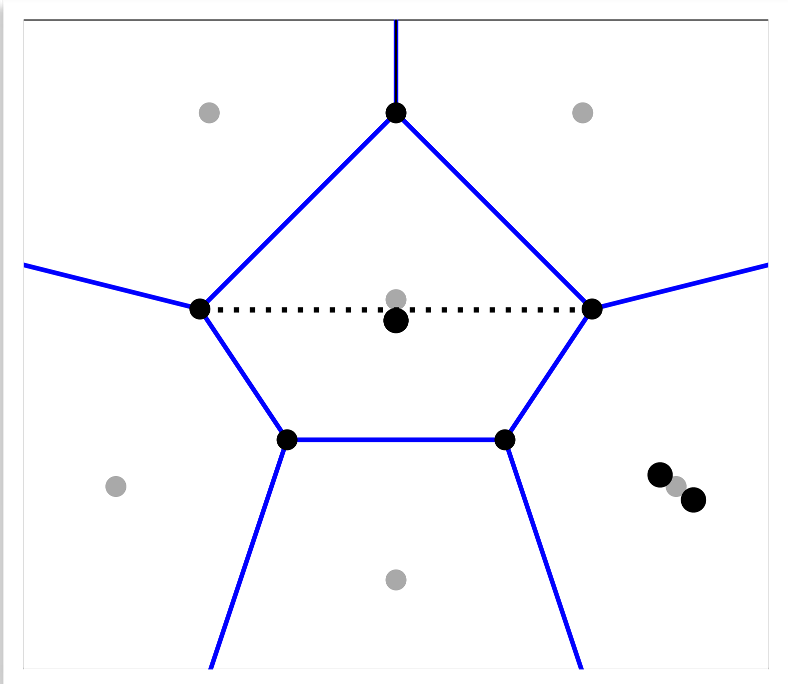

Lemma 1. *If* $V(W)$ *contains a cell that is not point symmetric, then Barney wins.*

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The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

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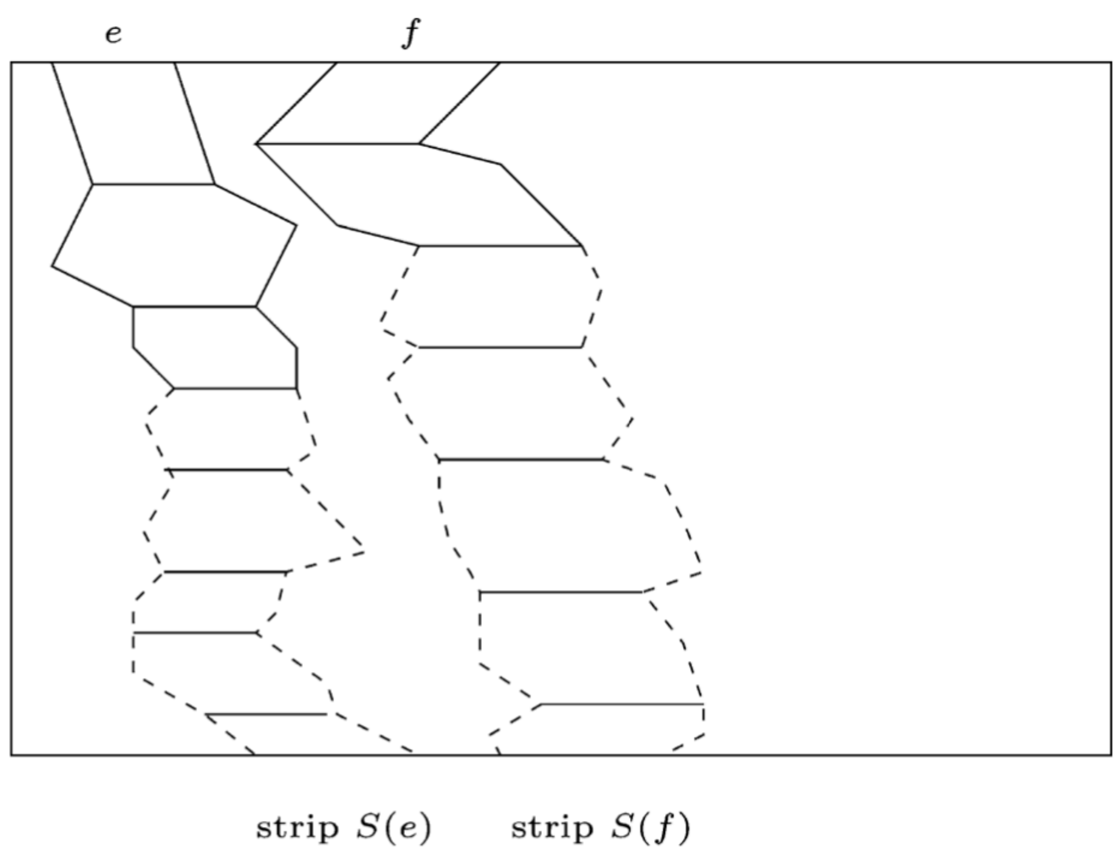

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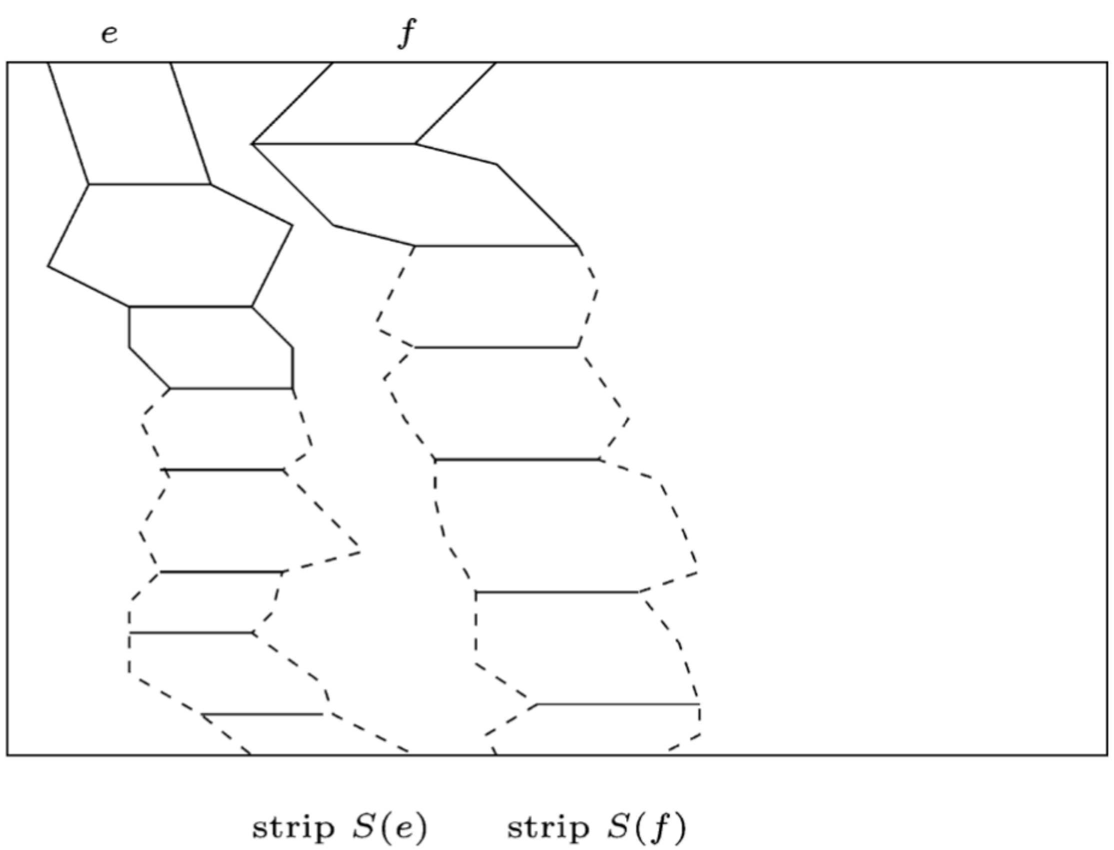

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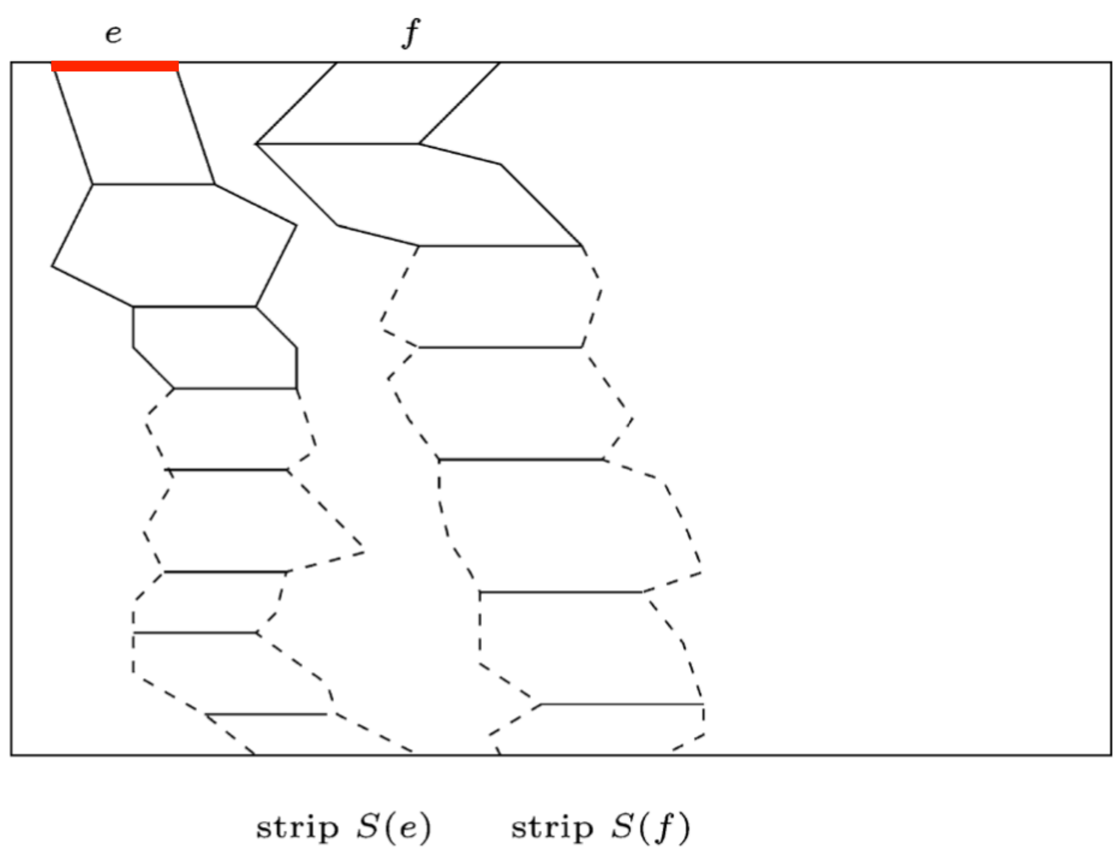

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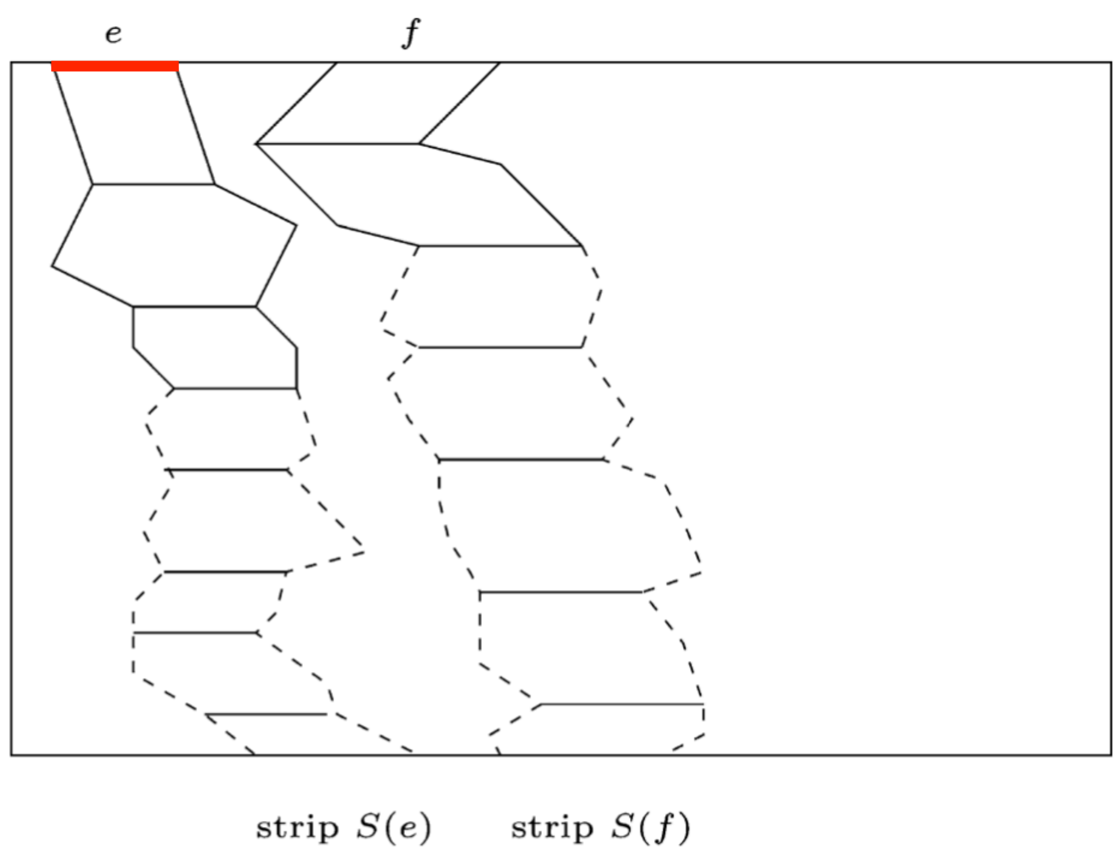

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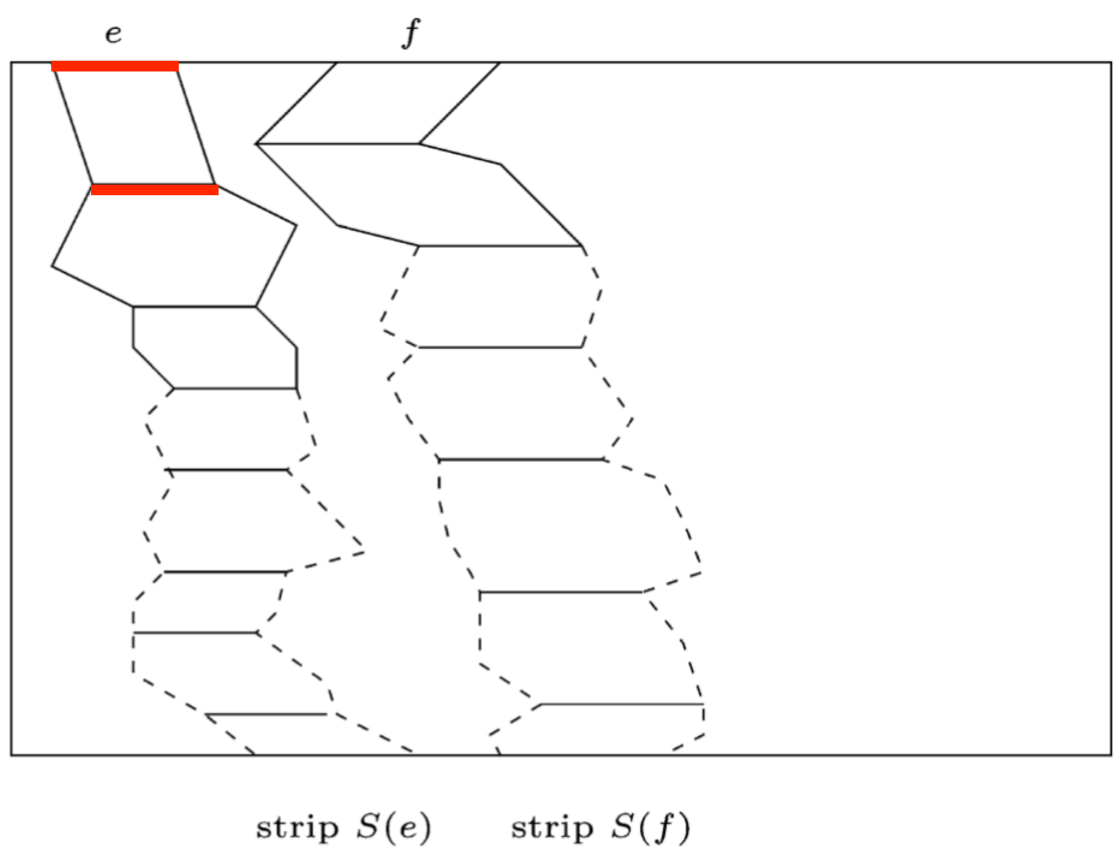

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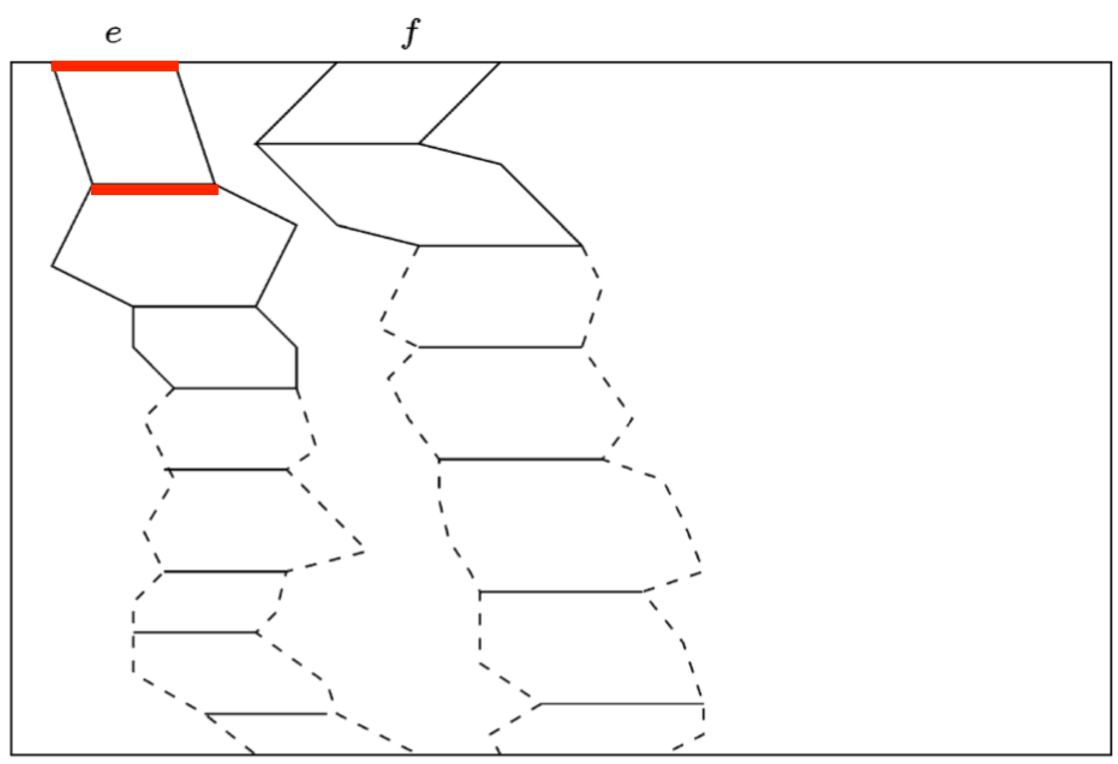

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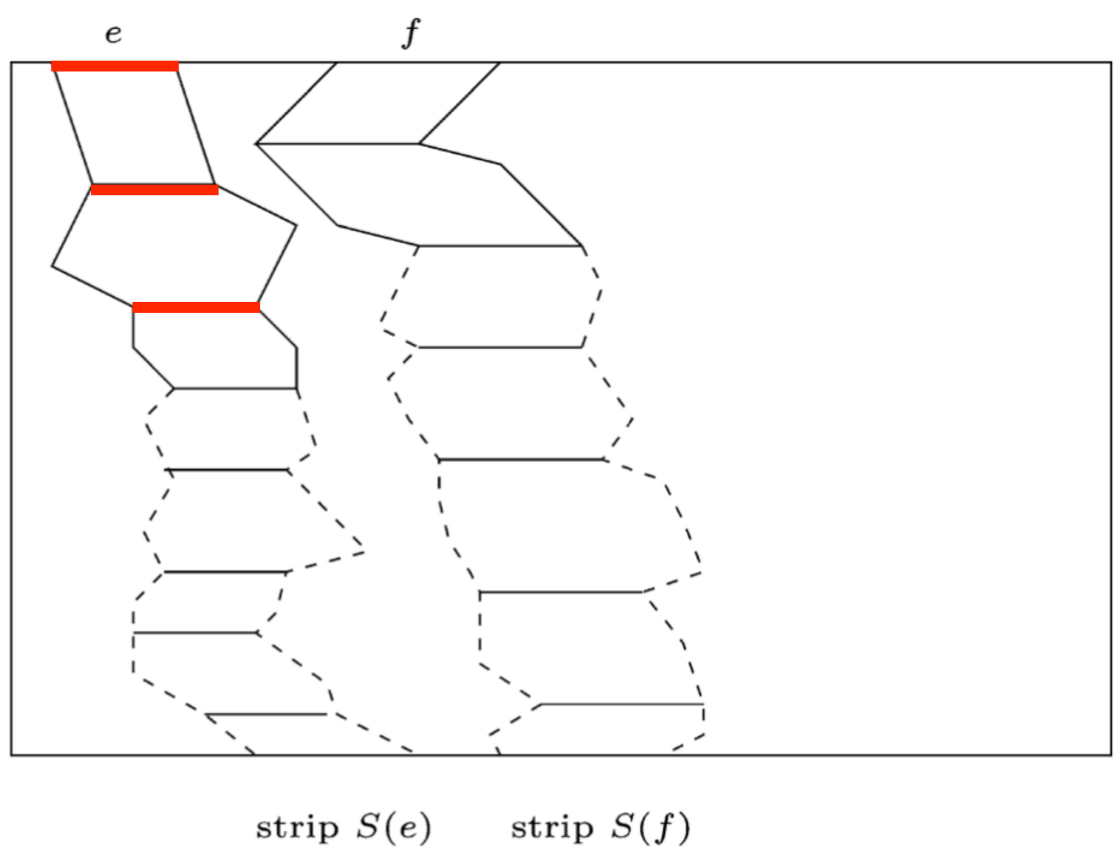

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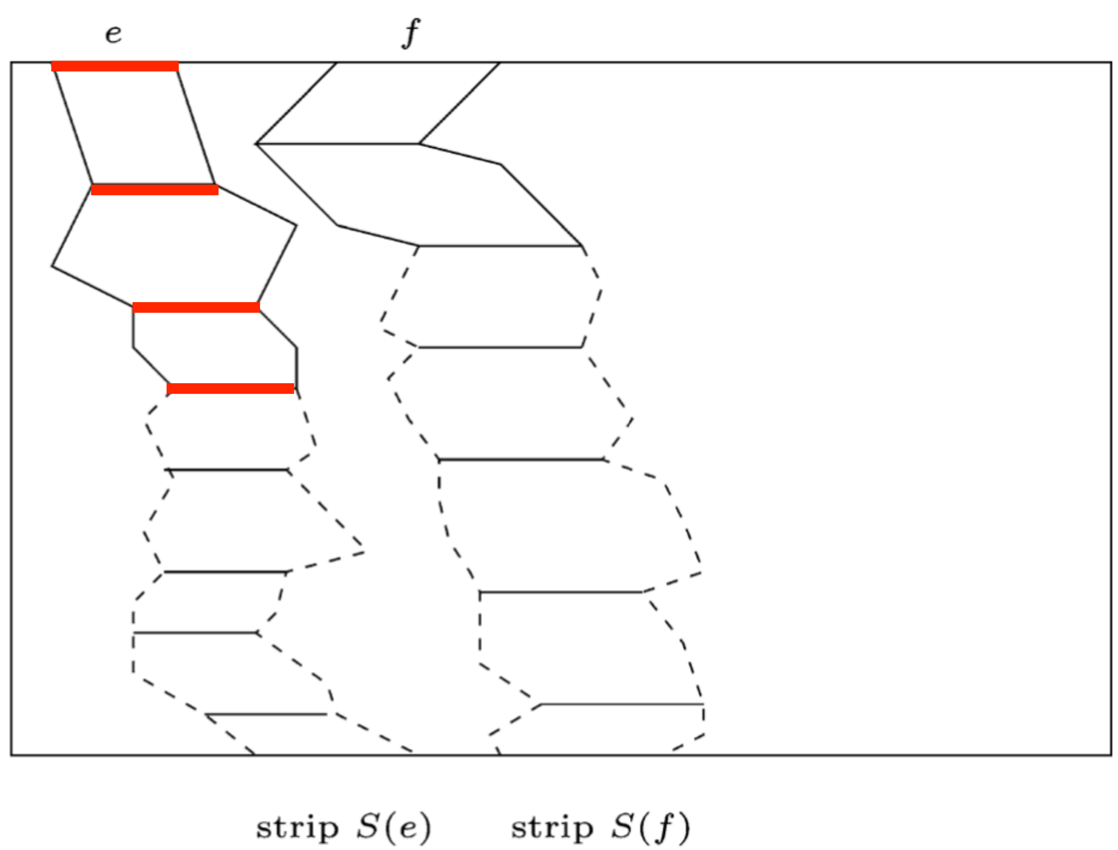

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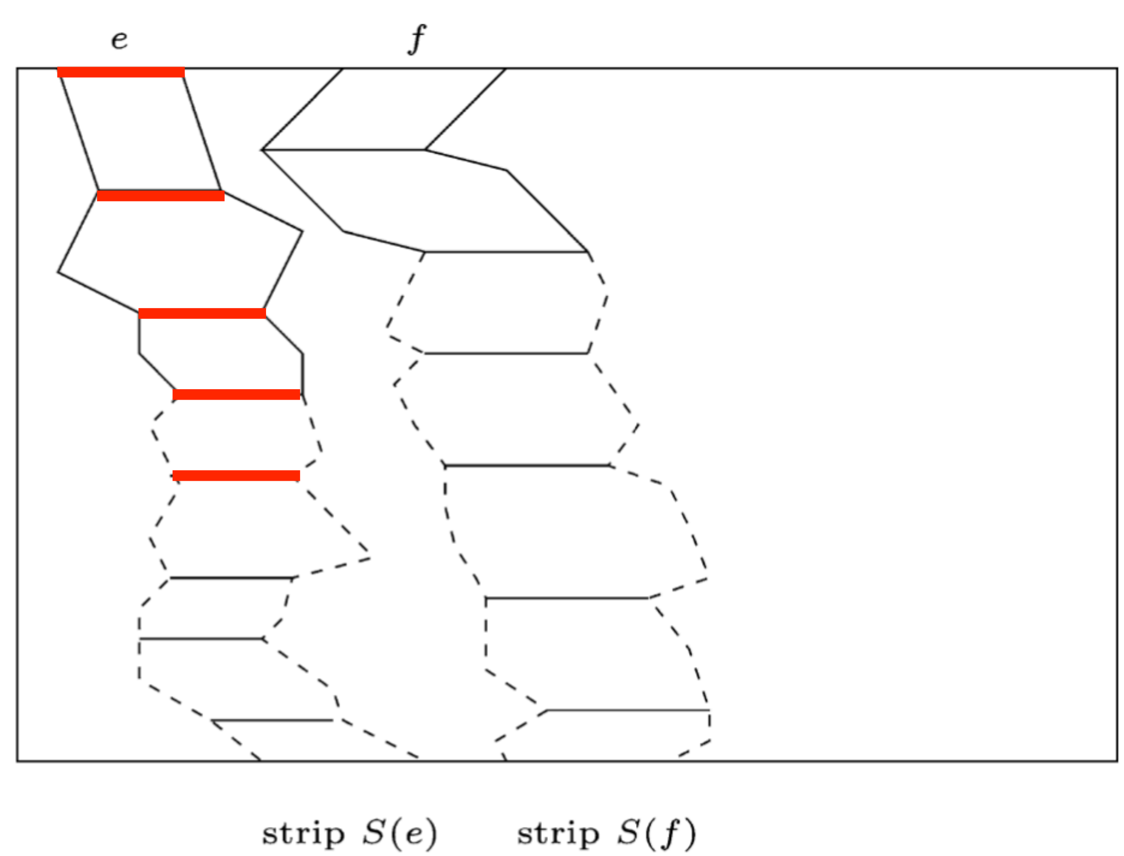

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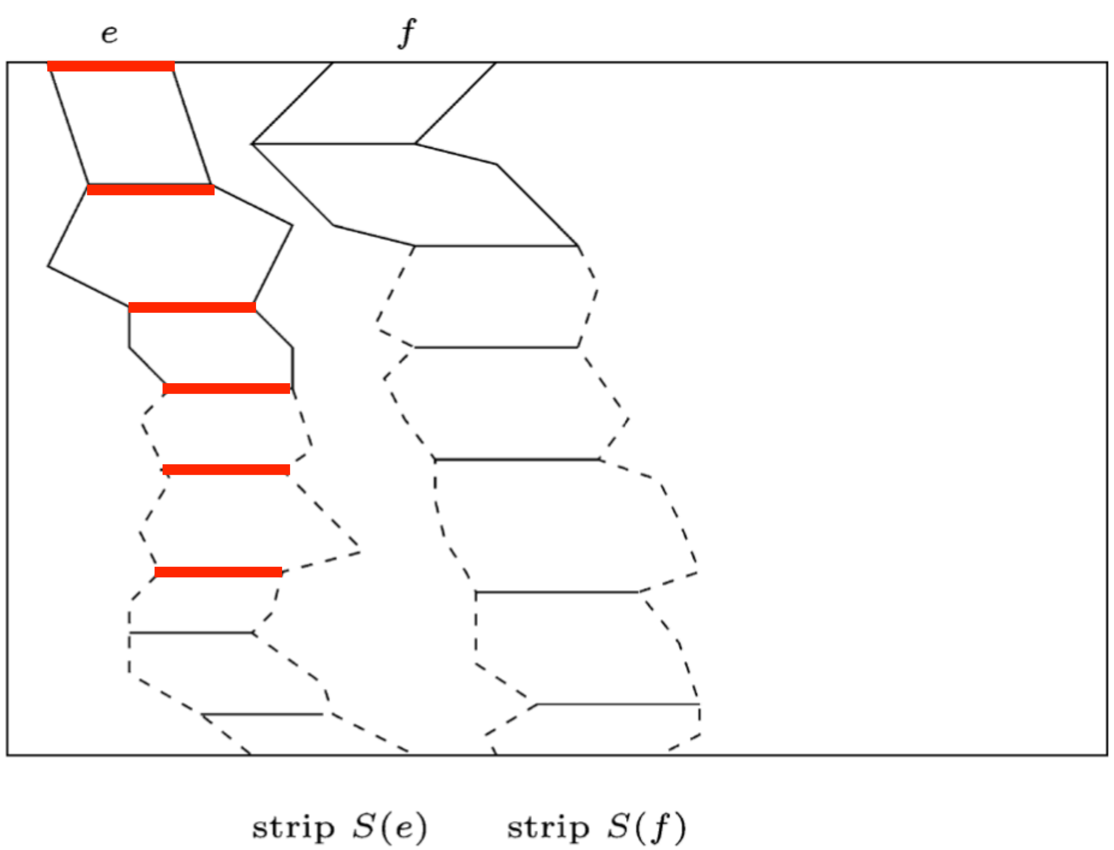

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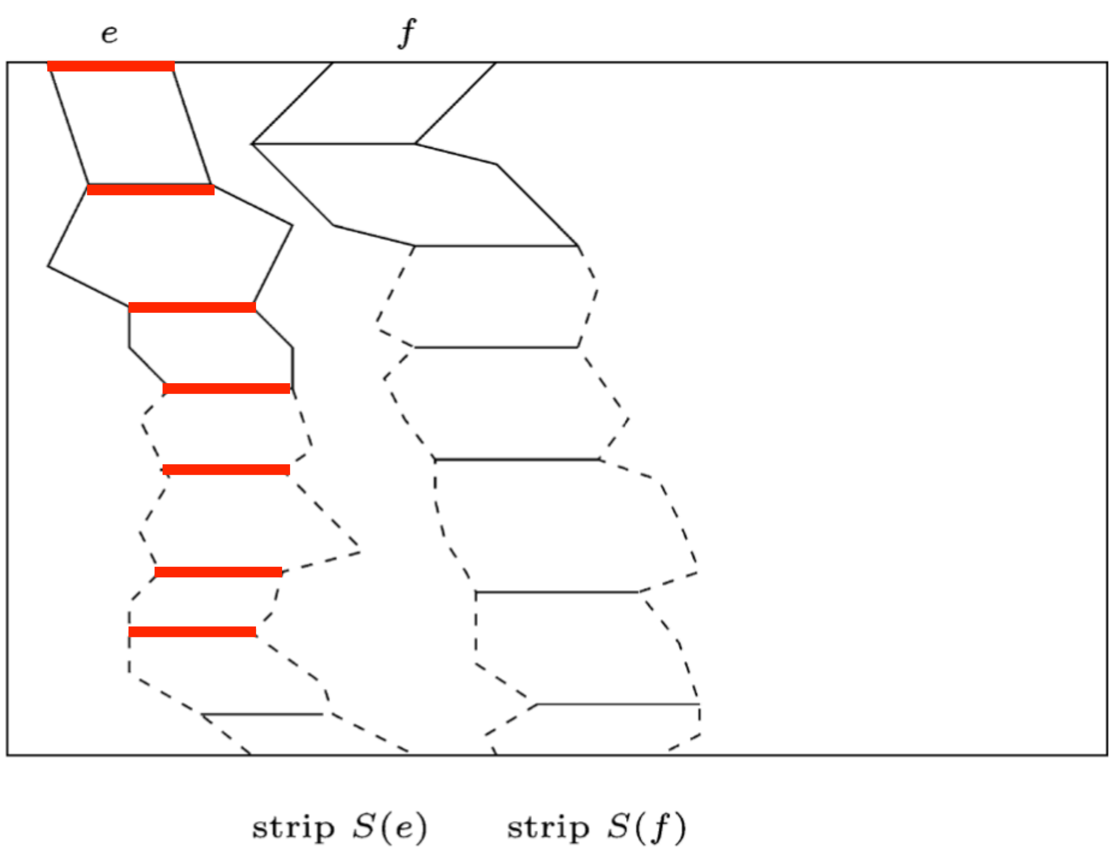

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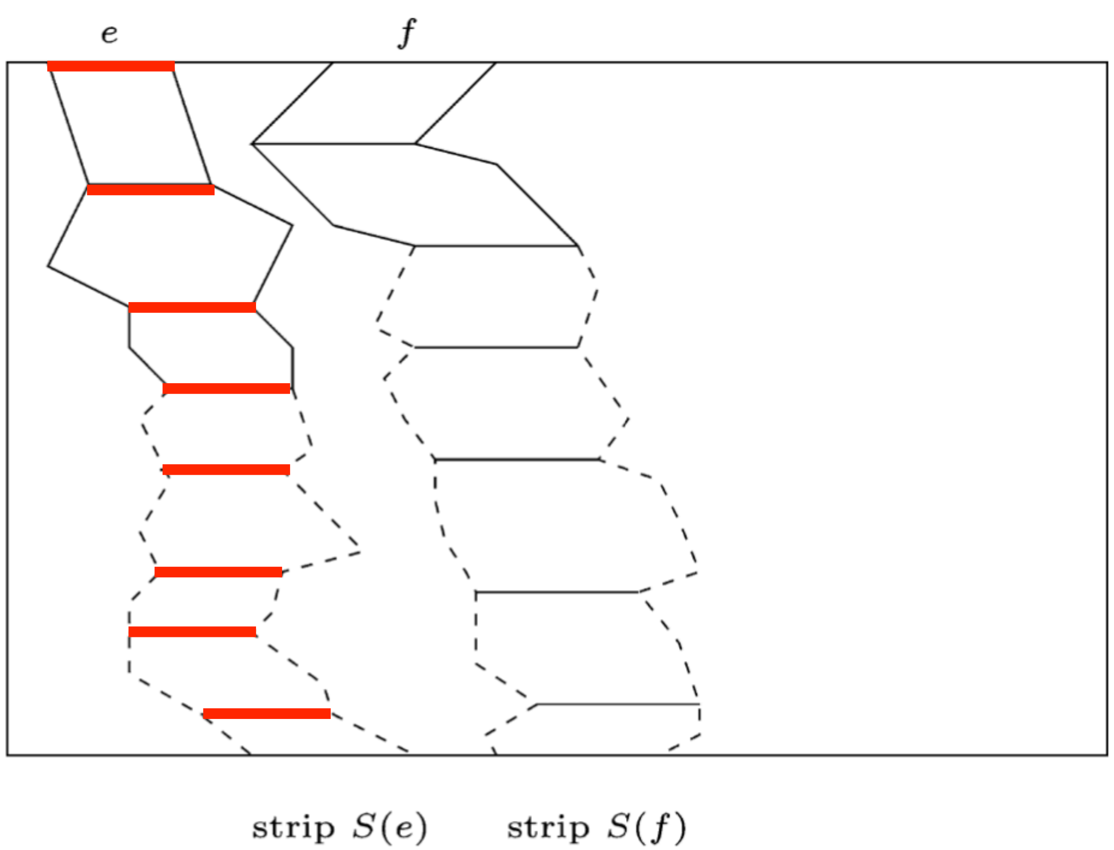

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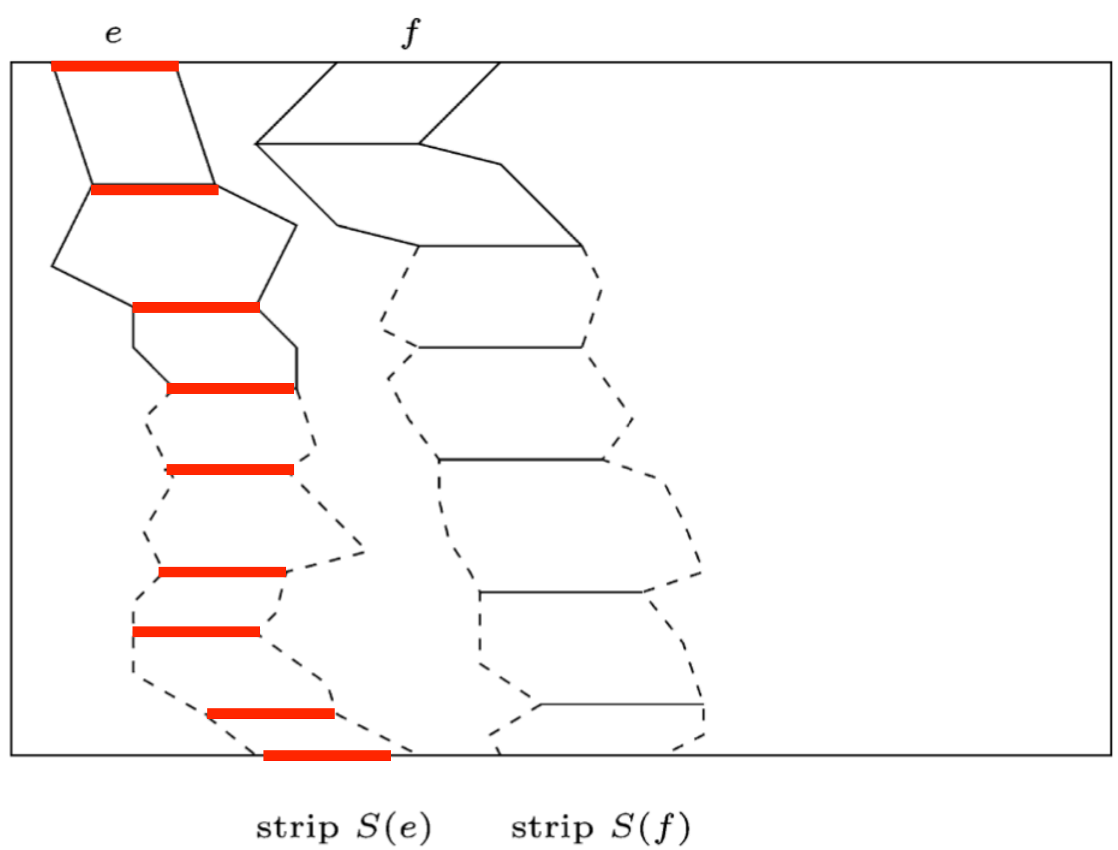

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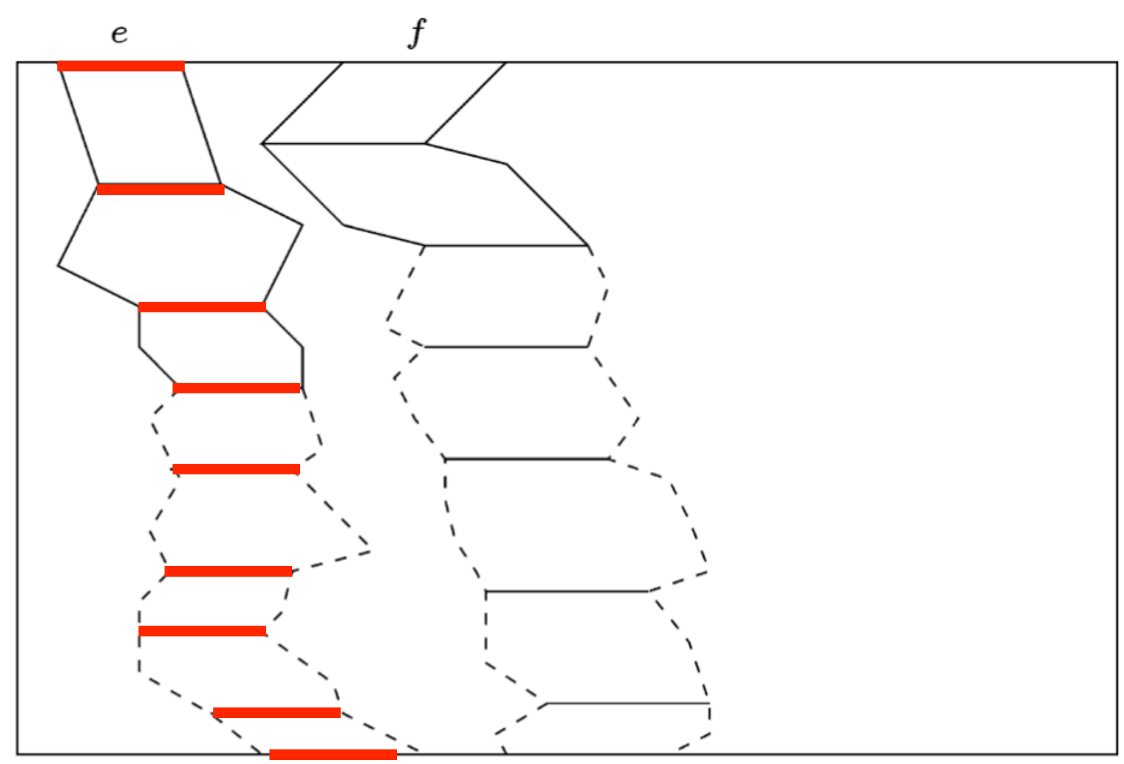

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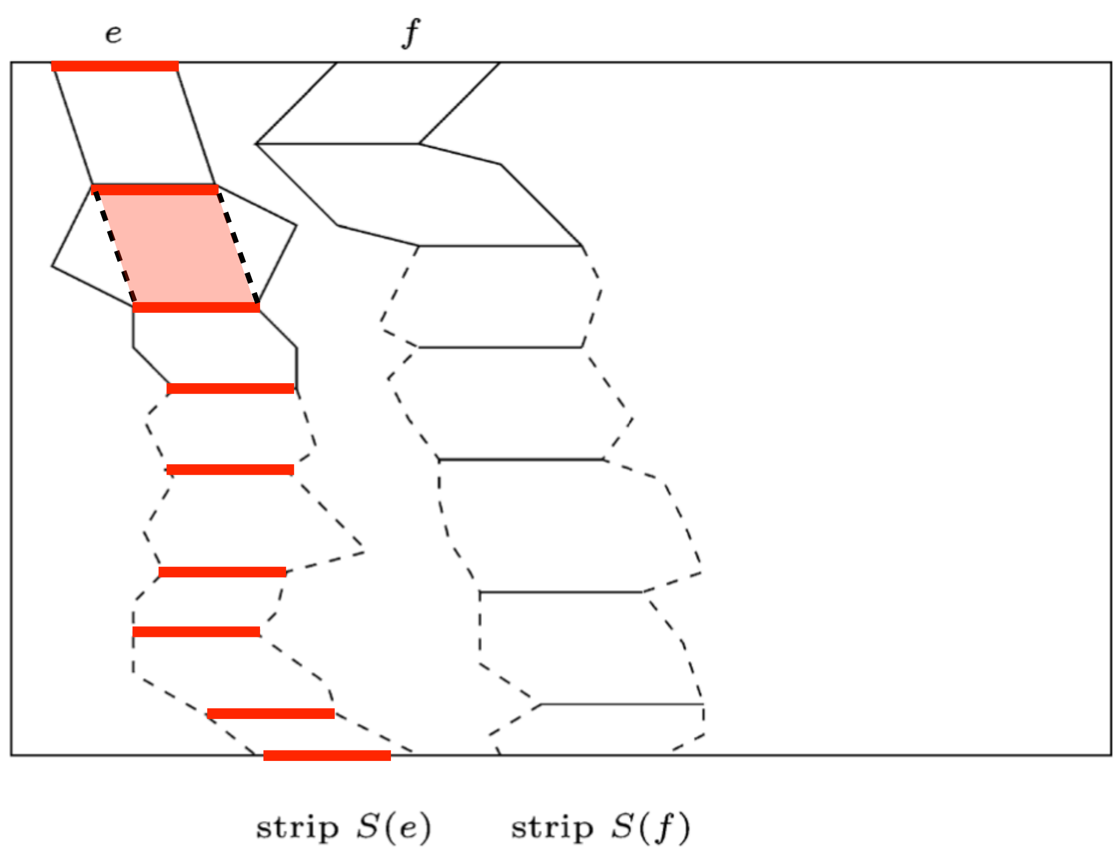

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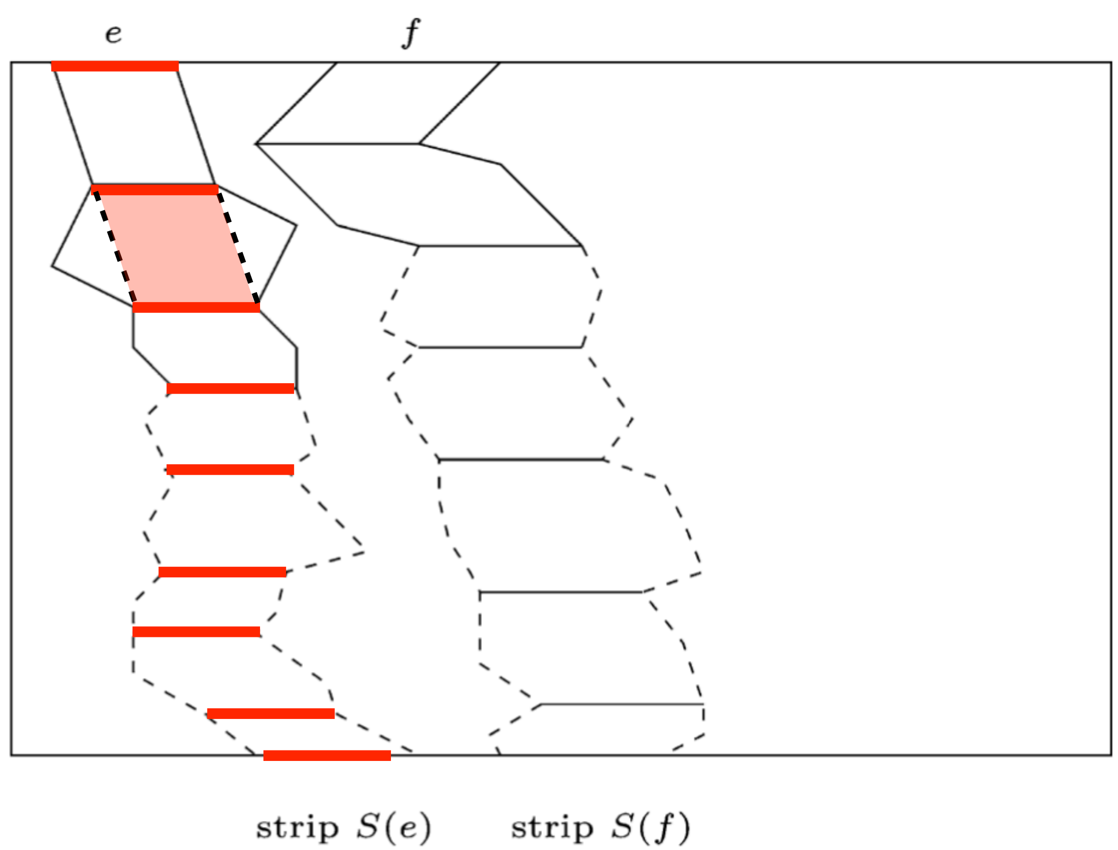

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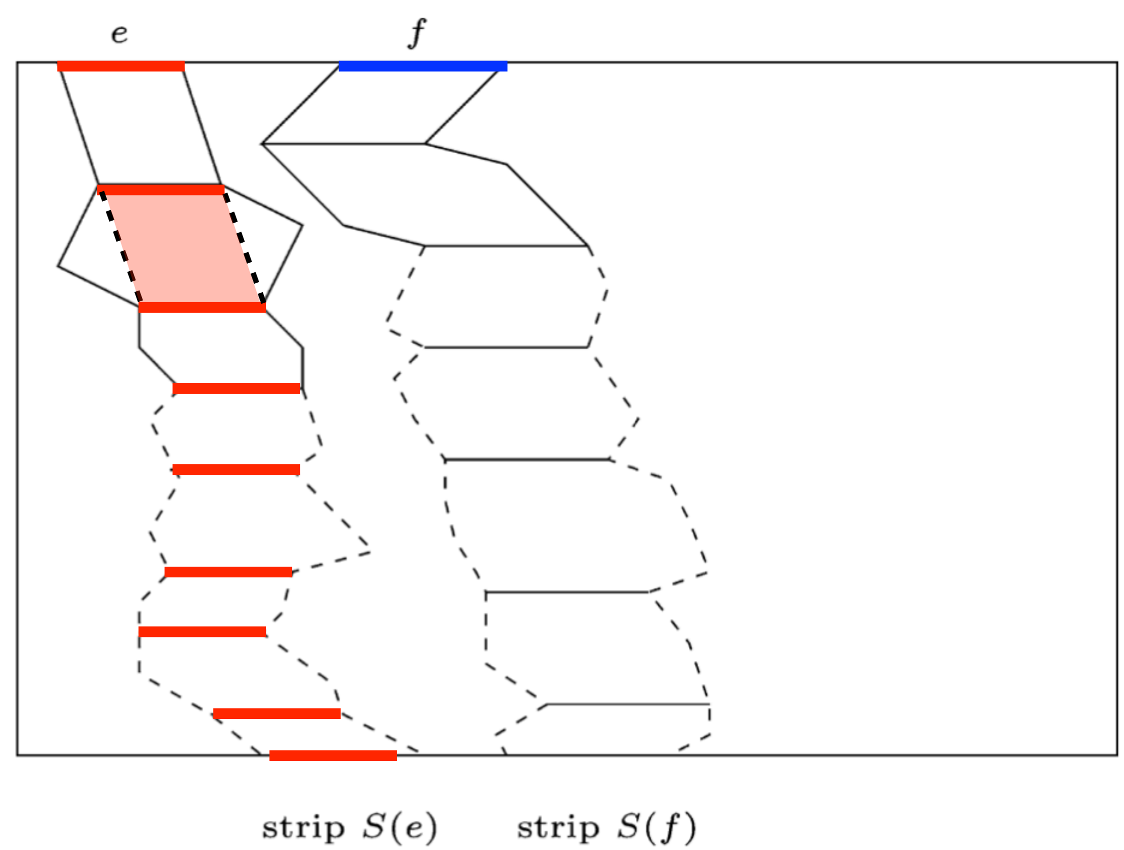

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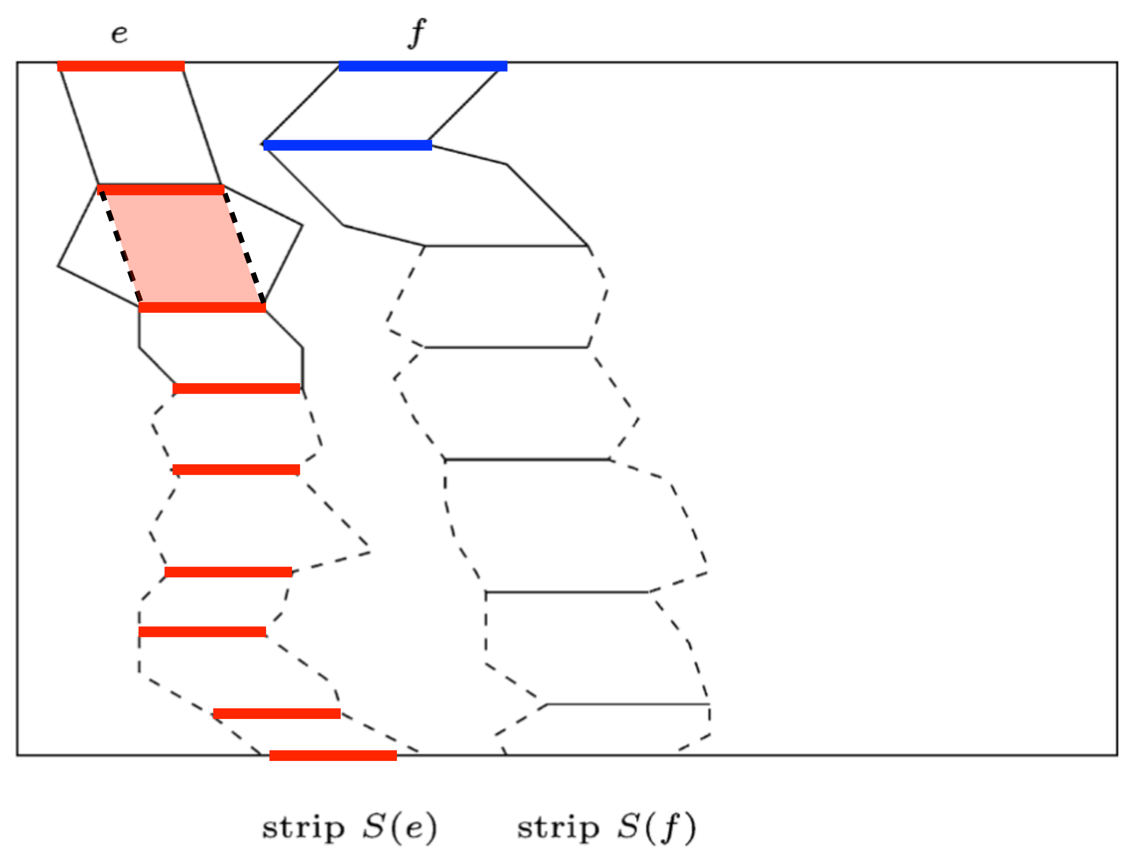

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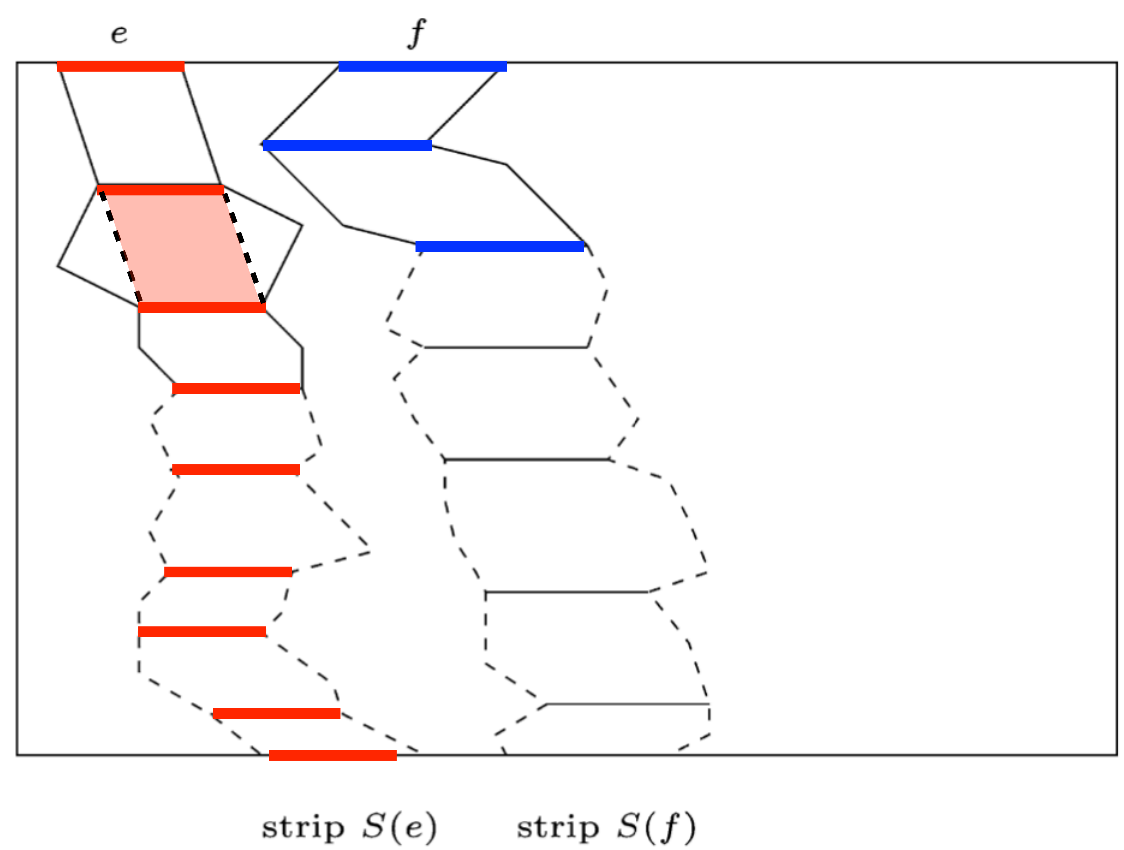

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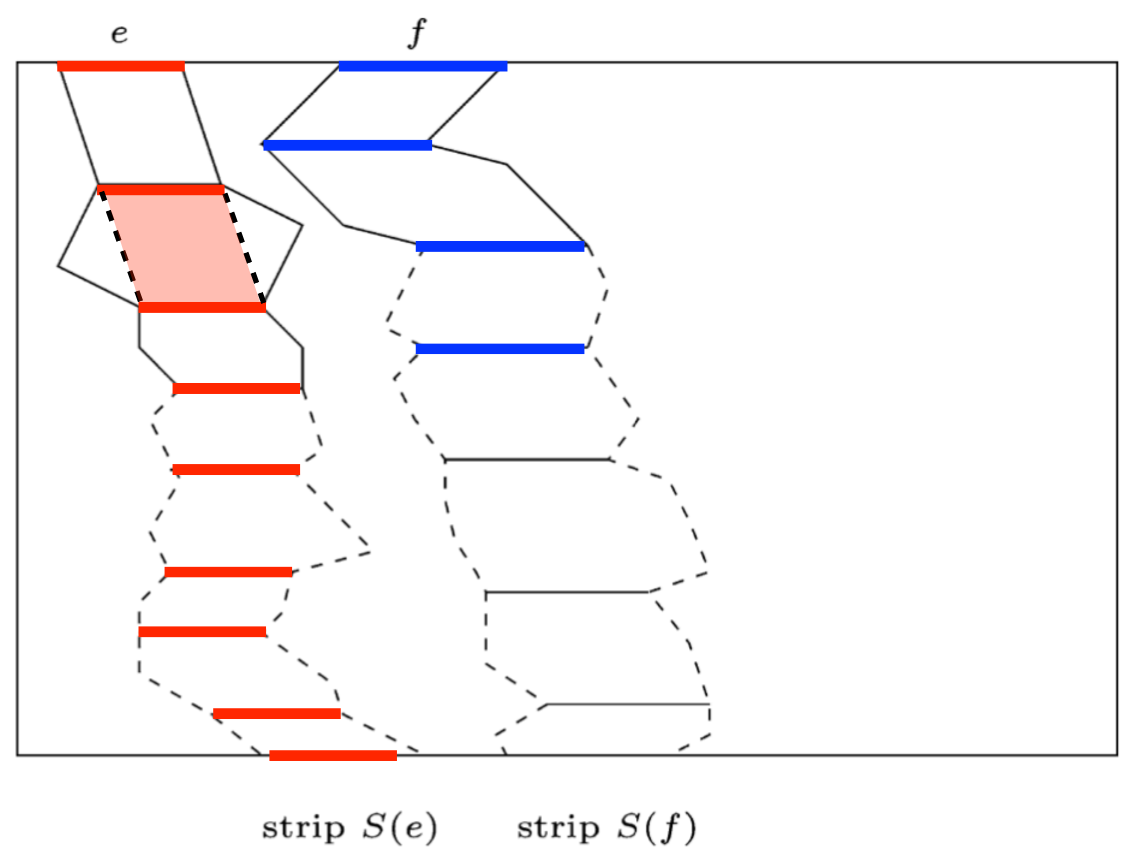

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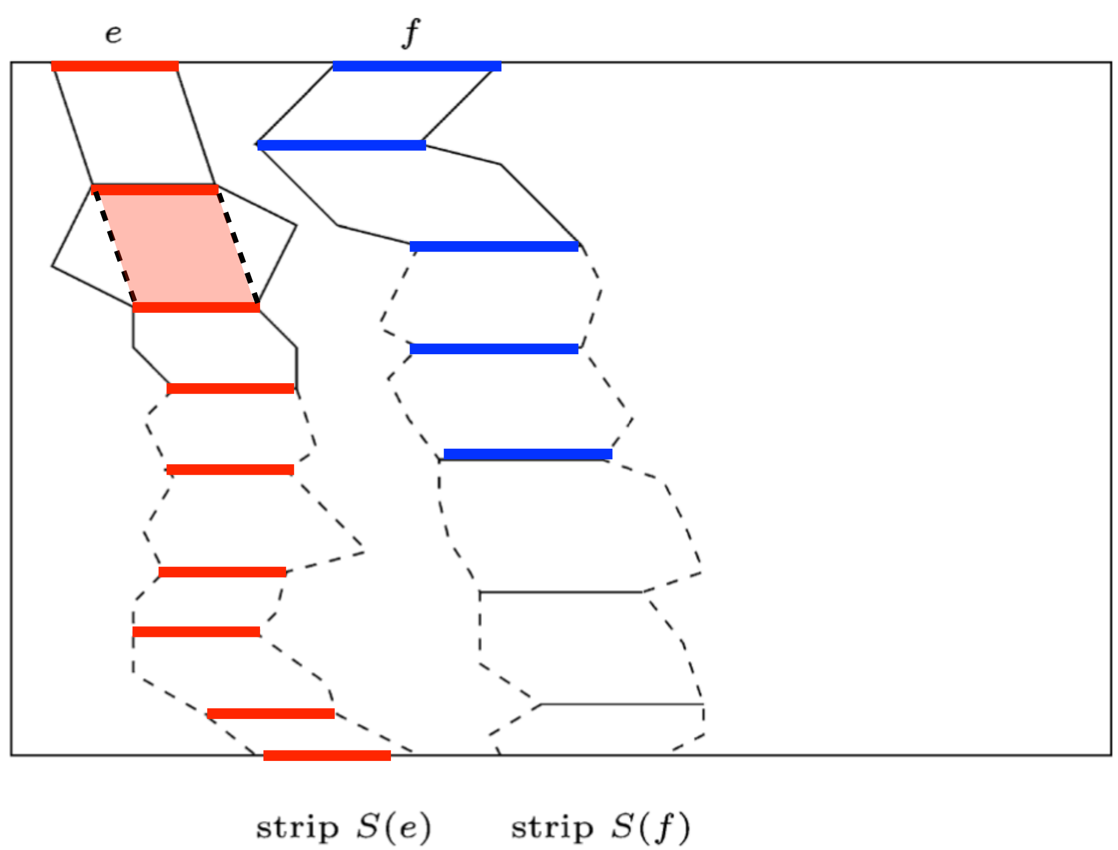

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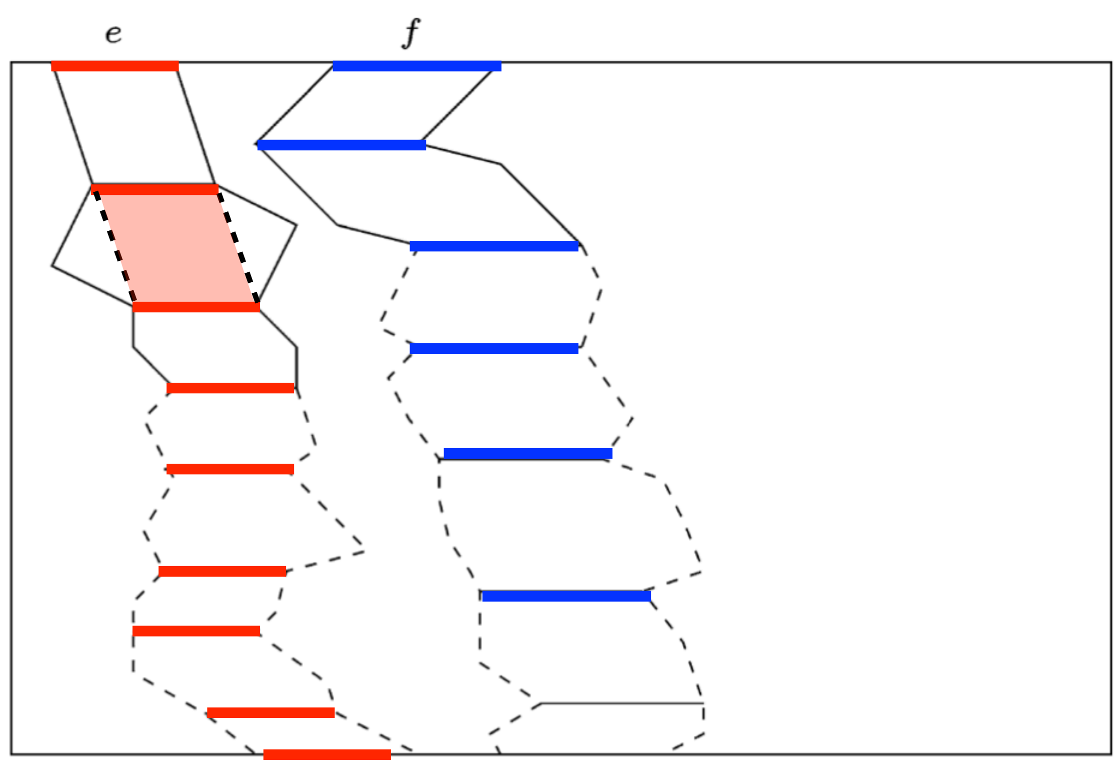

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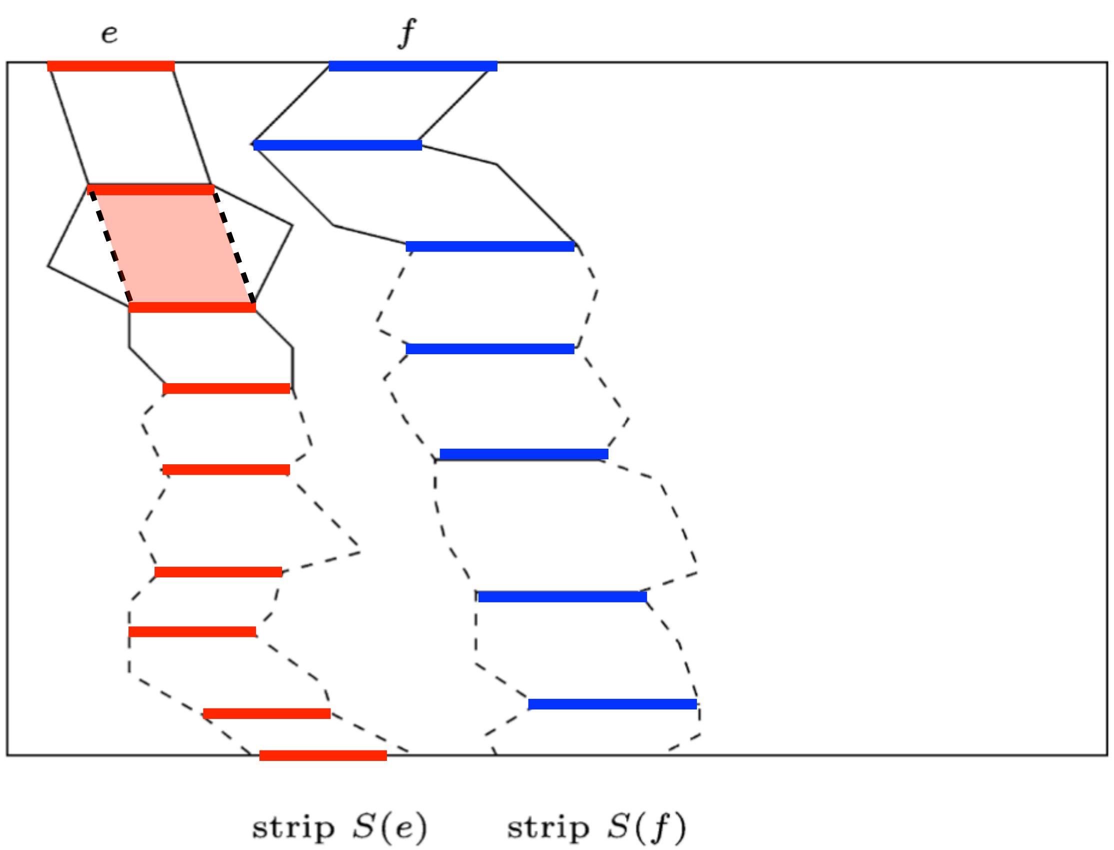

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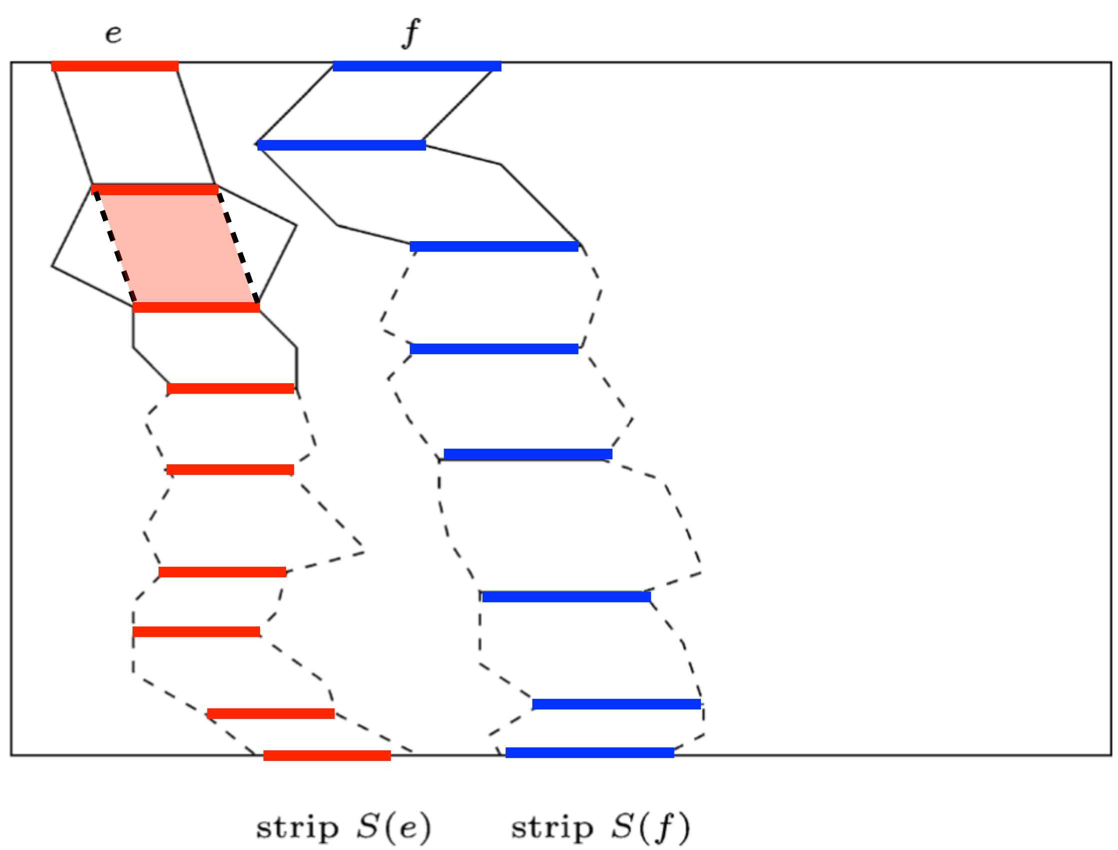

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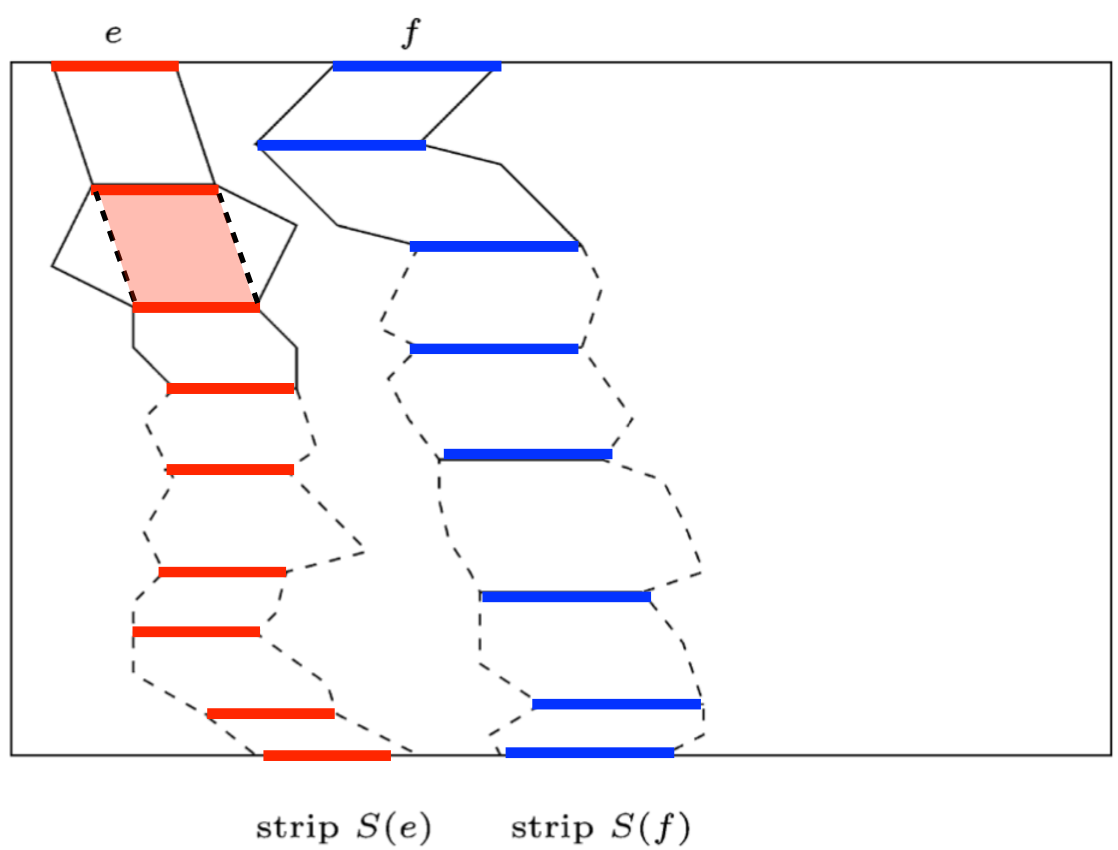

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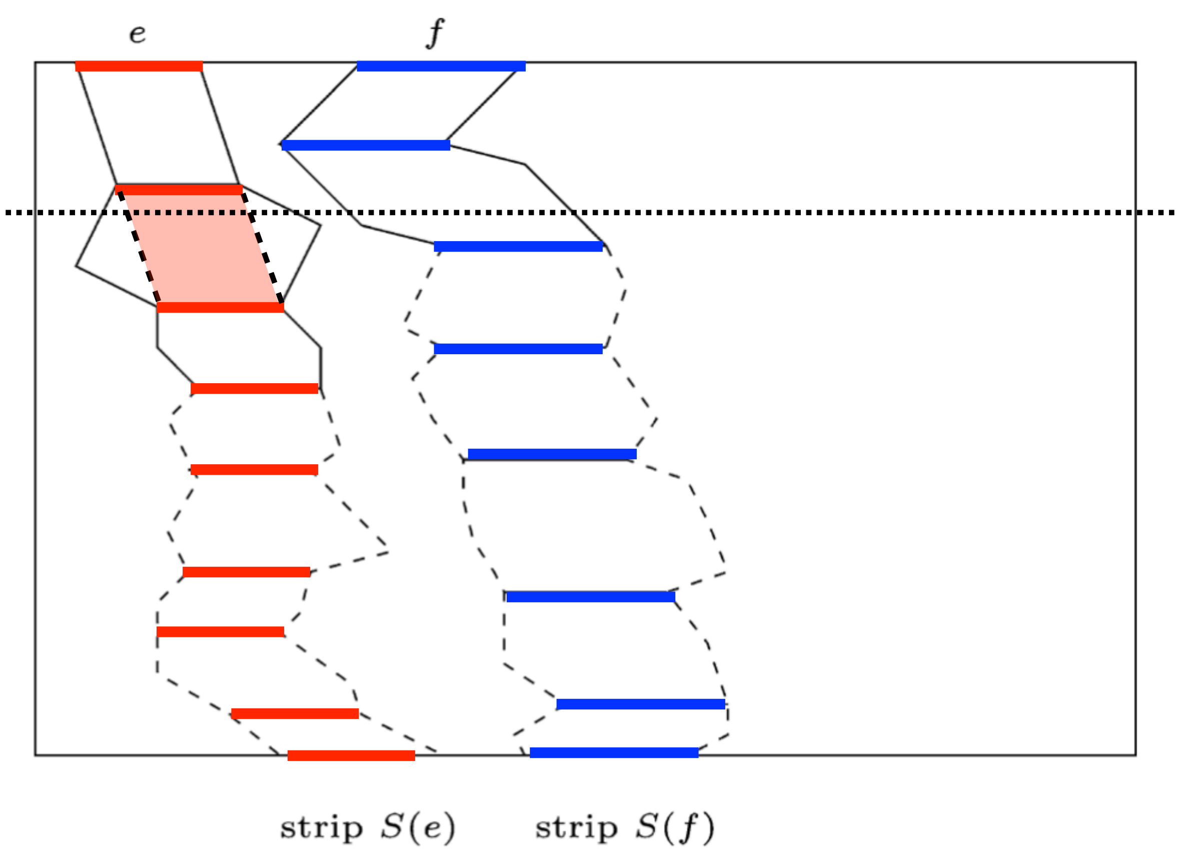

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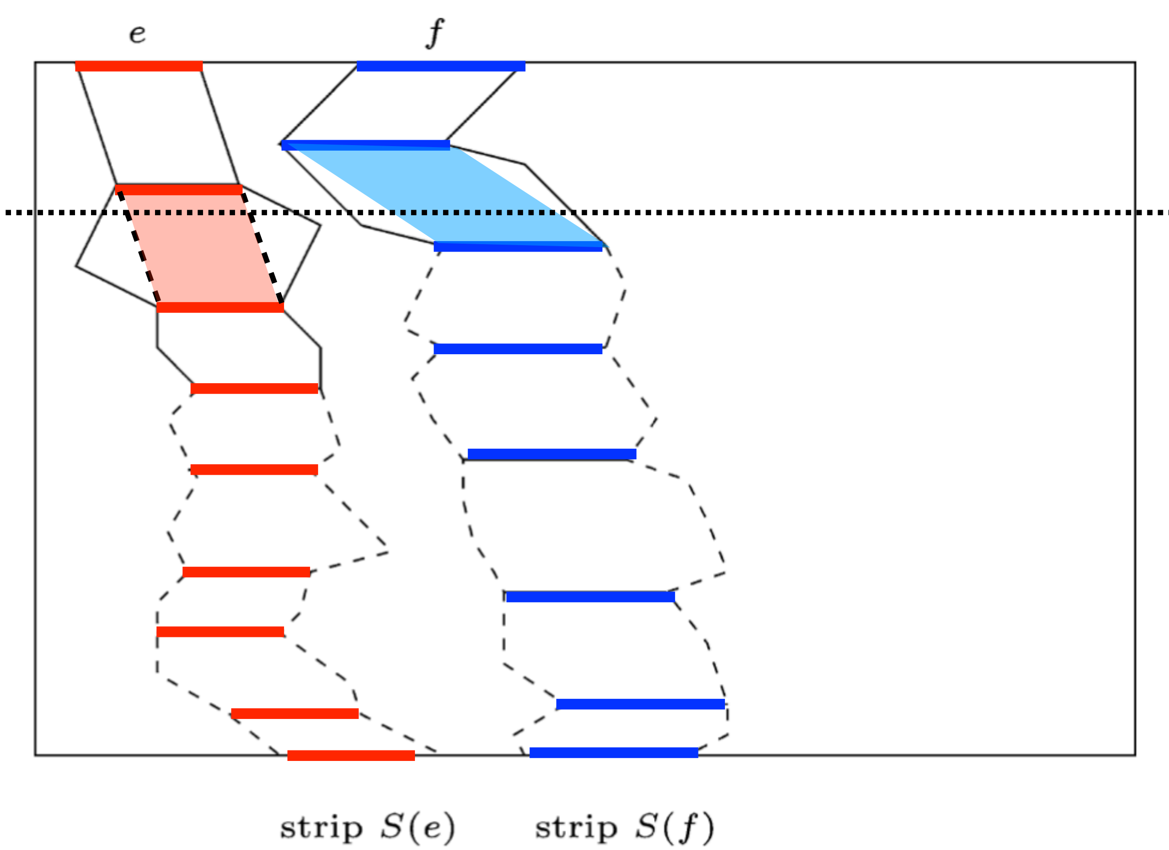

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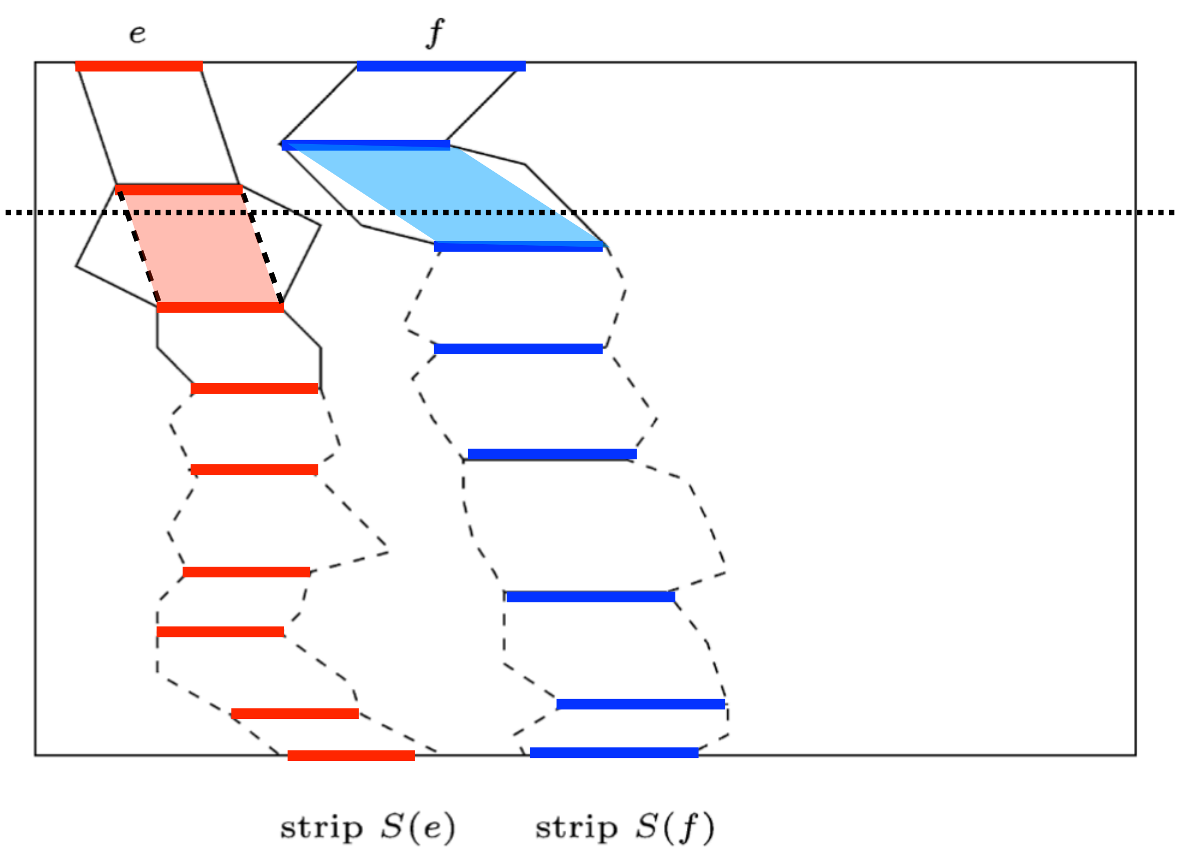

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- **We get a rectangular grid.**

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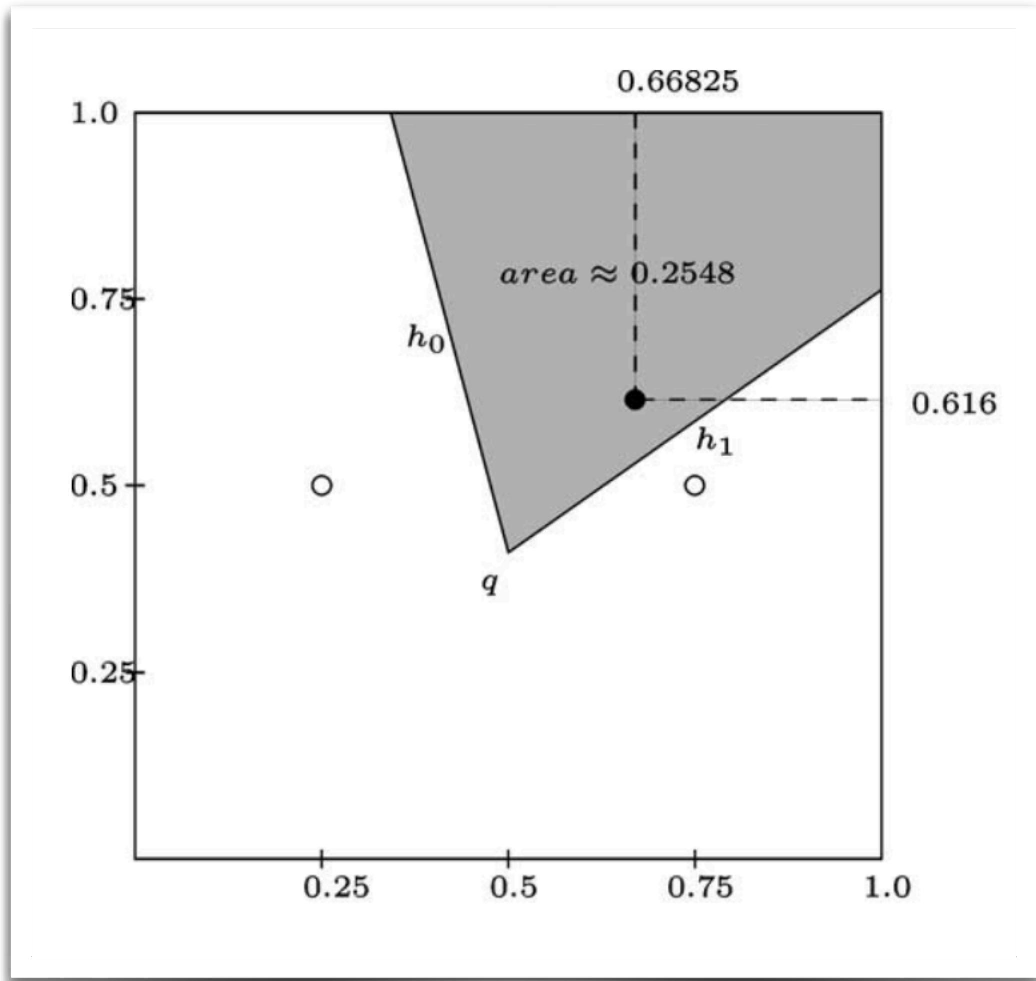

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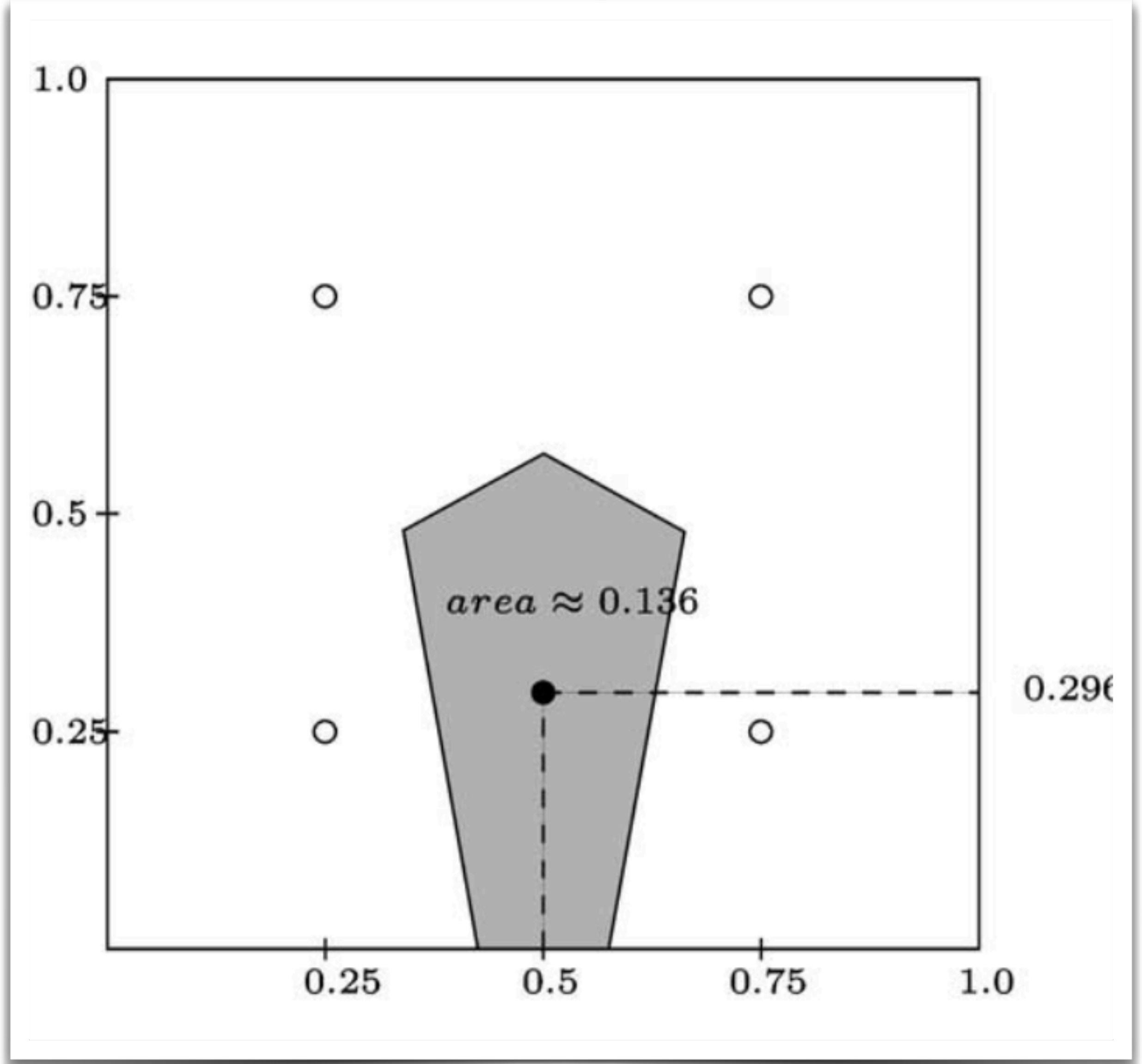

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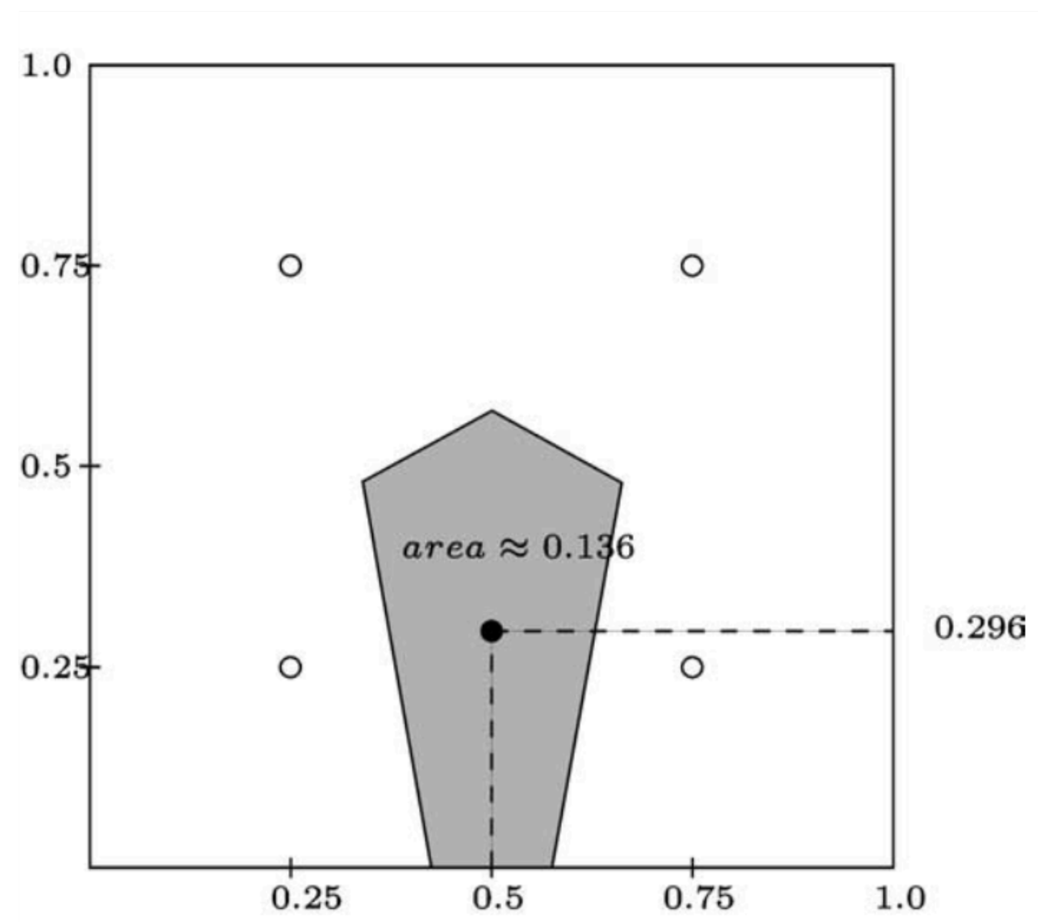

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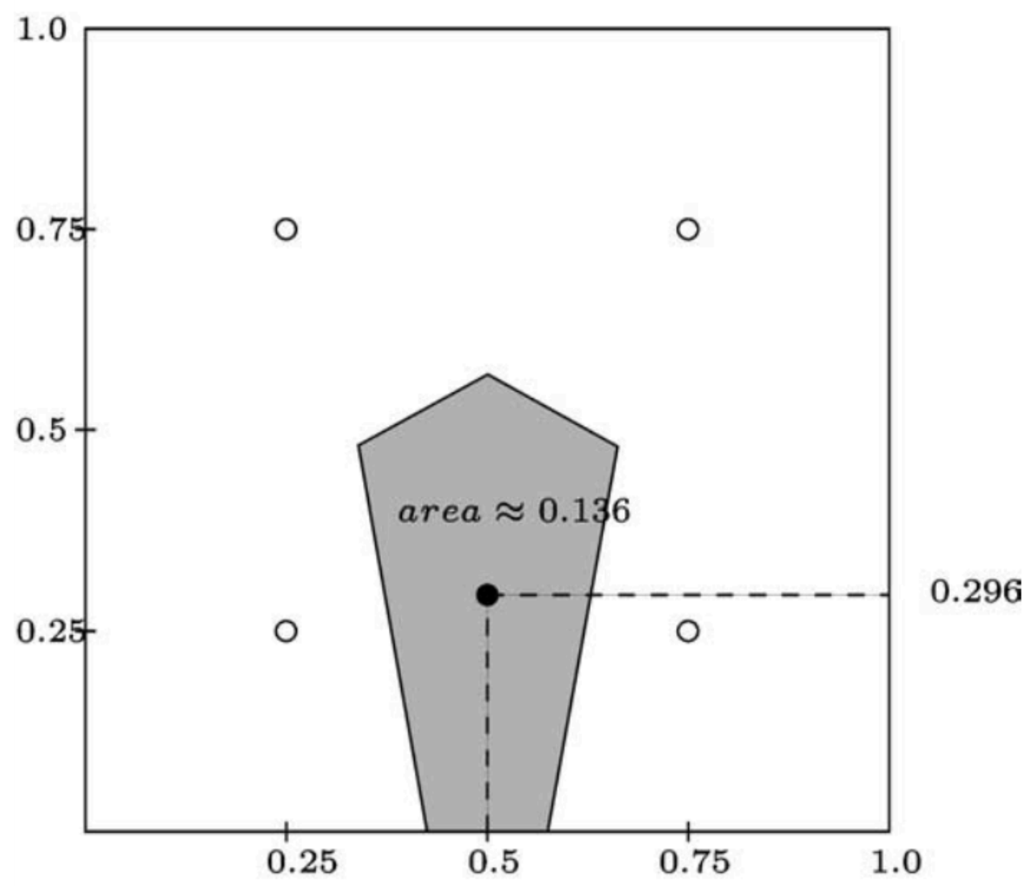

Corollary 2. *If* $n \geqslant 3$, *then Wilma can only win by placing her points in a* $1 \times n$ *grid.*

The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

Lemma 6. *Let $n \geqslant 3$. Barney can win if $\rho > \sqrt{2}/n$; otherwise, he loses.*

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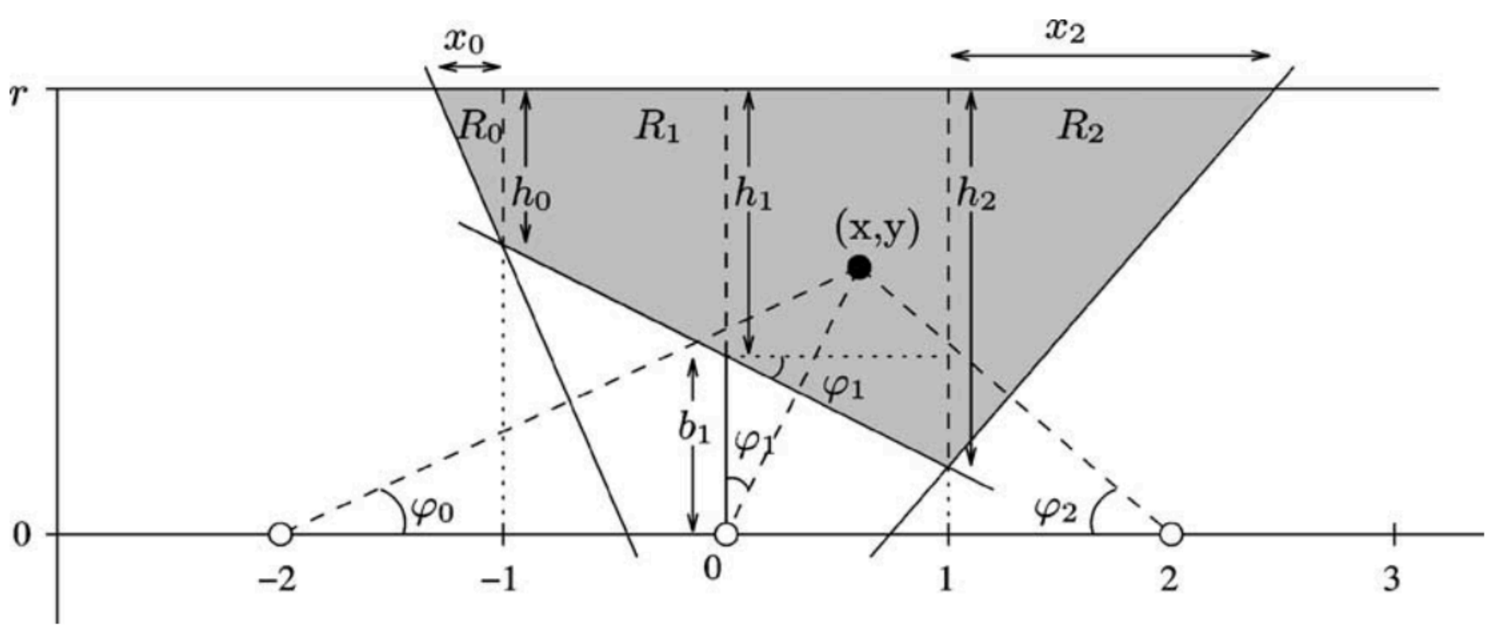

Fig. 3. Wilma has placed at least three points on a line.

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Lemma 6. *Let $n \geqslant 3$. Barney can win if $\rho > \sqrt{2}/n$; otherwise, he loses.*

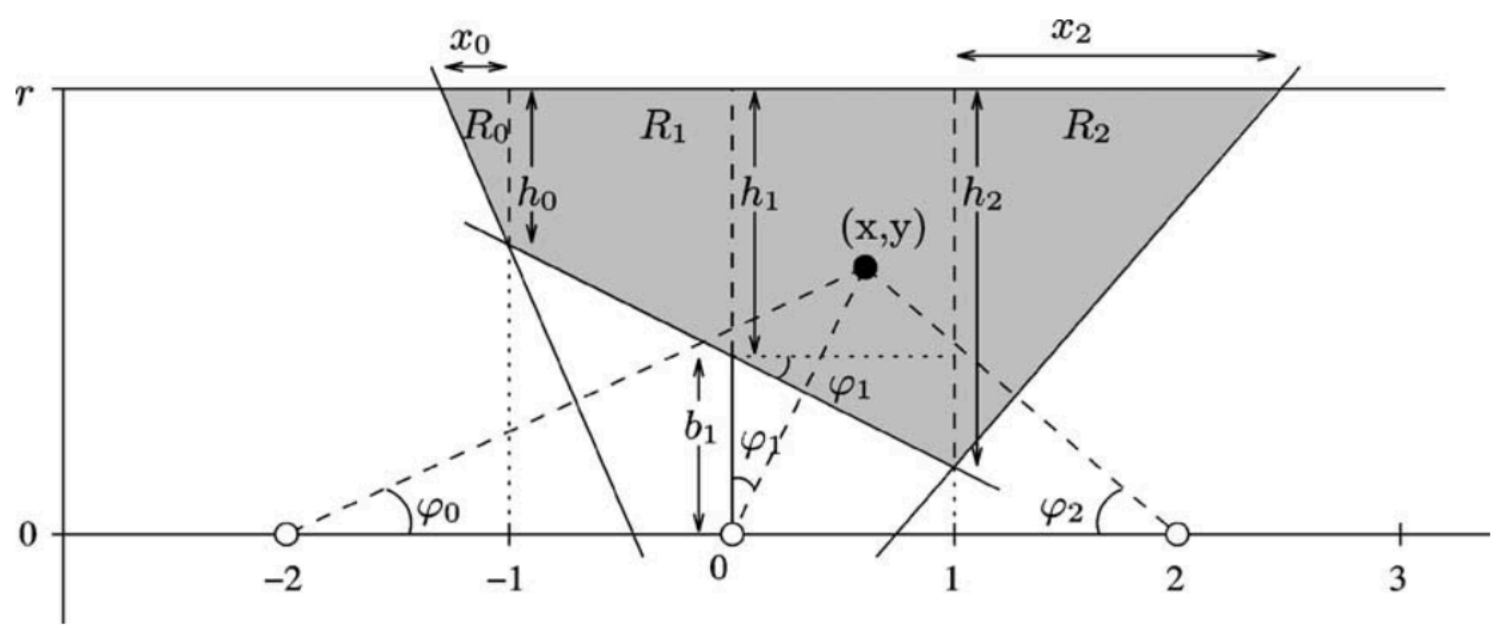

Fig. 3. Wilma has placed at least three points on a line.

Theorem 7. *If $n \geqslant 3$ and $\rho > \sqrt{2}/n$, or $n = 2$ and $\rho > \sqrt{3}/2$, then Barney wins. In all other cases, Wilma wins.*

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The One-Round Voronoi Game Replayed [Fekete and Meijer 2003/2005]

General polygons instead of rectangles?!

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Theorem 8. *For a polygon with holes, it is NP-hard to maximize the area Barney can claim, even if all of Wilma's points have been placed.*

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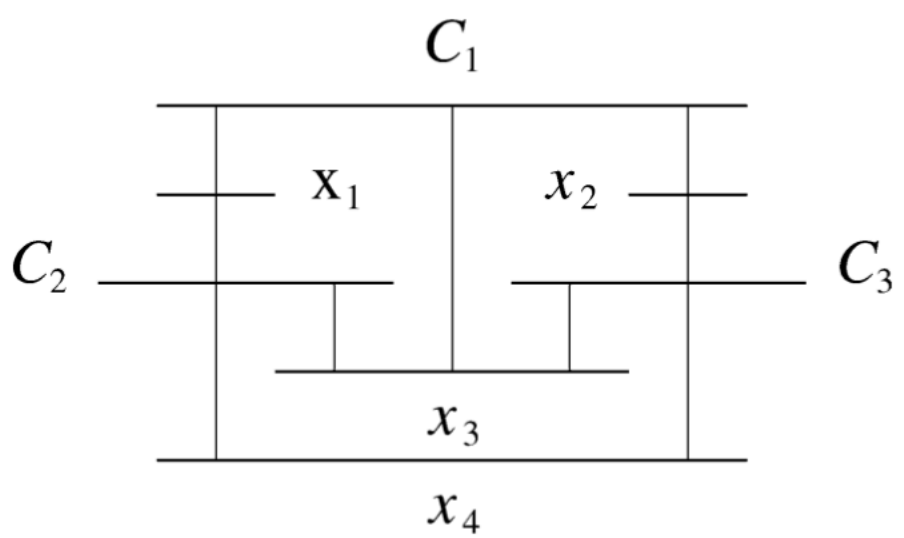

Fig. 4. A geometric representation of the variable-clause incidence graph G_I for the Planar 3SAT instance $I = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_4)$.

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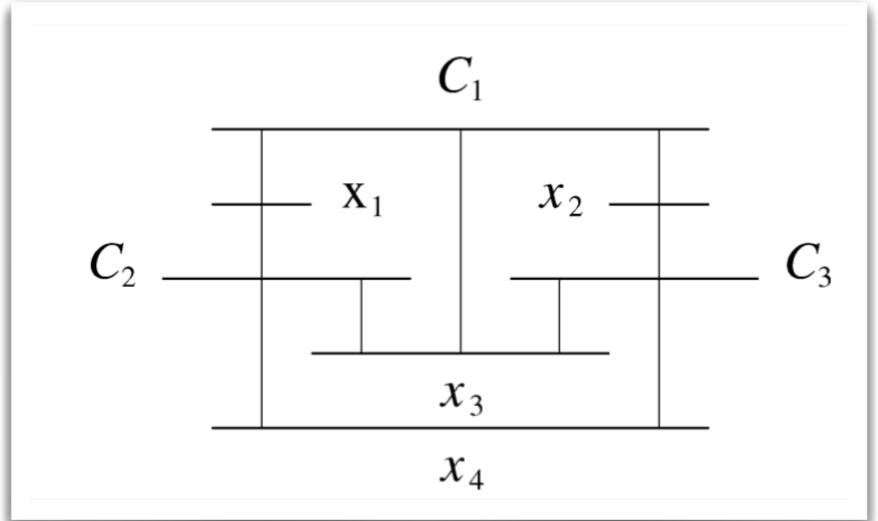

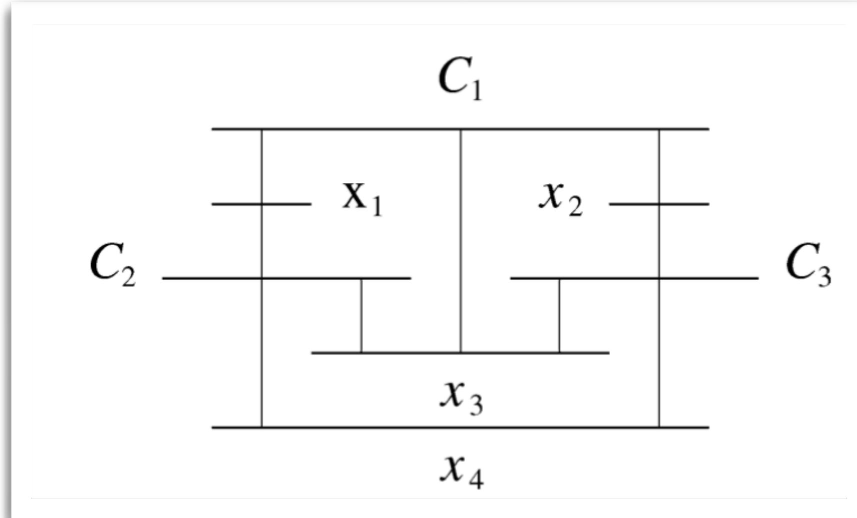
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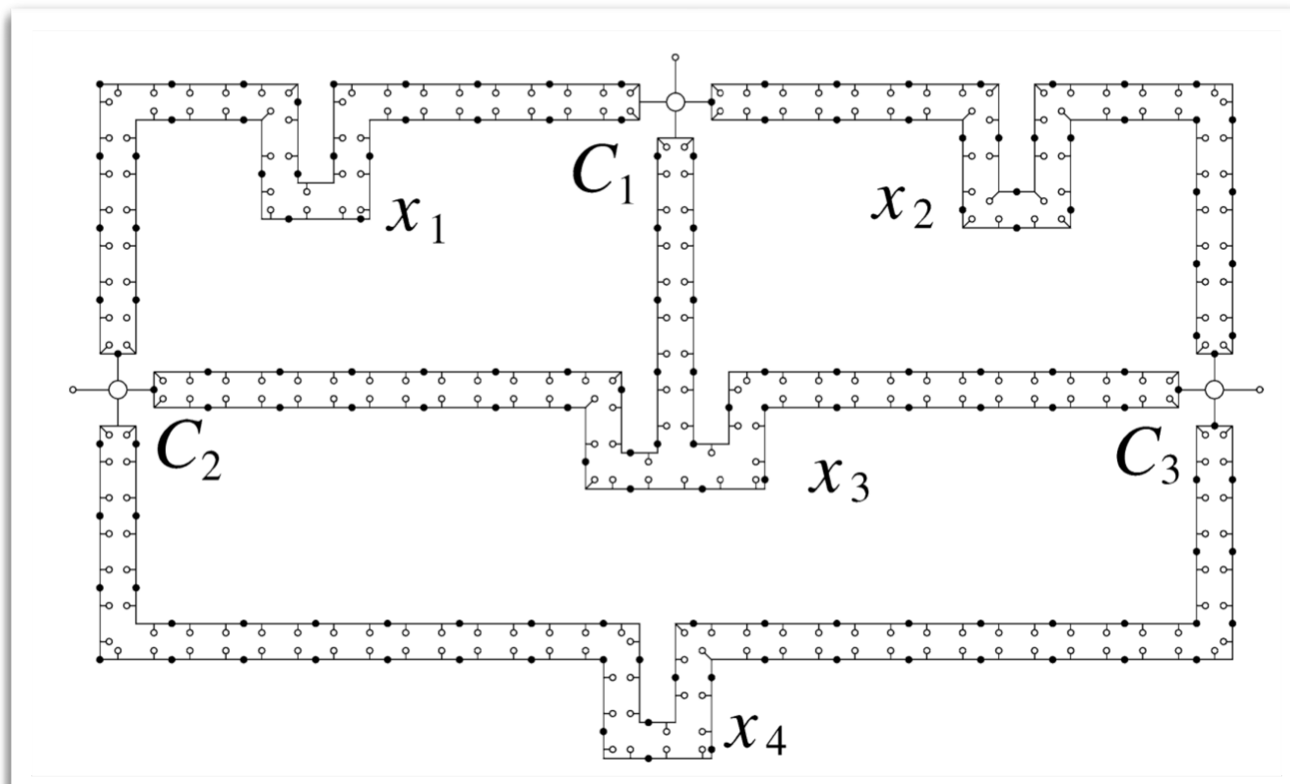
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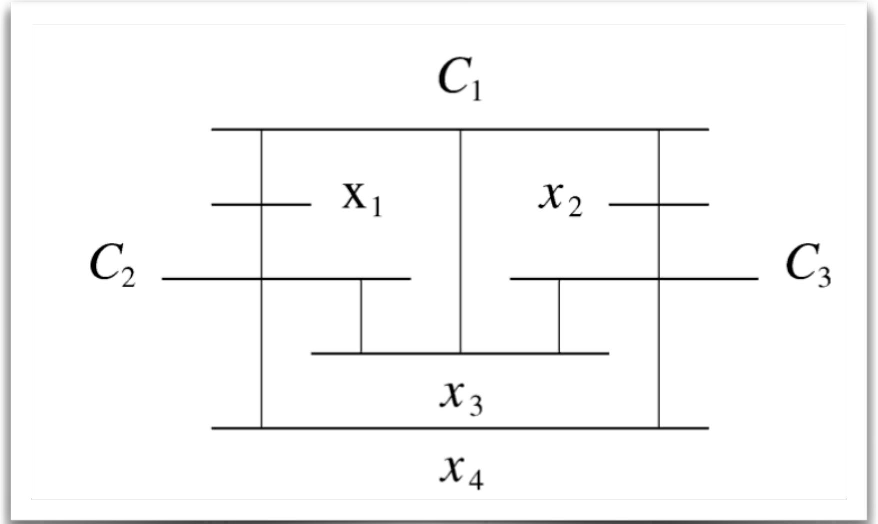

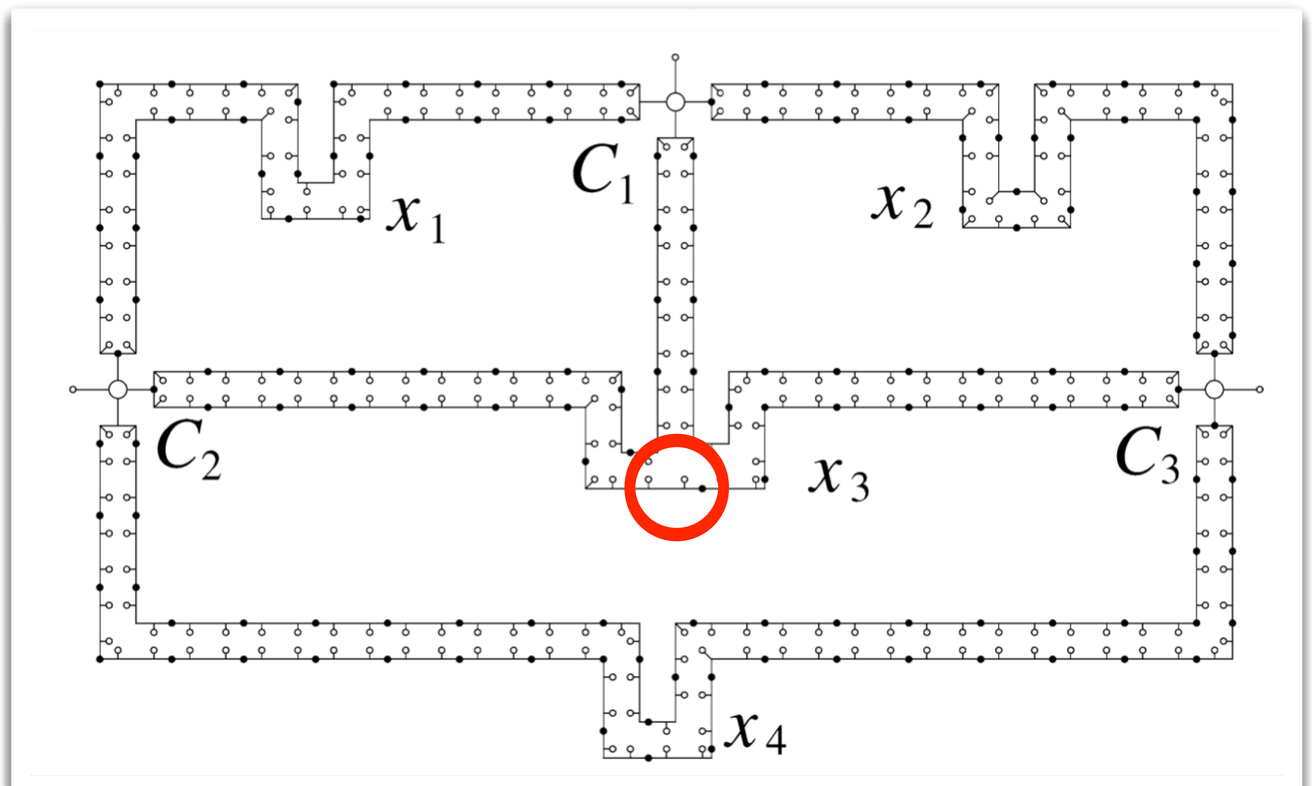

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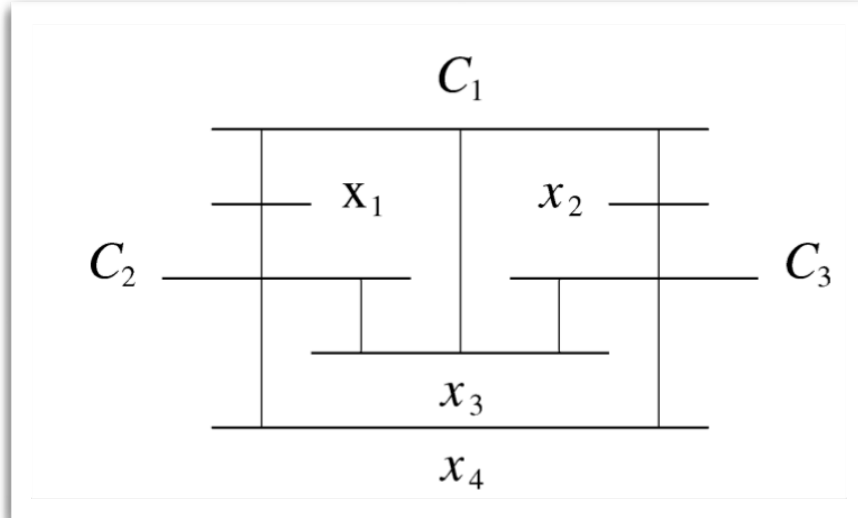

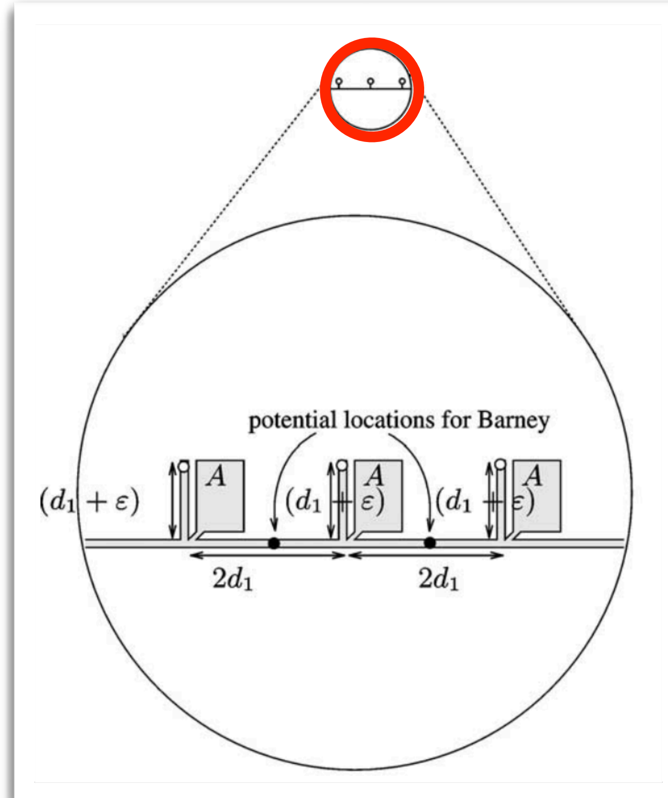

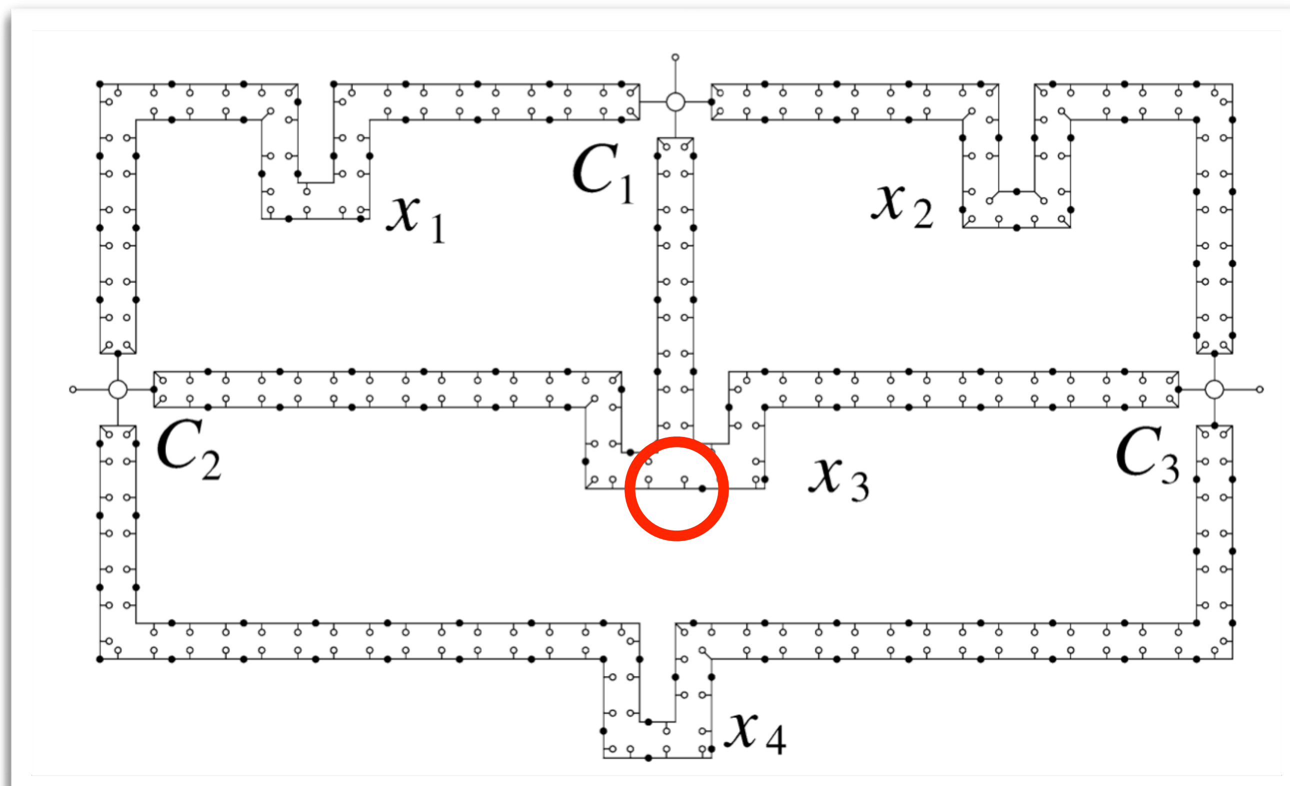

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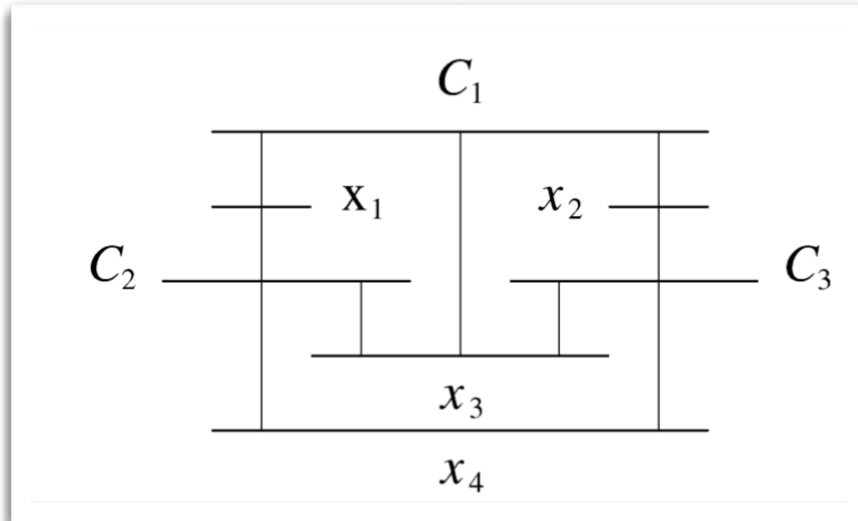

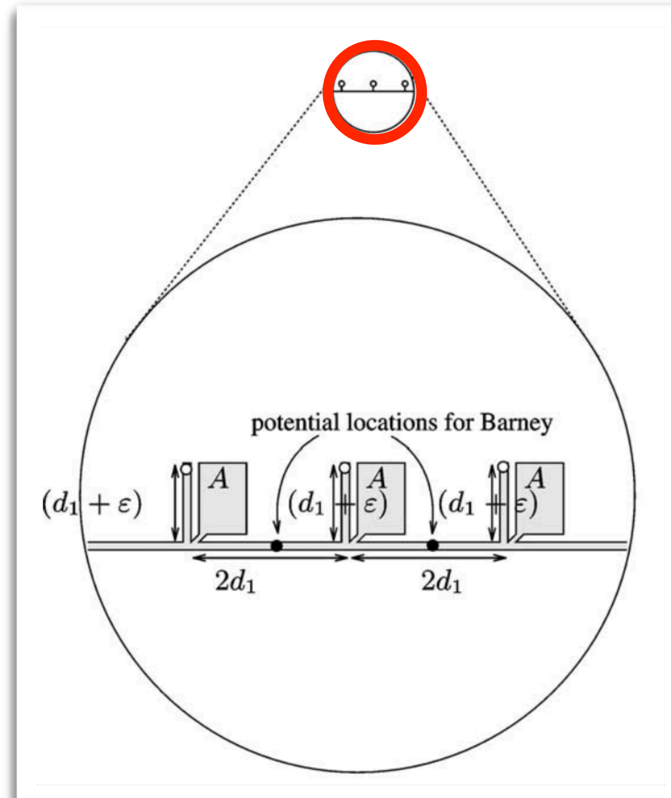

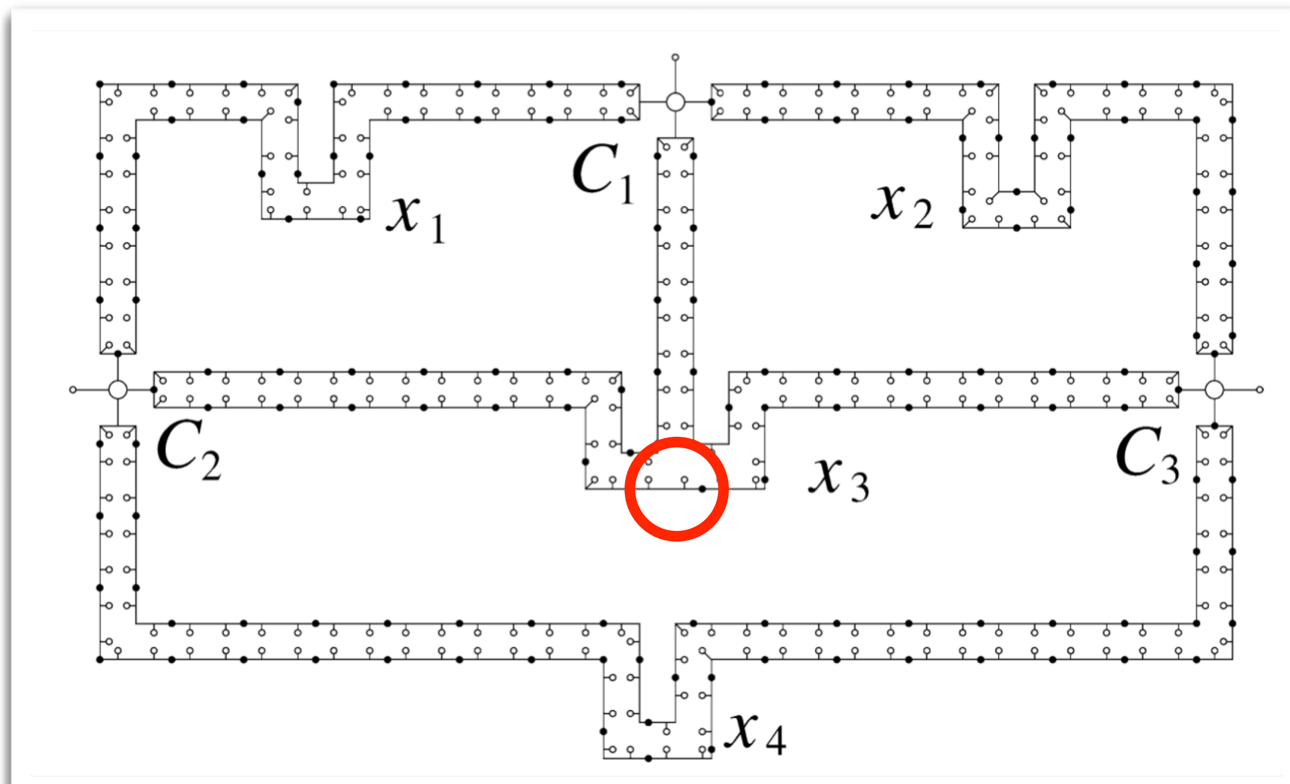

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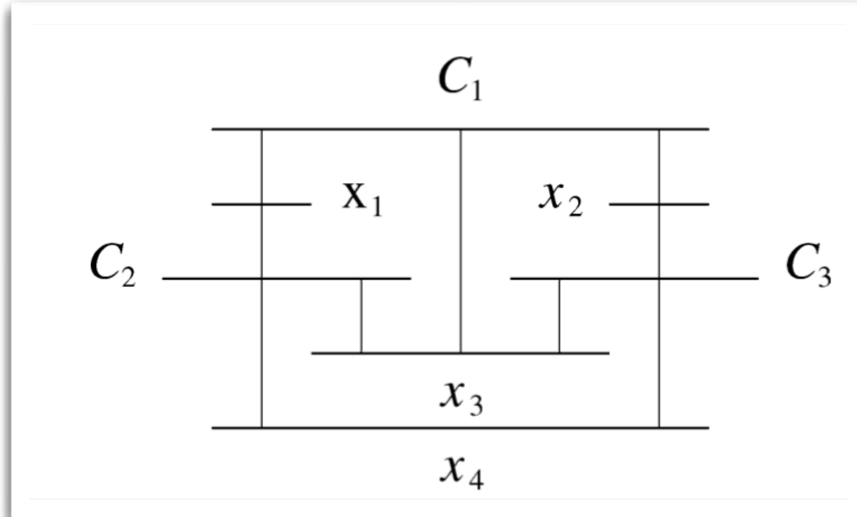

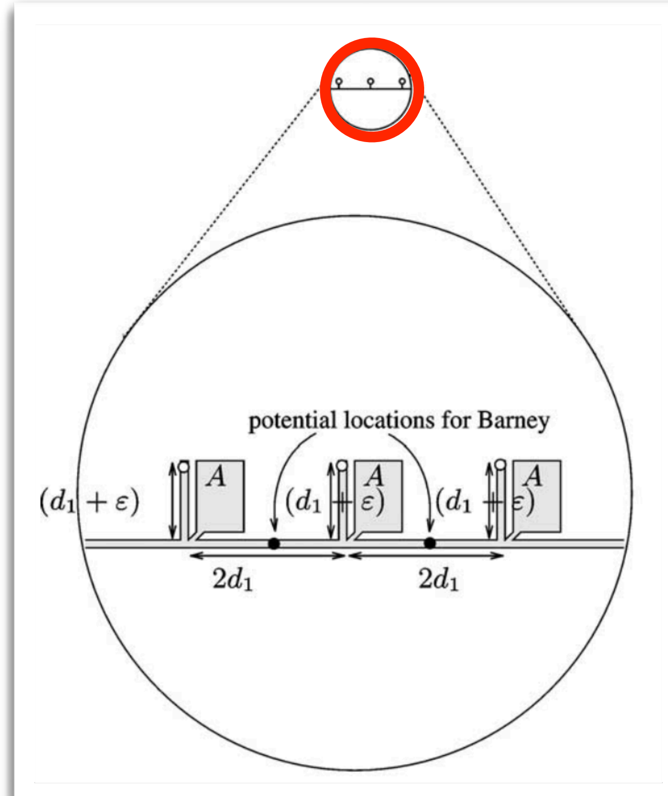

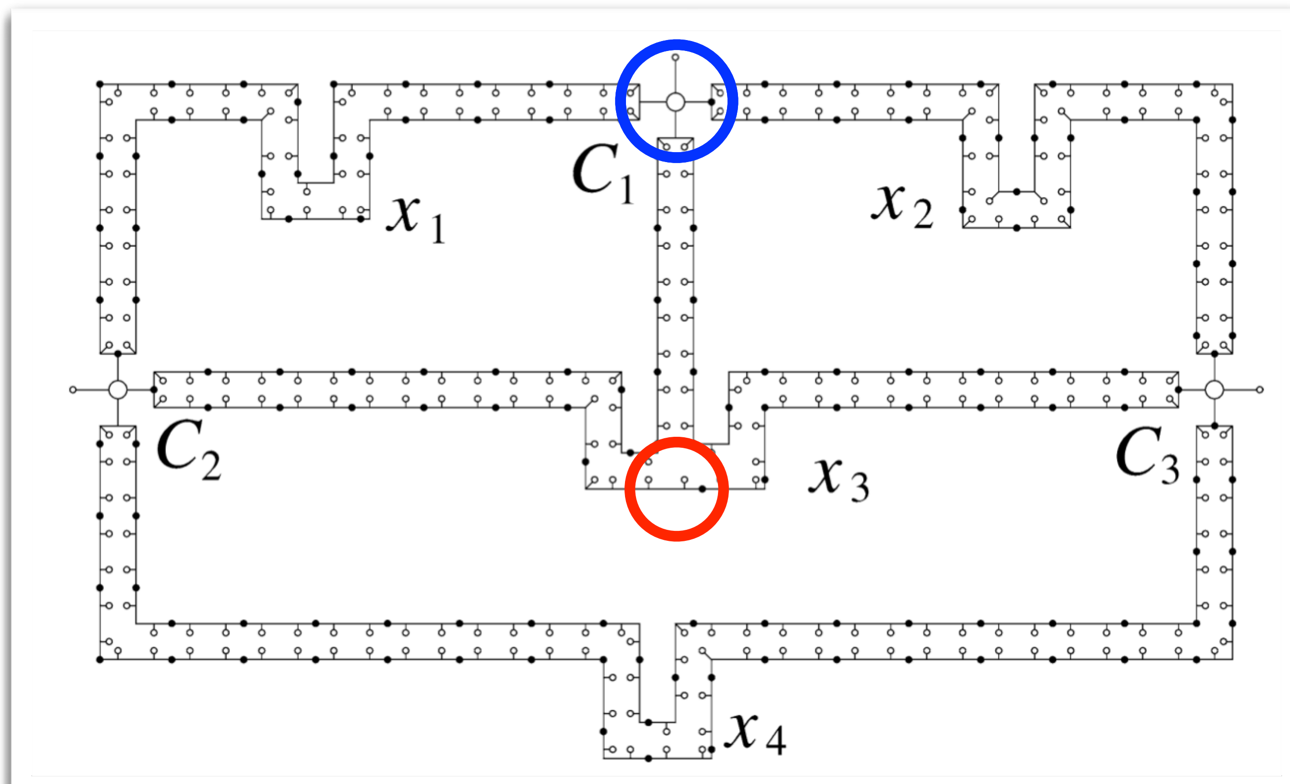

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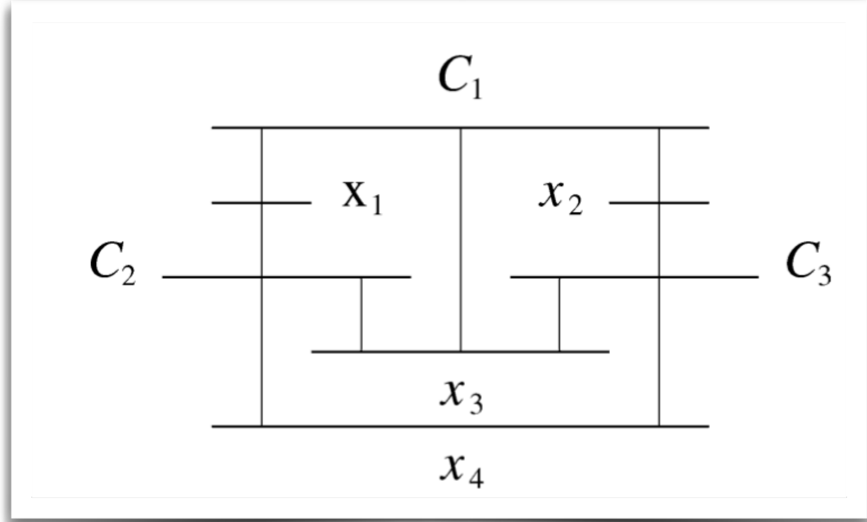

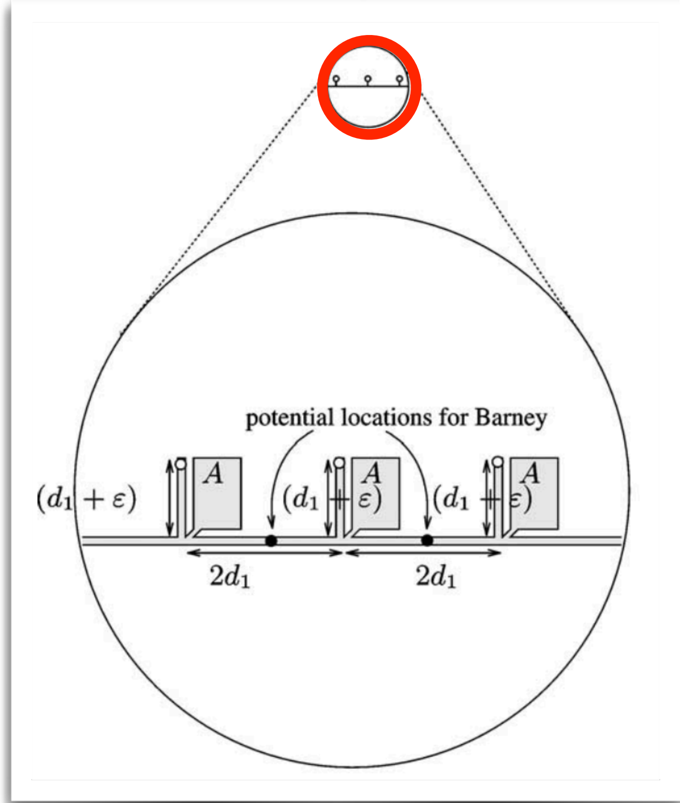

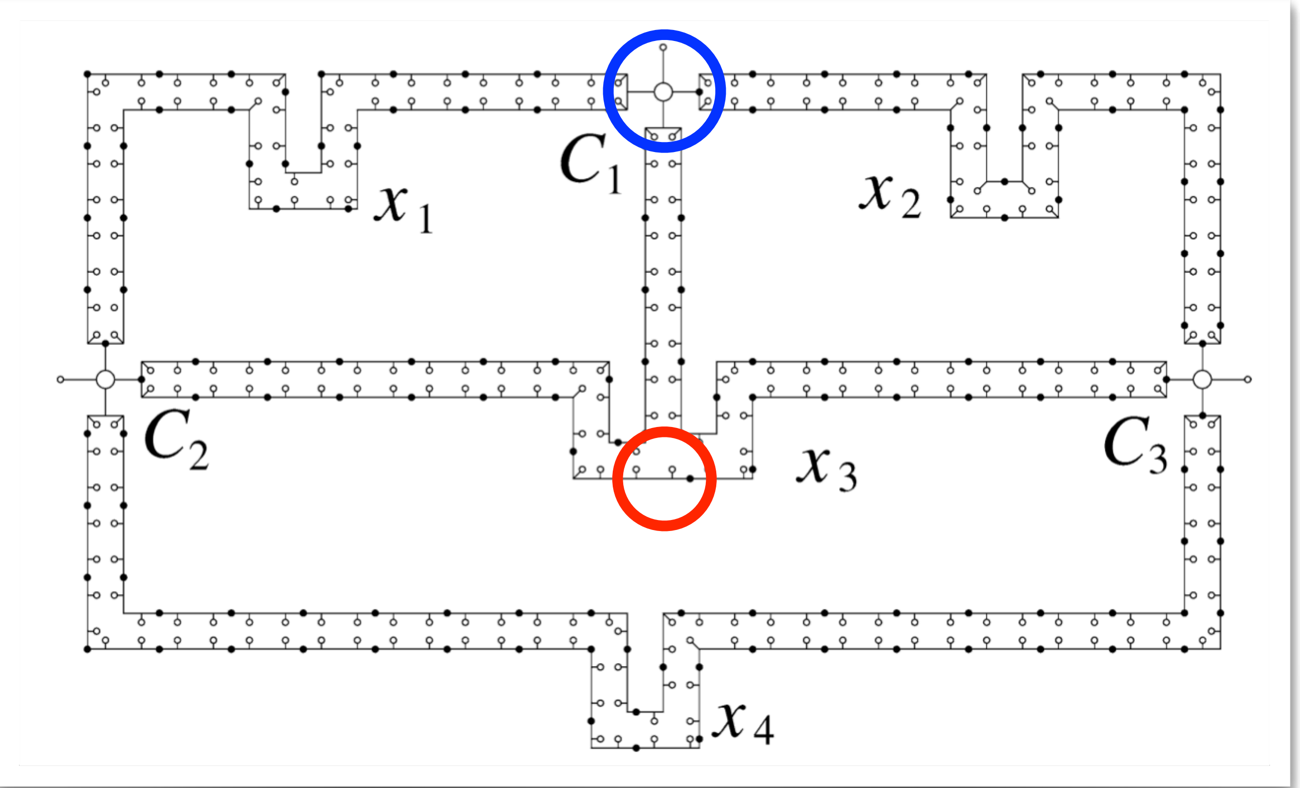

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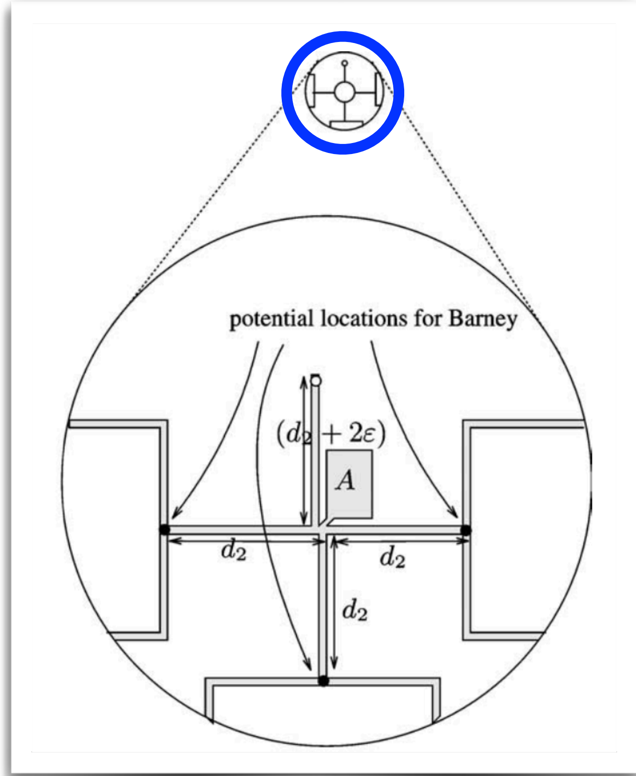

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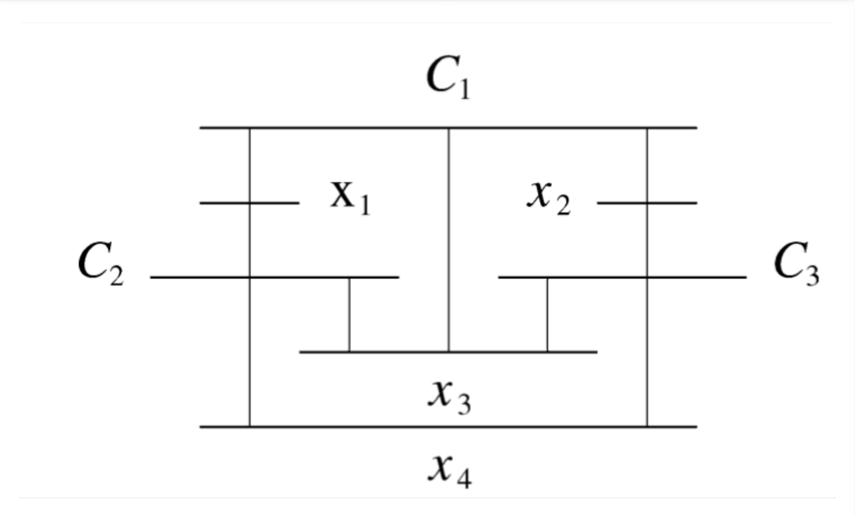
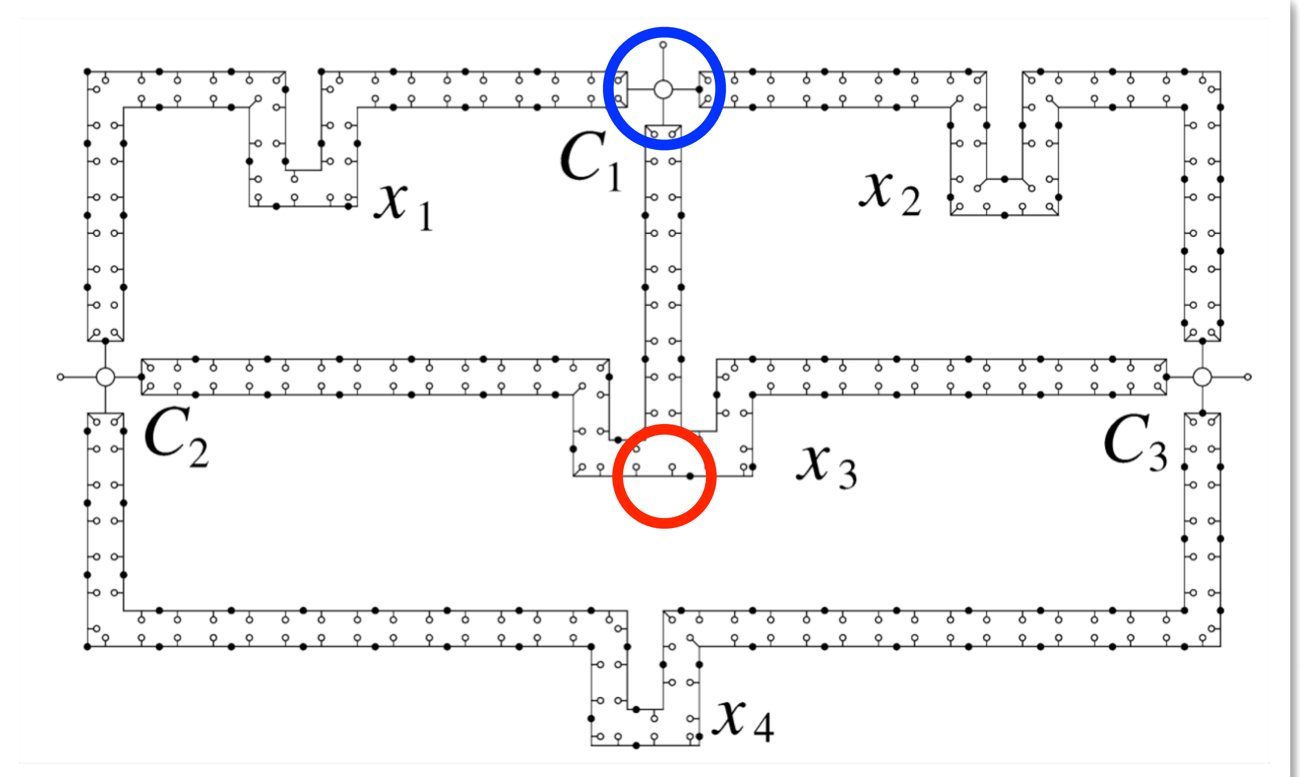
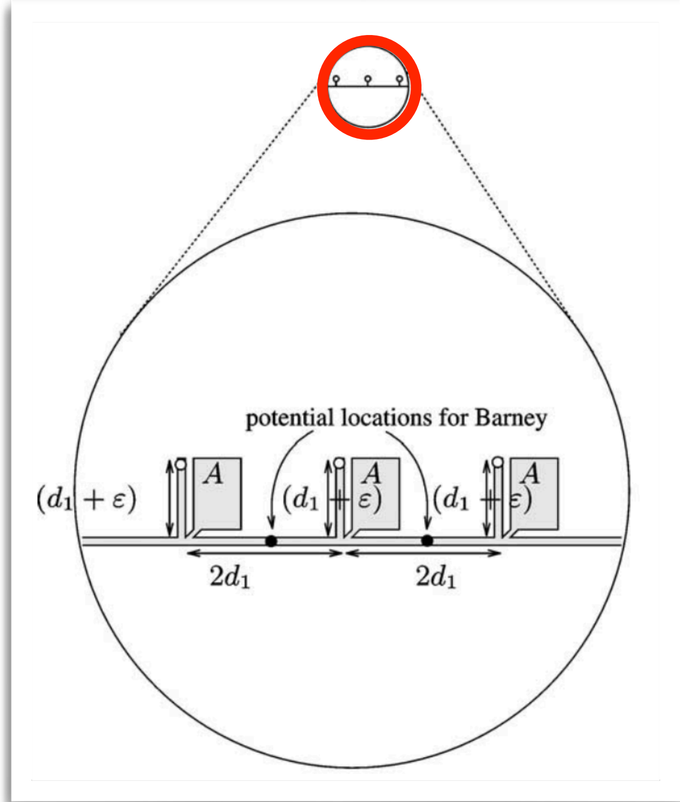
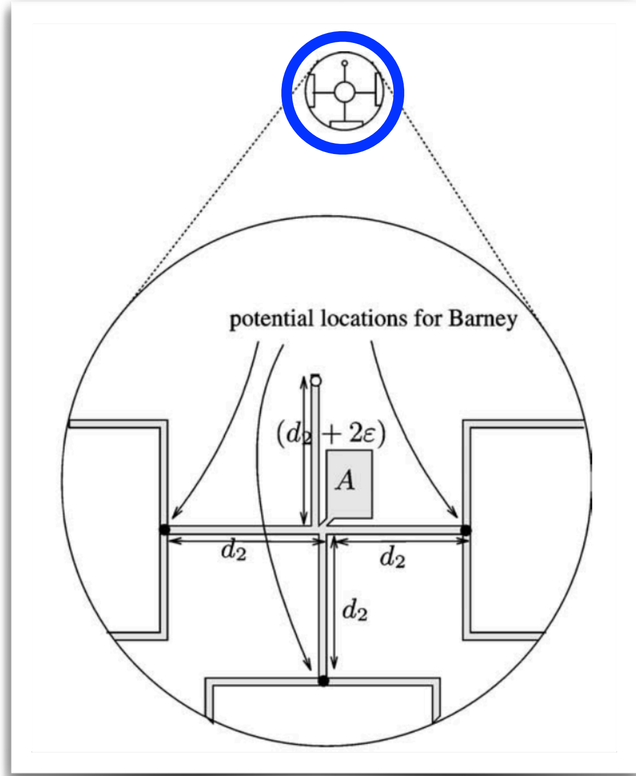






- For each variable, choose *true* or *false* by picking all even or all odd black positions.
- For each clause, a satisfying truth assignment picks additional area.



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Locating a Single Site [Cheong, Efrat and Har-Peled 2007]

Discrete Comput Geom 37:545–563 (2007)
DOI: 10.1007/s00454-007-1328-5

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Finding a Guard that Sees Most and a Shop that Sells Most*

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Abstract. We present a near-quadratic time algorithm that computes a point inside a simple polygon P in the plane having approximately the largest visibility polygon inside P, and a near-linear time algorithm for finding the point that will have approximately the largest Voronoi region when added to an n-point set in the plane. We apply the same technique to find the translation that approximately maximizes the area of intersection of two polygonal regions in near-quadratic time, and the rigid motion doing so in near-cubic time.

1. Introduction

We consider two problems where our goal is to find a point x such that the area of the region $V(x)$ "controlled" by x is as large as possible. In the first problem we are given a simple polygon P, and $V(x)$ is the *visibility polygon* of x, that is, the region of points y inside P such that the segment xy does not intersect the boundary of P. In the second problem we are given a set of points T, and $V(x)$ is the *Voronoi cell* of x in the Voronoi diagram of the set $T \cup \{x\}$, that is, the set of points that are closer to x than to any point in T.

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Theorem 3.3. *Given a set T of n points in the plane and a parameter $\delta > 0$, one can deterministically compute, in time $O(n/\delta^4 + n\log n)$, a point x_{app} such that $\mu(x_{app}) \geq (1-\delta)\mu_{opt}$.*

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The One-Round Manhattan Game [Byrne, Fekete, Kalcsics and Kleist 2021]

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Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game *

Thomas Byrne[1] [0000−0003−0548−4086], Sándor P. Fekete[2] [0000−0002−9062−4241], Jörg Kalcsics[1] [0000−0002−5013−3448], and Linda Kleist[2] [0000−0002−3786−916X]

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Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

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Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria for the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge is that facility location problems often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: e.g., the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

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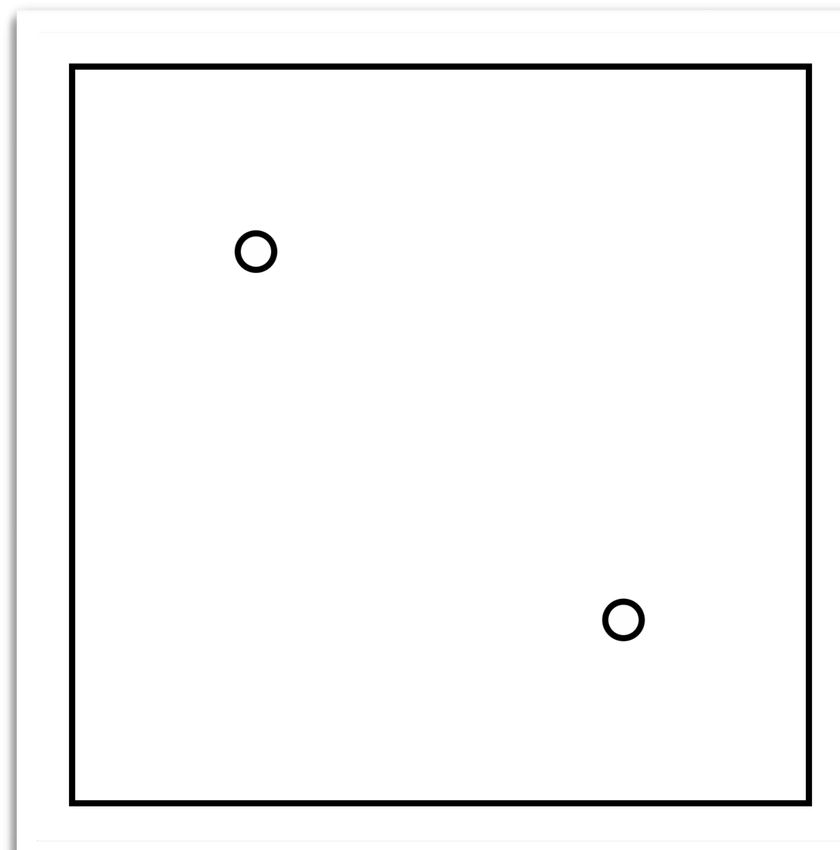

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Thomas Byrne[1][0000−0003−0548−4086], Sándor P. Fekete[2][0000−0002−9062−4241], Jörg Kalcsics[1][0000−0002−5013−3448], and Linda Kleist[2][0000−0002−3786−916X]

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Abstract. We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* configurations (in which the Voronoi cells of individual facilities are equalized with respect to geometric properties) is richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

1 Introduction

Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria for the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge is that facility location problems often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: e.g., the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

* A full version can be found at arXiv: 2011.13275 [6].

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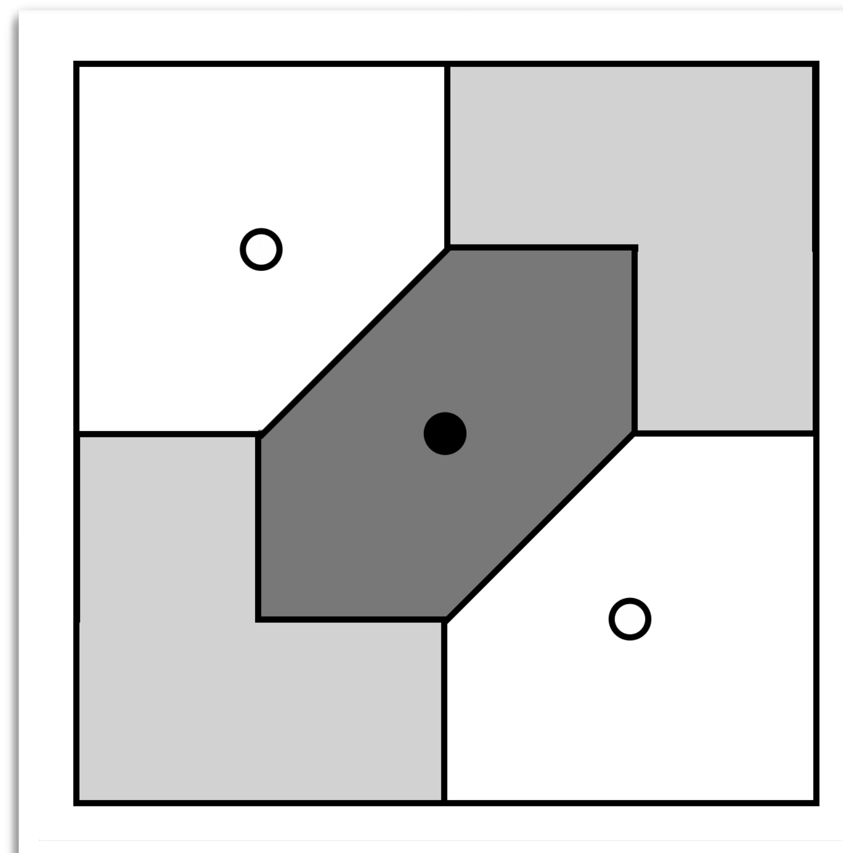

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The One-Round Manhattan Game [Byrne, Fekete, Kalcsics and Kleist 2021]

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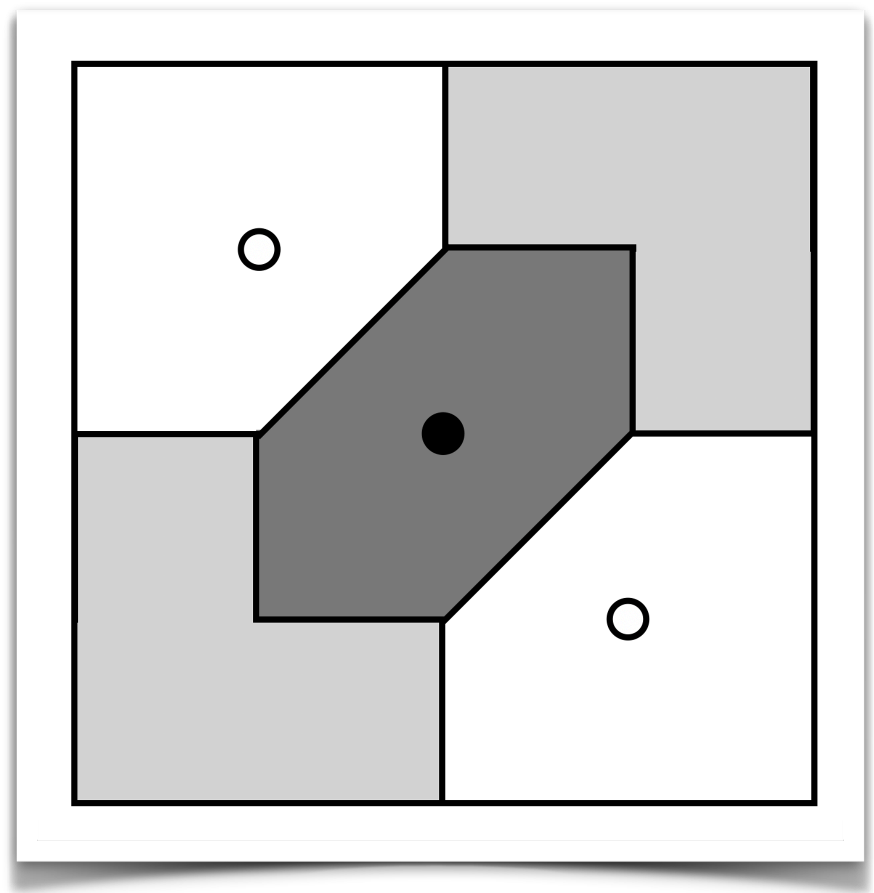

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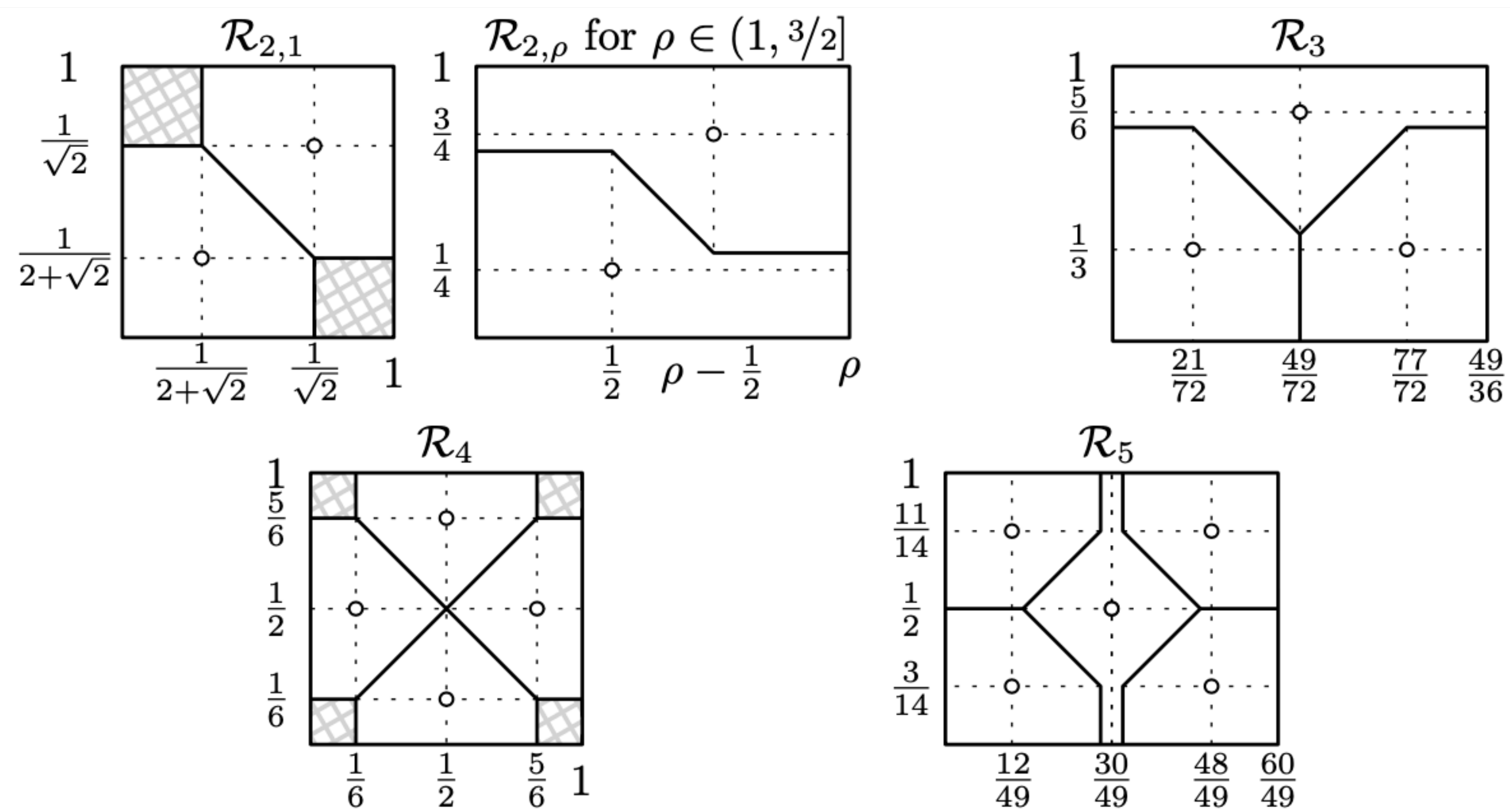

Fig. 4. Non-grid examples of balanced point sets of cardinality 2, 3, 4, and 5.

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The One-Round Manhattan Game [Byrne, Fekete, Kalcsics and Kleist 2021]

Theorem 15. *White has a winning strategy for placing n points in a $(1 \times \rho)$ rectangle with $\rho \geq 1$ if and only if $\rho \geq n$; otherwise Black has a winning strategy. Moreover, if $\rho \geq n$, the unique winning strategy for White is to place a $1 \times n$ grid.*

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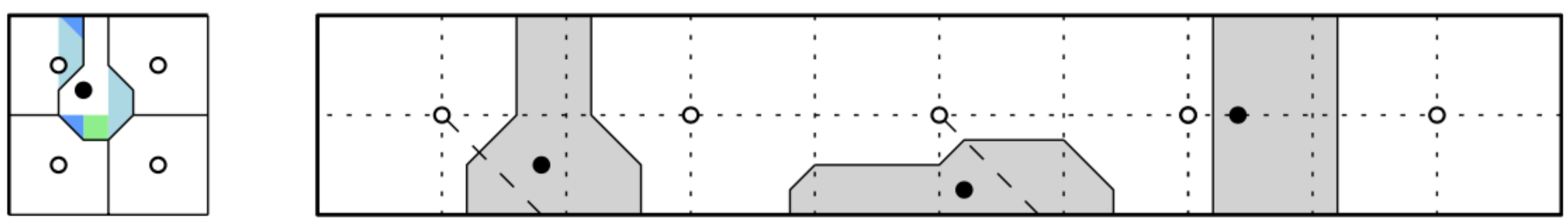

Fig. 9. Illustration of the proof of Theorem 15. (Left) A black winning point in a 2×2 grid. (Right) Every black cell has an area $\leq {}^1\!/_{2n} \cdot \mathcal{A}(R)$. Moreover, only $n-1$ locations result in cells of that size.

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Voronoi Game in Graphs [Teramoto, Demaine and Uehara 2011]

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Traveling salesmen in the presence of competition

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Abstract

We propose the "competing salesmen problem" (CSP), a two-player competitive version of the classical traveling salesman problem. This problem arises when considering two competing salesmen instead of just one. The concern for a shortest tour is replaced by the necessity to reach any of the customers before the opponent does.

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Theorem 1. *The decision problem whether player* I *can win in* CSP(1, 1) *is* PSPACE-*complete, even for the special case of bipartite graphs, with both players starting at distance* 2 *from each other.*

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The Voronoi game on graphs and its complexity

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[2]Computer Science and Artificial Intelligence Lab, the Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.
[3]School of Information Science, Japan Advanced Institute of Science and Technology, Ishikawa, Japan.

Abstract

The Voronoi game is a two-person game which is a model for a competitive facility location. The game is played on a continuous domain, and only two special cases (one-dimensional case and one-round case) are well investigated. We introduce the *discrete* Voronoi game in which the game arena is given as a graph. We first analyze the game when the arena is a large complete k-ary tree, and give an optimal strategy. When both players play optimally, the first player wins when k is odd, and the game ends in a tie for even k. Next we show that the discrete Voronoi game is intractable in general. Even for the one-round case in which the strategy adopted by the first player consist of a fixed single node, deciding whether the second player can win is NP-complete. We also show that deciding whether the second player can win is PSPACE-complete in general.

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A preliminary version was presented at the 2nd IEEE Symposium on Computational Intelligence and Games (CIG 2006). A part of this work was done while the third author was visiting MIT, USA.

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G_{pos}(POS DNF):

Input: A positive DNF formula A (that is, a DNF formula containing no negative literal).

Rule: Two players alternately choose some variable of A which has not been chosen yet. The game ends after all variables of A have been chosen. The first player wins if and only if A is true when all variables chosen by the first player are set to 1 and all variables chosen by the second player are set to 0. (In other words, the first player wins if and only if he takes every variable of some disjunct.)

Output: Determine whether the first player has the winning strategy for A.

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Theorem 4 *The discrete Voronoi game is* PSPACE*-complete in general.*

Thank you for today!