Computational Geometry
Chapter 4: Voronoi Diagrams

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1. Introduction and Motivation
2. Definitions
3. Representing planar partitions
4. Properties
5. Fortune's algorithm
6. Variations
7. The Voronoi game
8. Summary and conclusions

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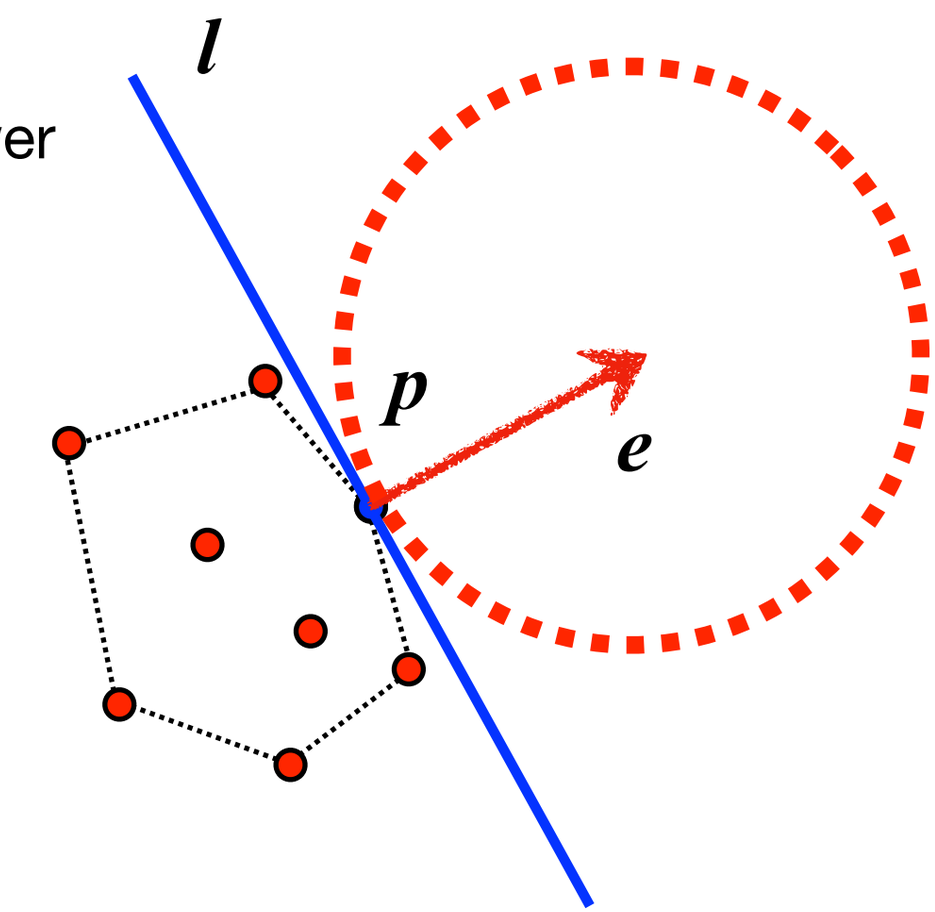
Lemma 4.13

$p \in \mathcal{P}$ lies on boundary of $\mathrm{conv}(\mathcal{P}) \Leftrightarrow V(p)$ unbounded.

Corollary 4.14:

Computing the Voronoi diagram for n points has a lower bound of $\Omega(n \log n)$.



Vironoi Man

Approach:

- Consider a moving „frontier“ between resolved and unresolved part.

Crucial issue:

- $p \in \mathcal{P}$ below ℓ can influence $Vor(p)$ above ℓ.

Observation:

- The separation between resolved and unresolved part for a point p and line ℓ is a curve consisting of points that have equal distance from p and ℓ.

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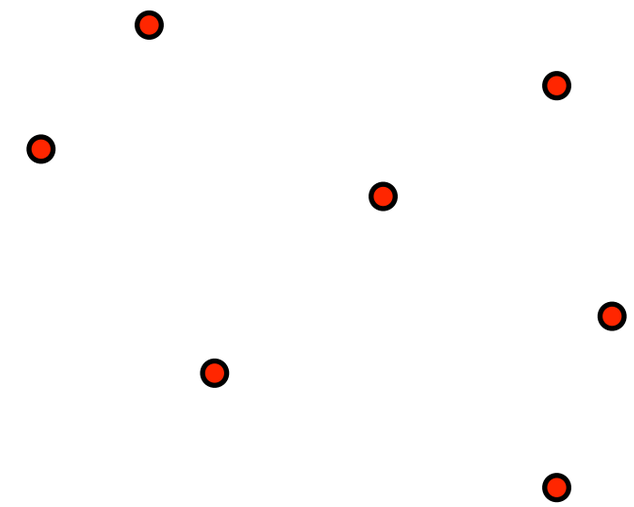

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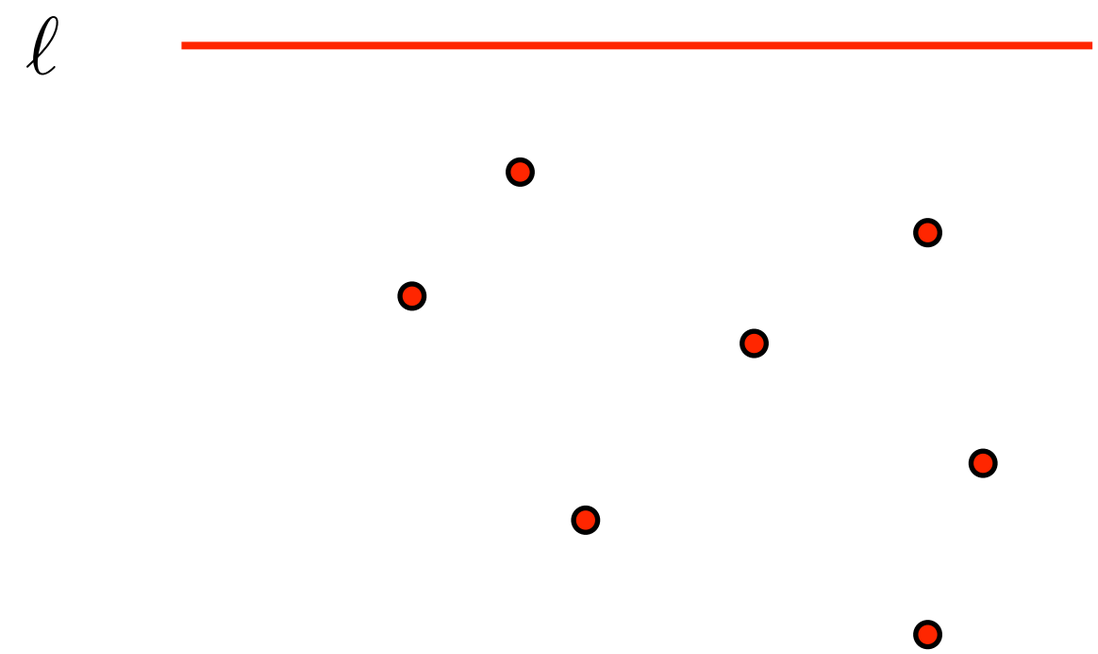

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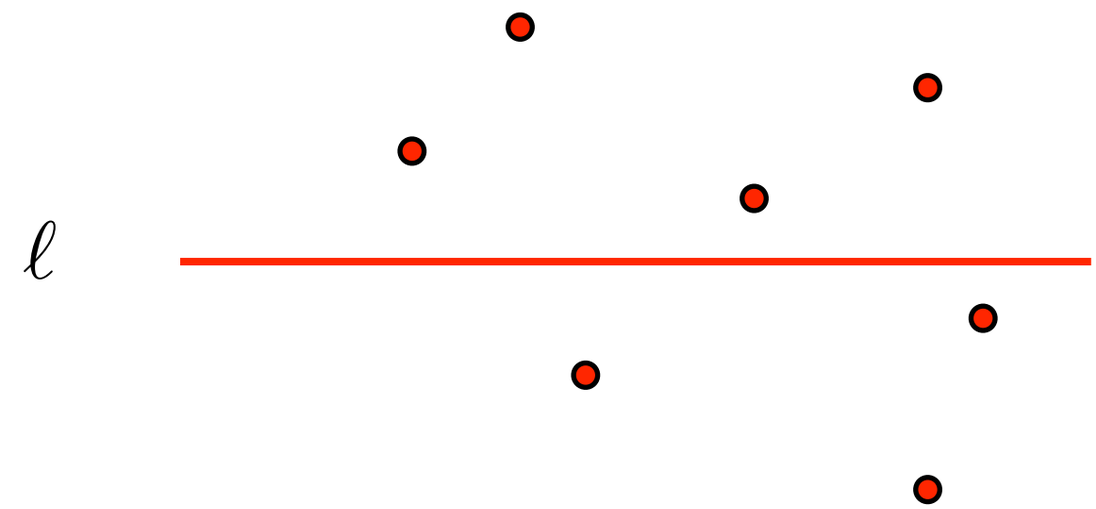

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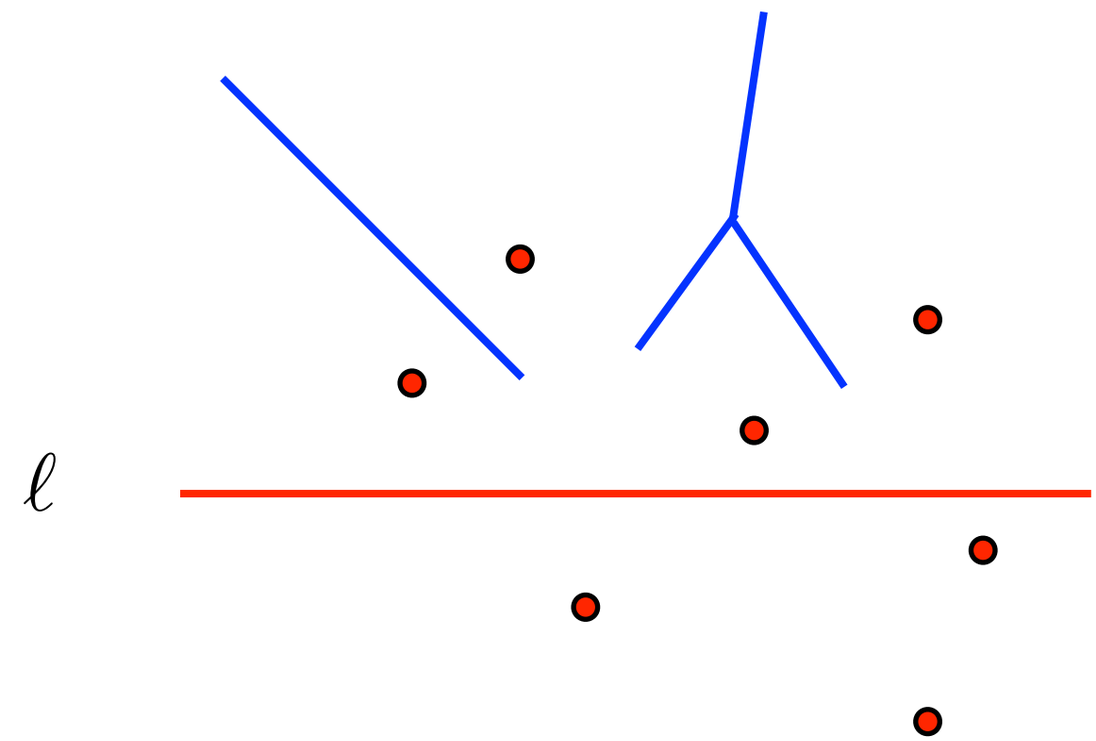

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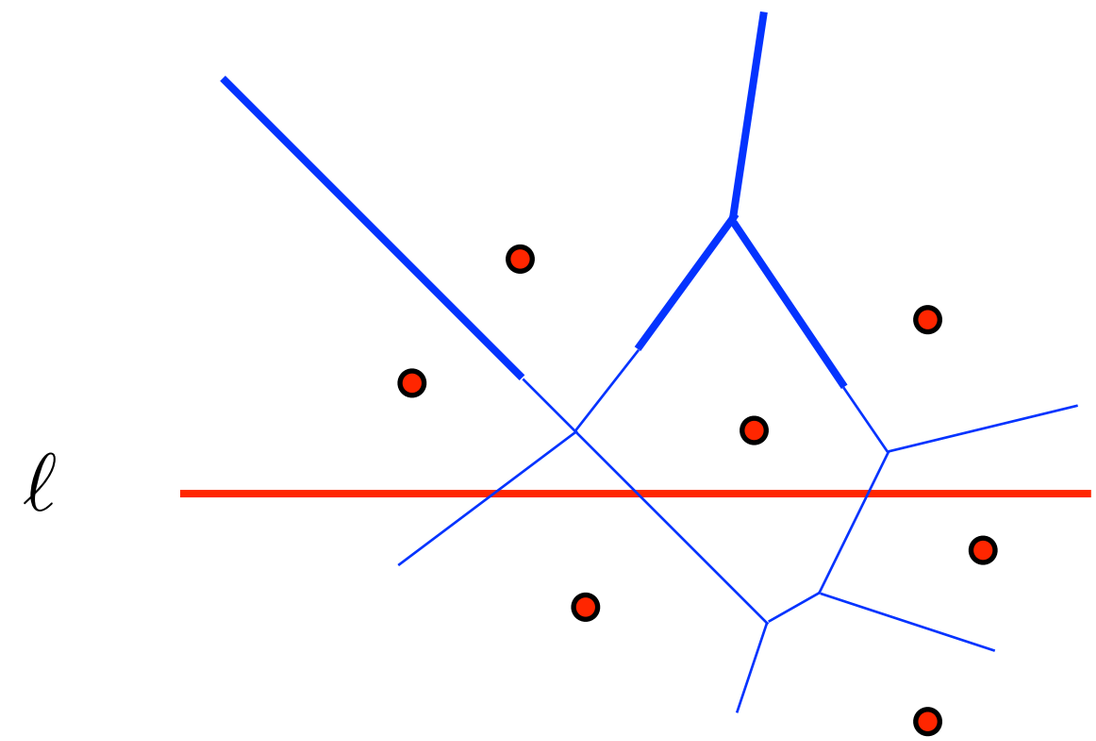

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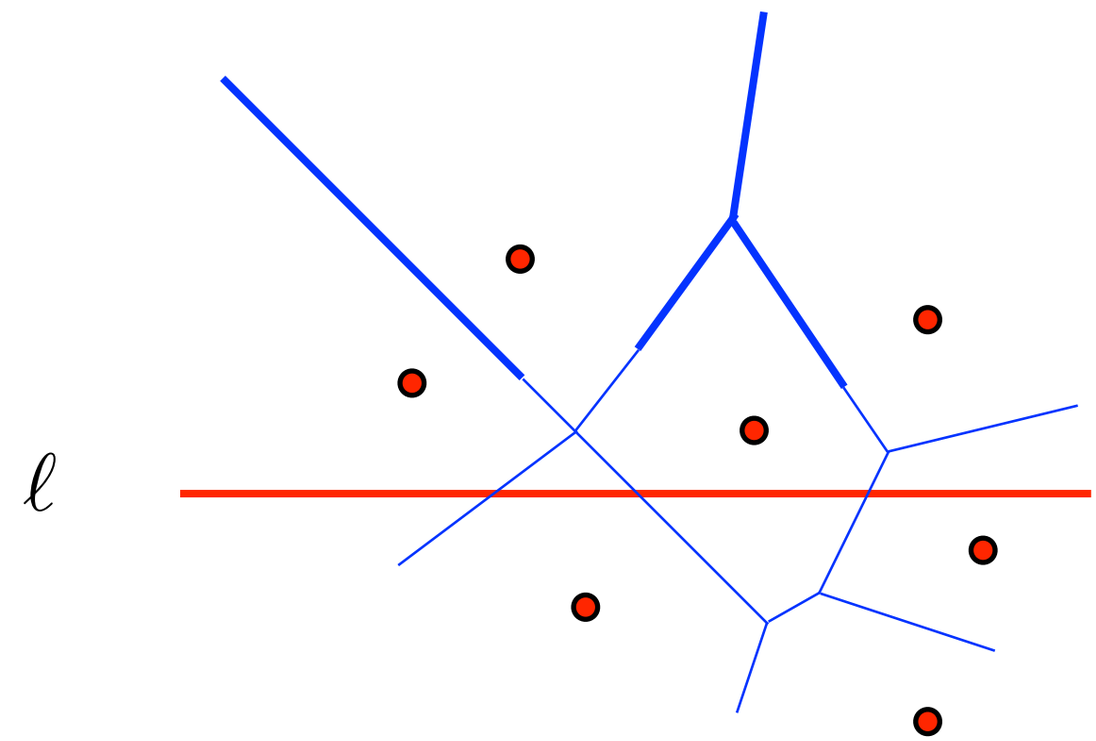

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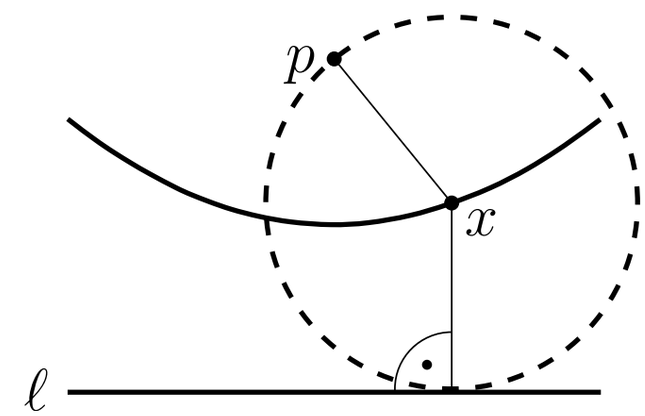

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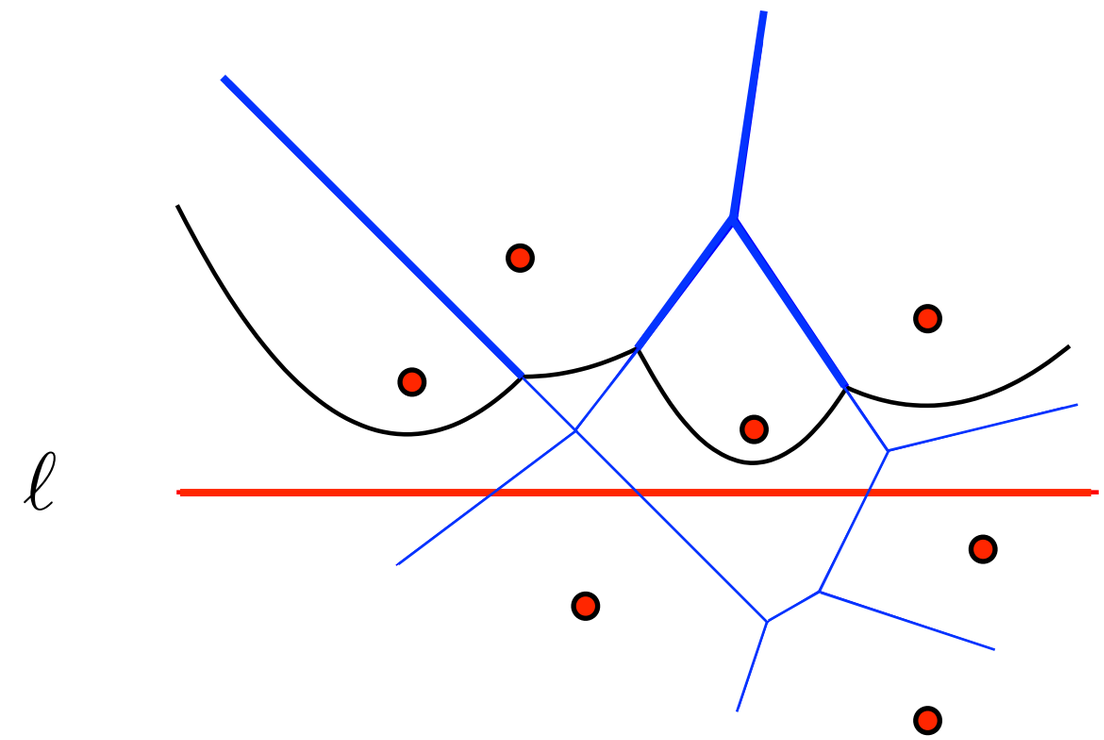

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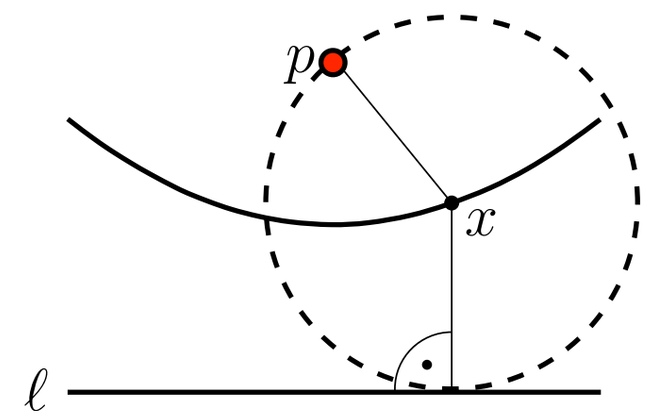

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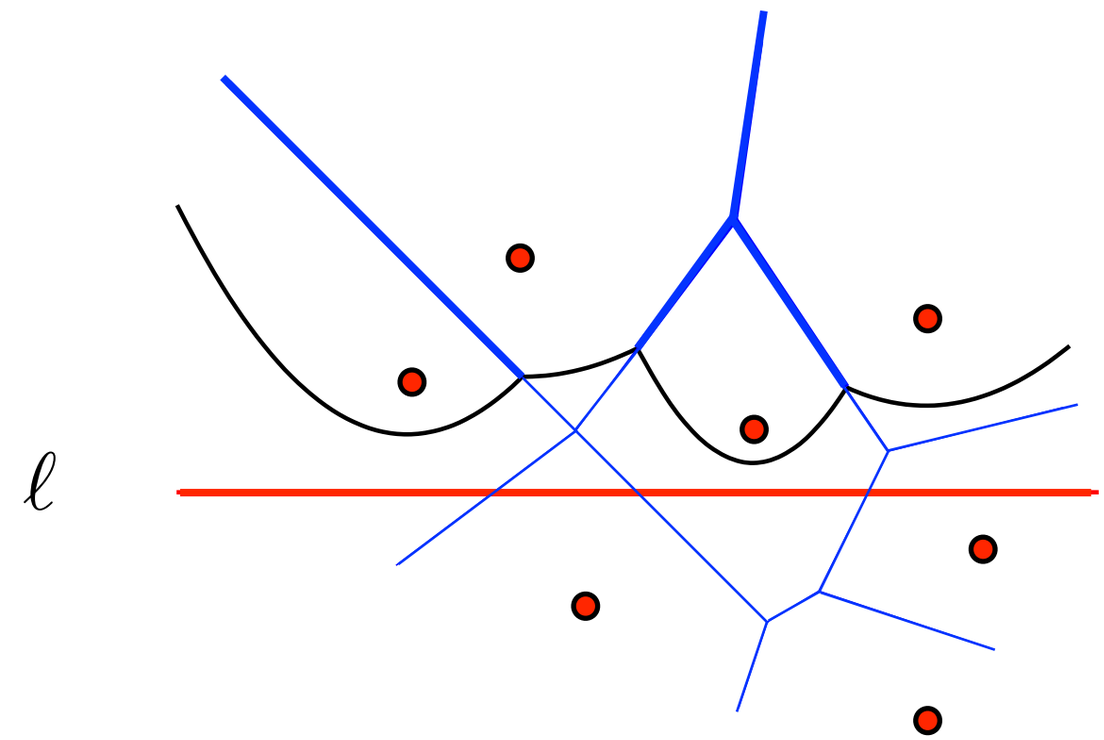

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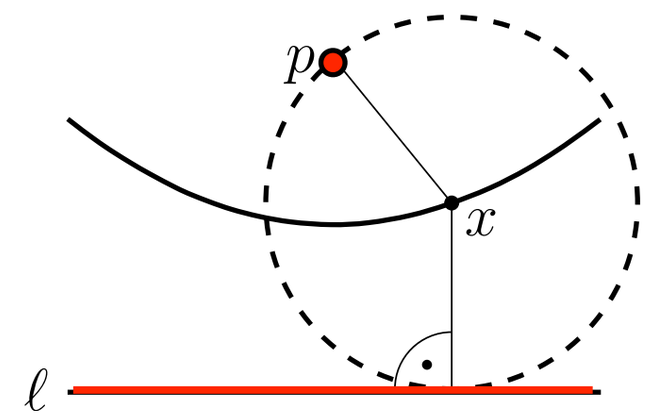

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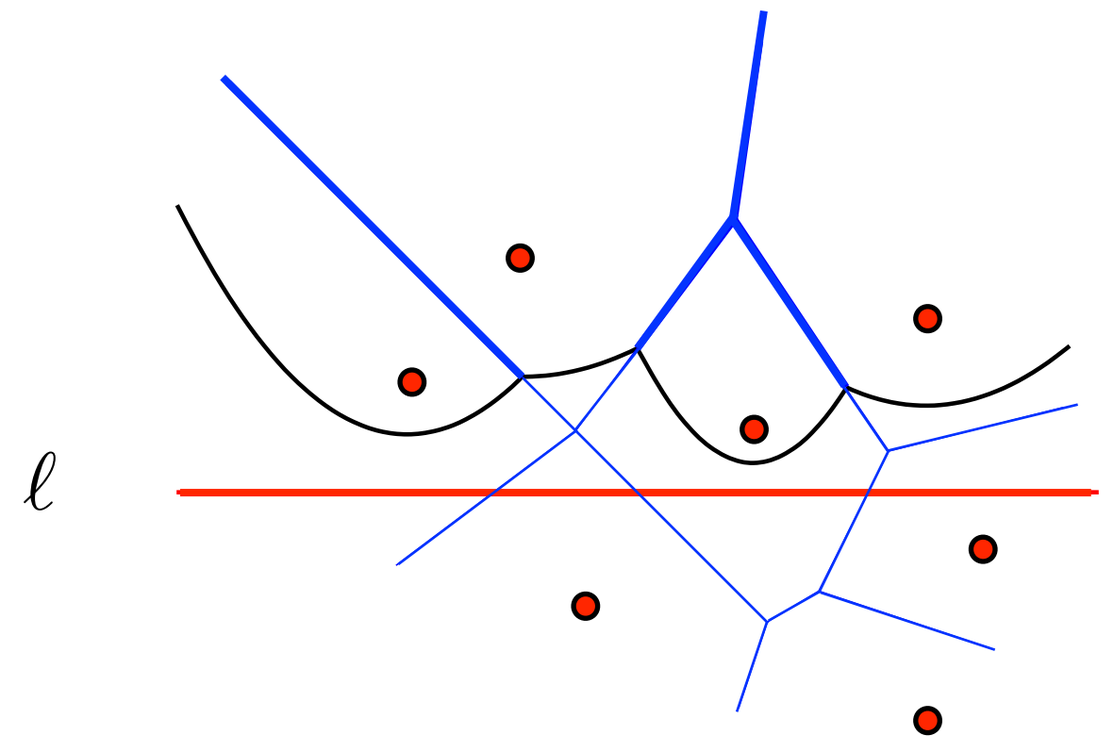

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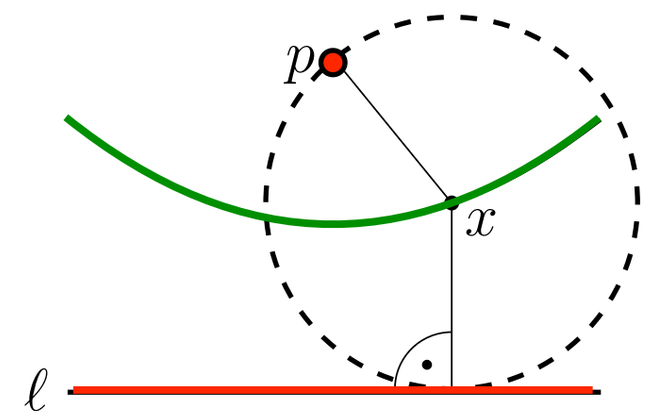

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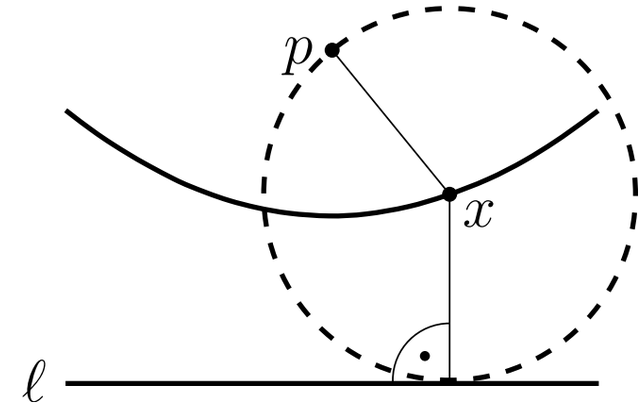
$$\{x \in \mathbb{R}^2 \mid d(x,p) = d(x,\ell)\}$$



Theorem 4.15:

The curve is a parabola (with *focus* p and *directrix* ℓ).

Proof:

Consider $p=(0,s)$ and $X=(x,0)$.
Then $C=(x,y)$ with

$$d_1^2 = x^2 + (y-s)^2$$
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So

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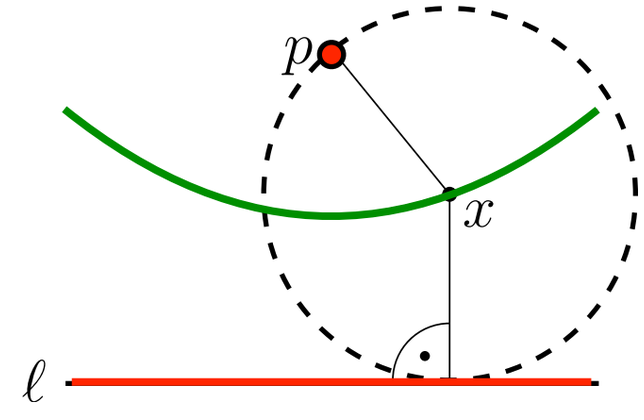

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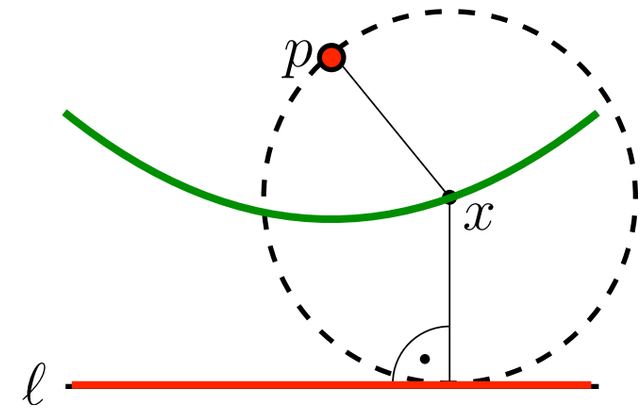

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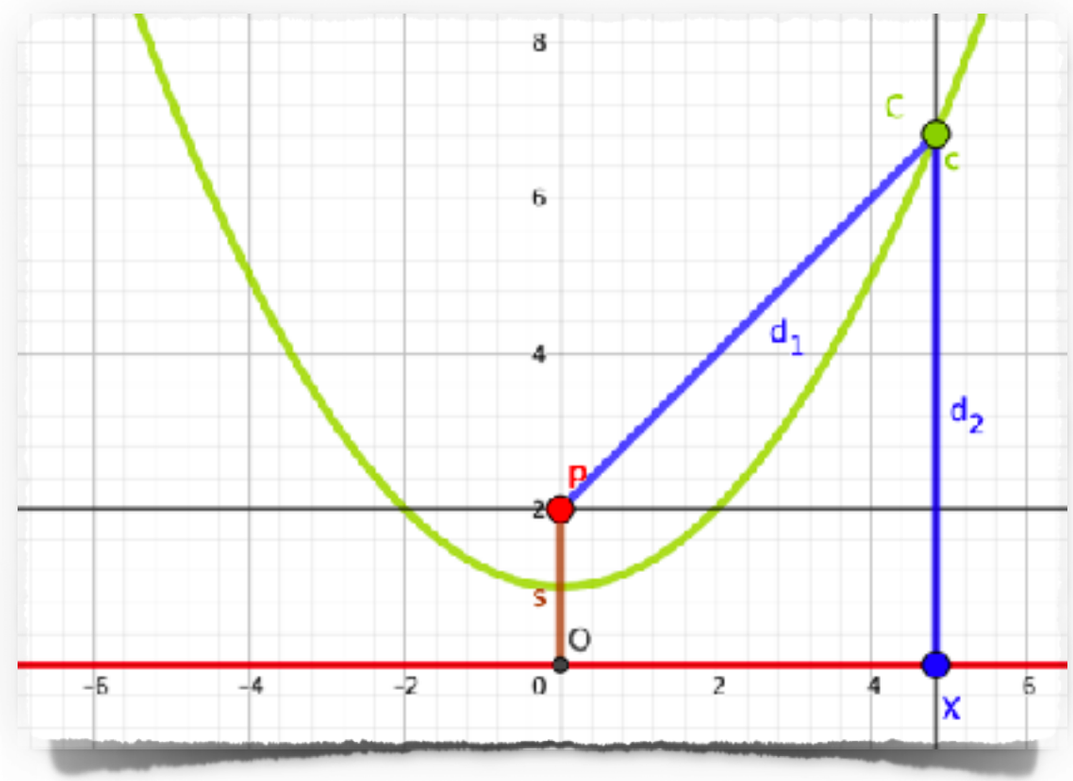

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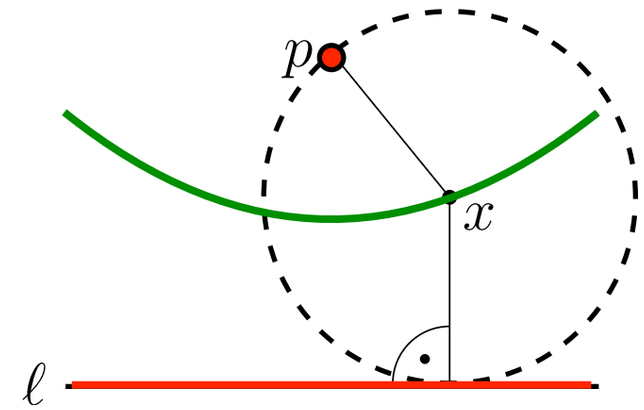

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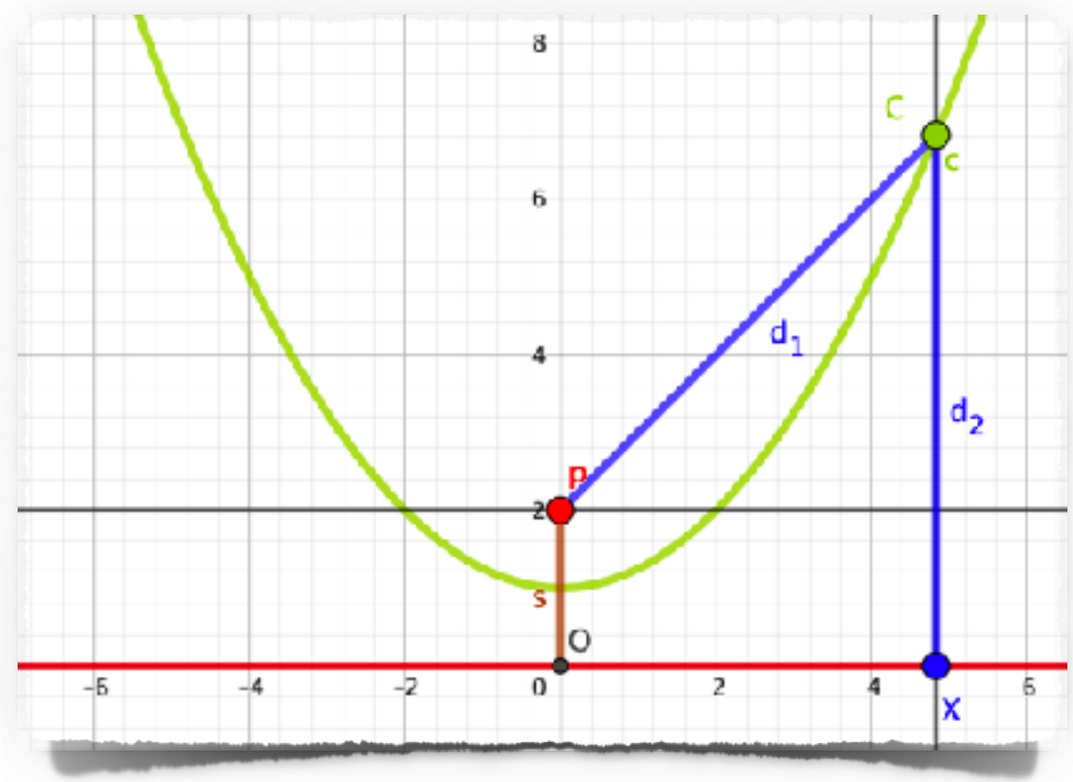

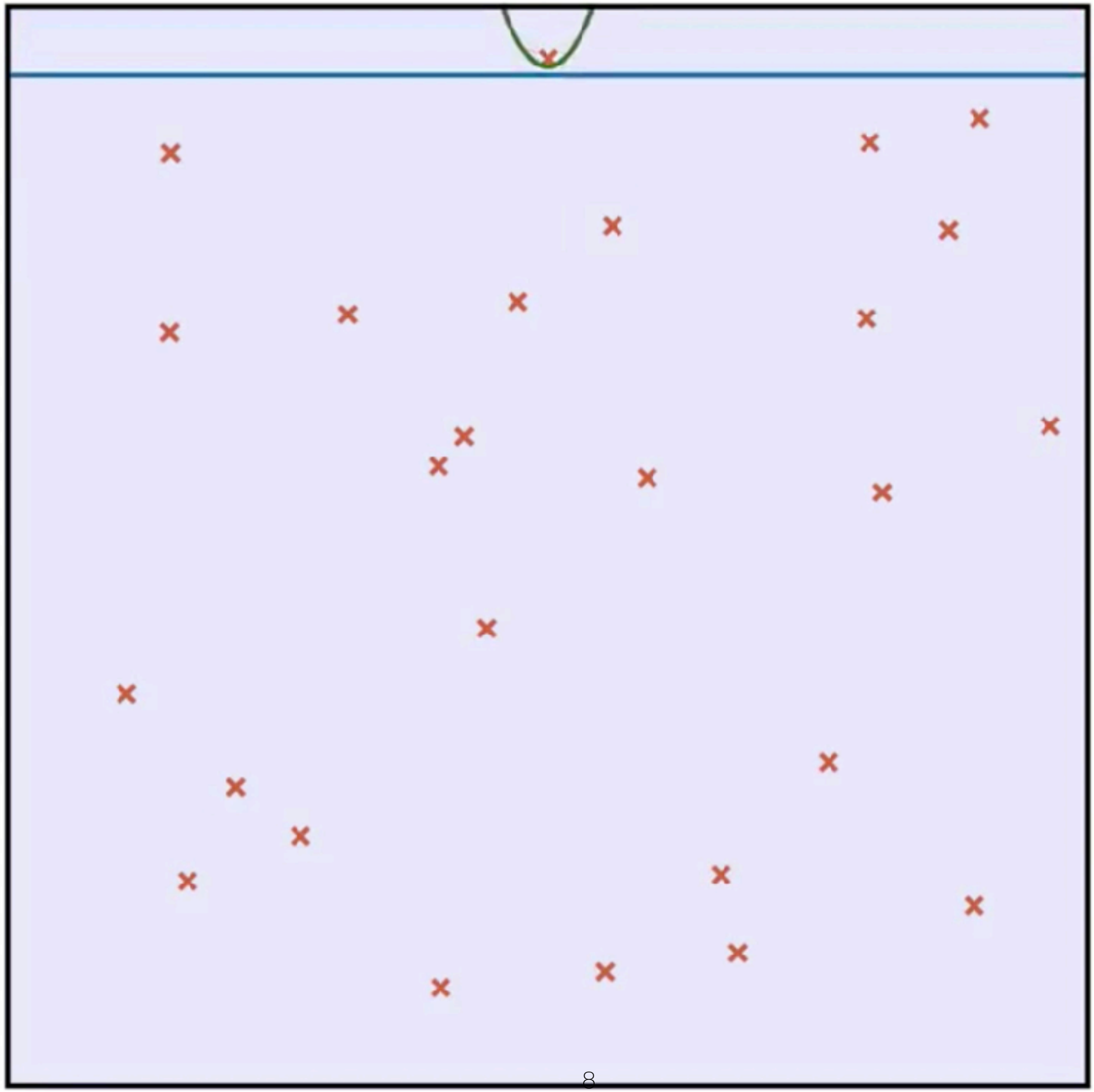

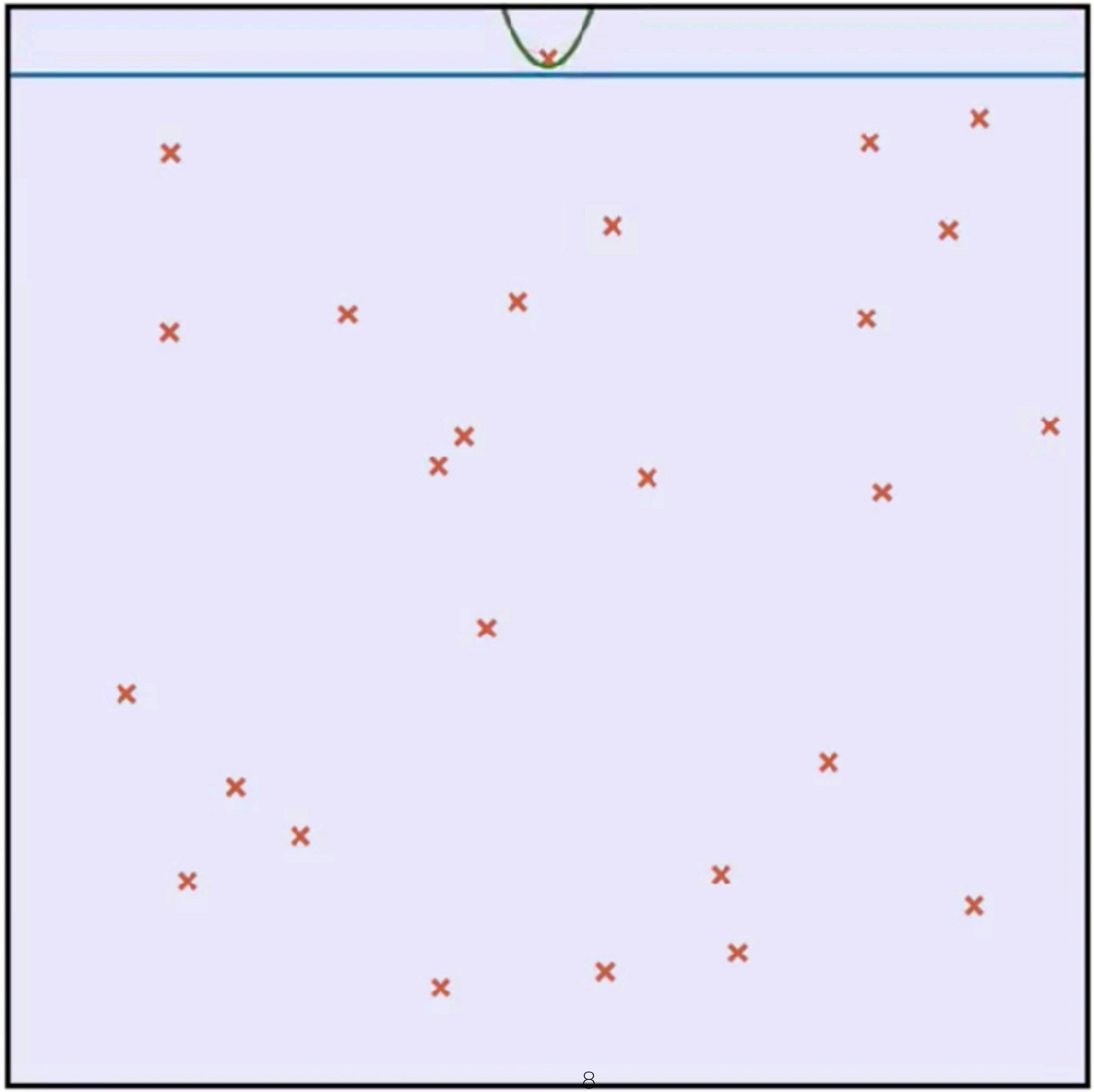

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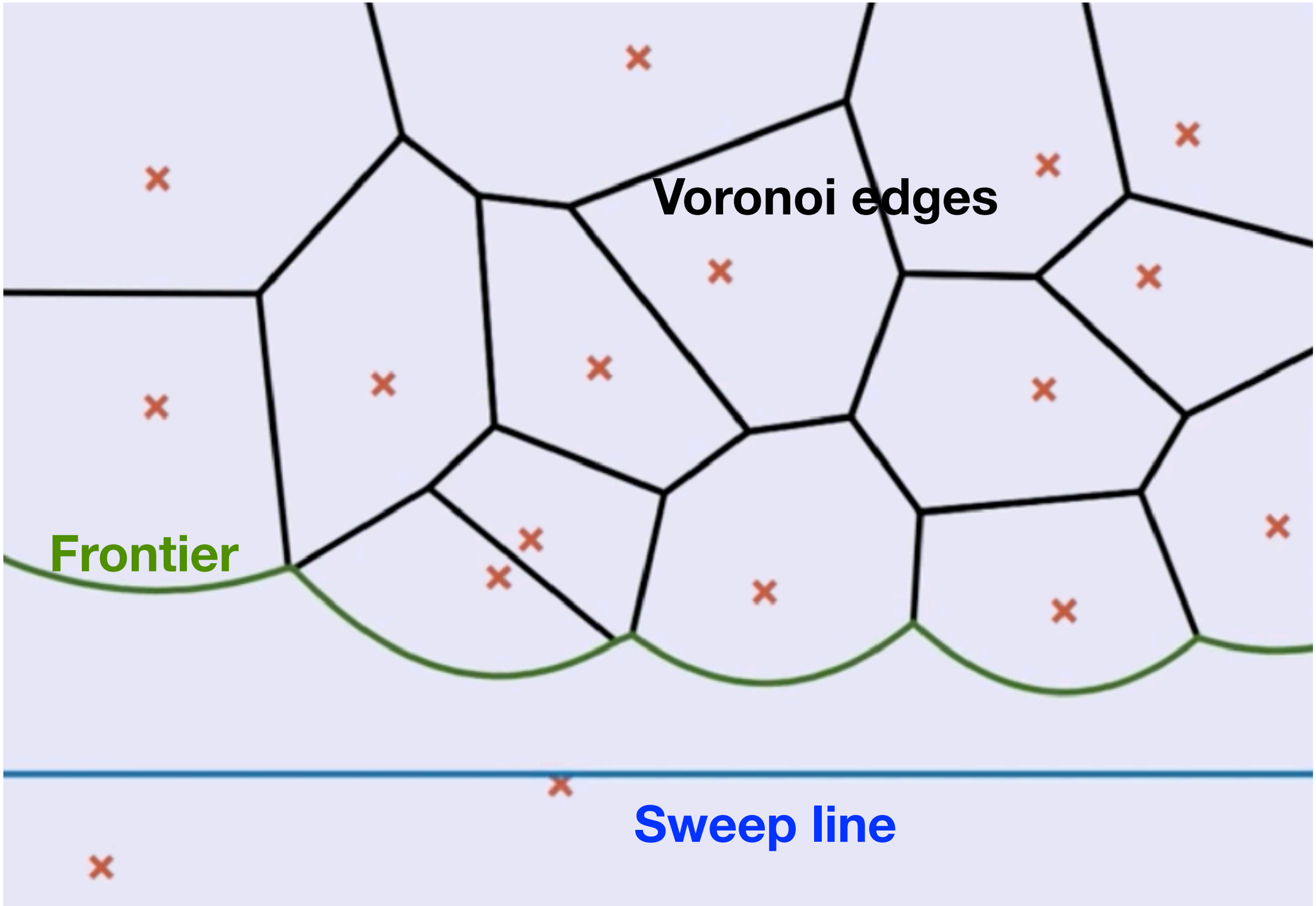

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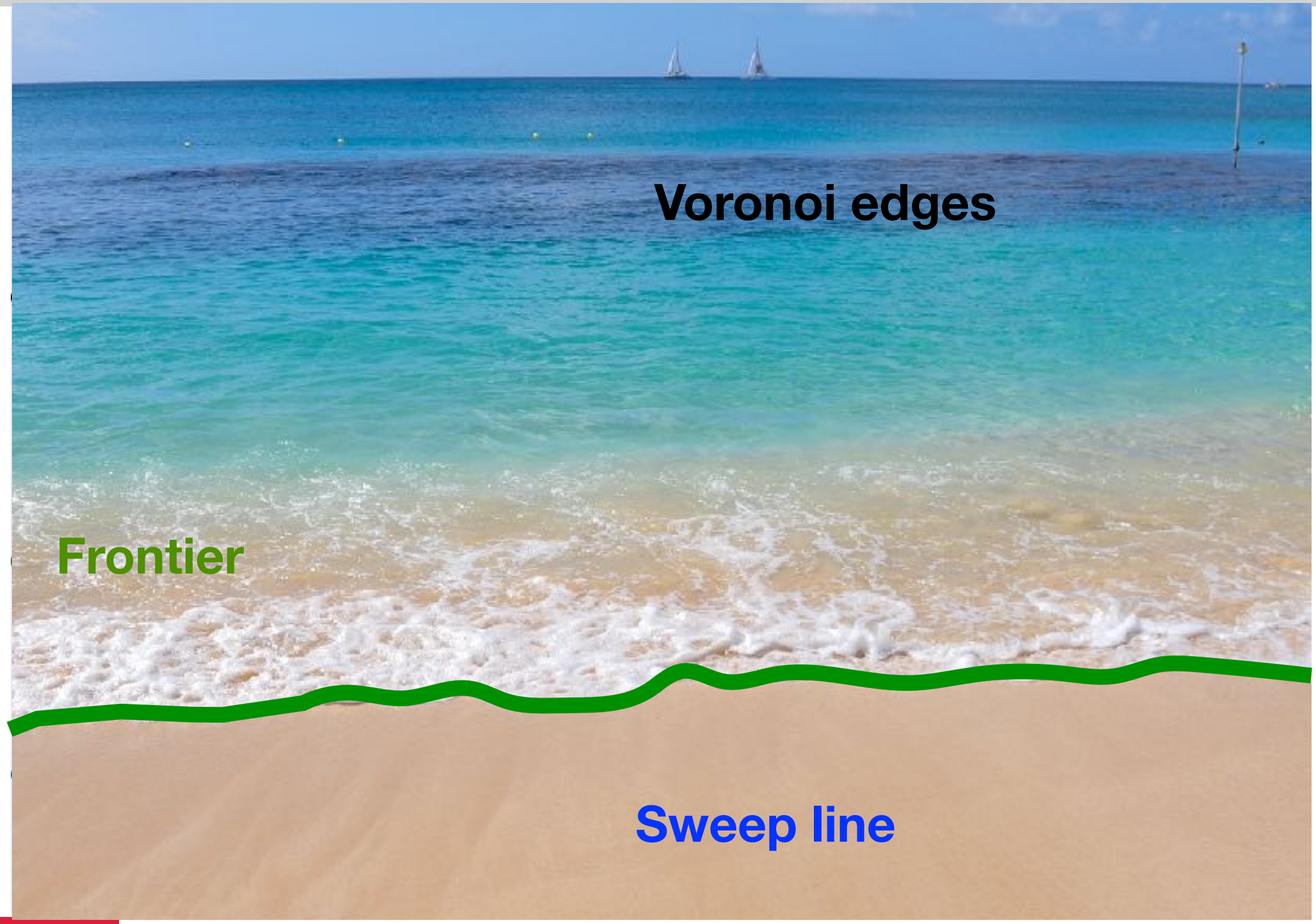

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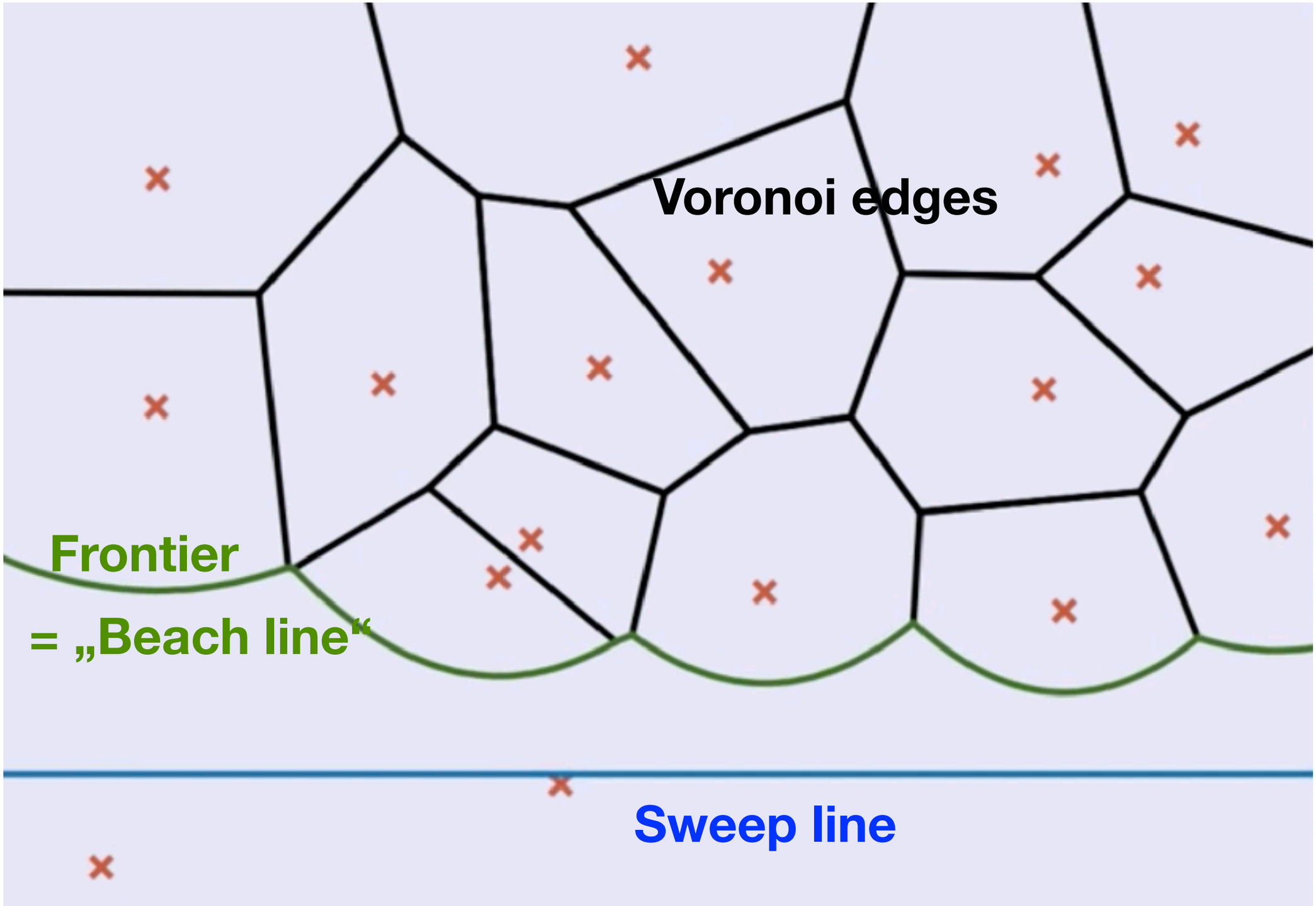

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- Let $x \in \mathbb{R}^2$ be above ℓ and $p \in \mathcal{P}$ below ℓ
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- Let $q \in \mathcal{P}$ be nearest site for x.
  If $d(x,q) \leq d(x,\ell)$, then q not below ℓ.
- $\{x \in \mathbb{R}^2 \mid d(x,q) \leq d(x,\ell)\}$
  is bounded by parabola.

Consequence:

- $\exists$ parabola β: x above β
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Beach line:

- $p_1, \ldots, p_k$ above ℓ $\rightarrow$ parabolas $\beta_1, \ldots, \beta_k$
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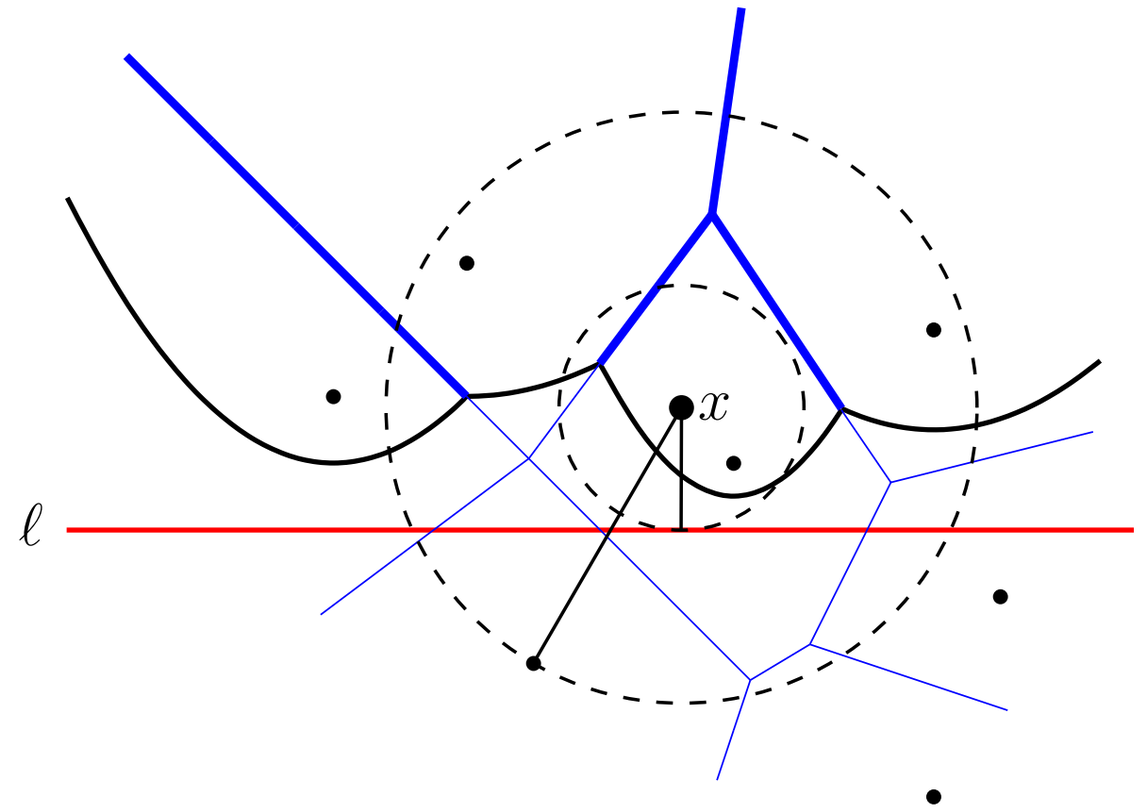

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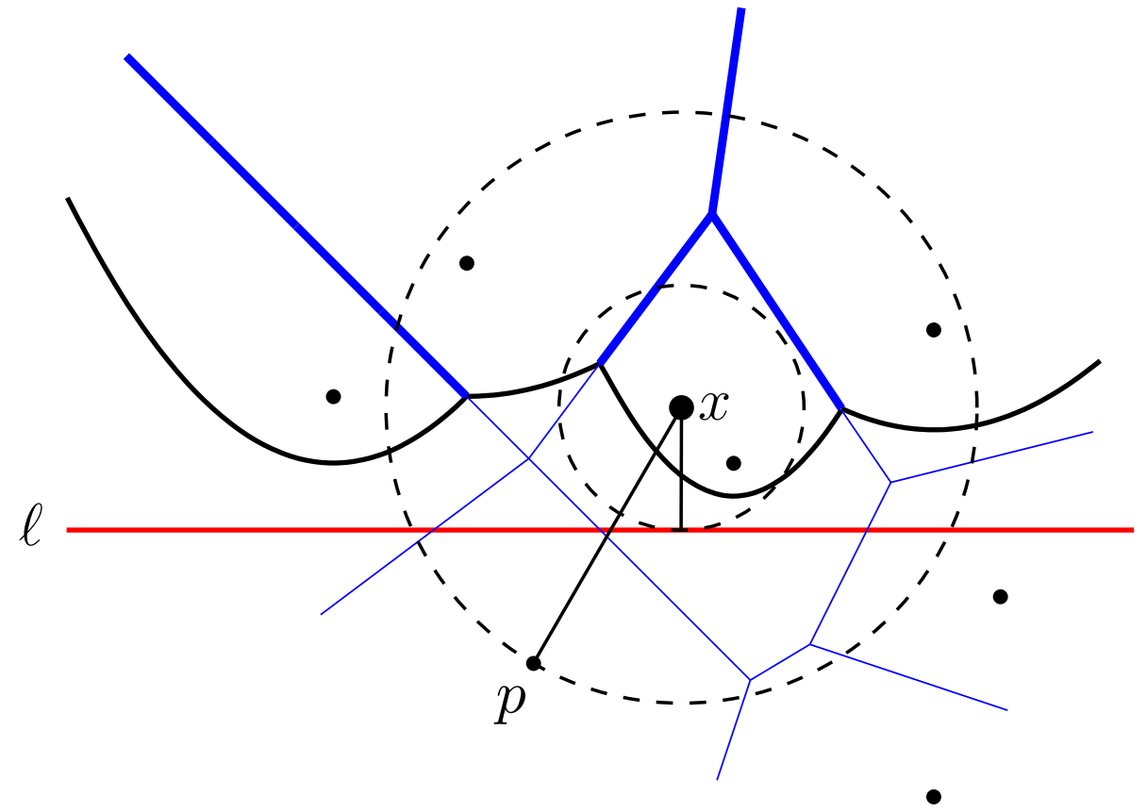

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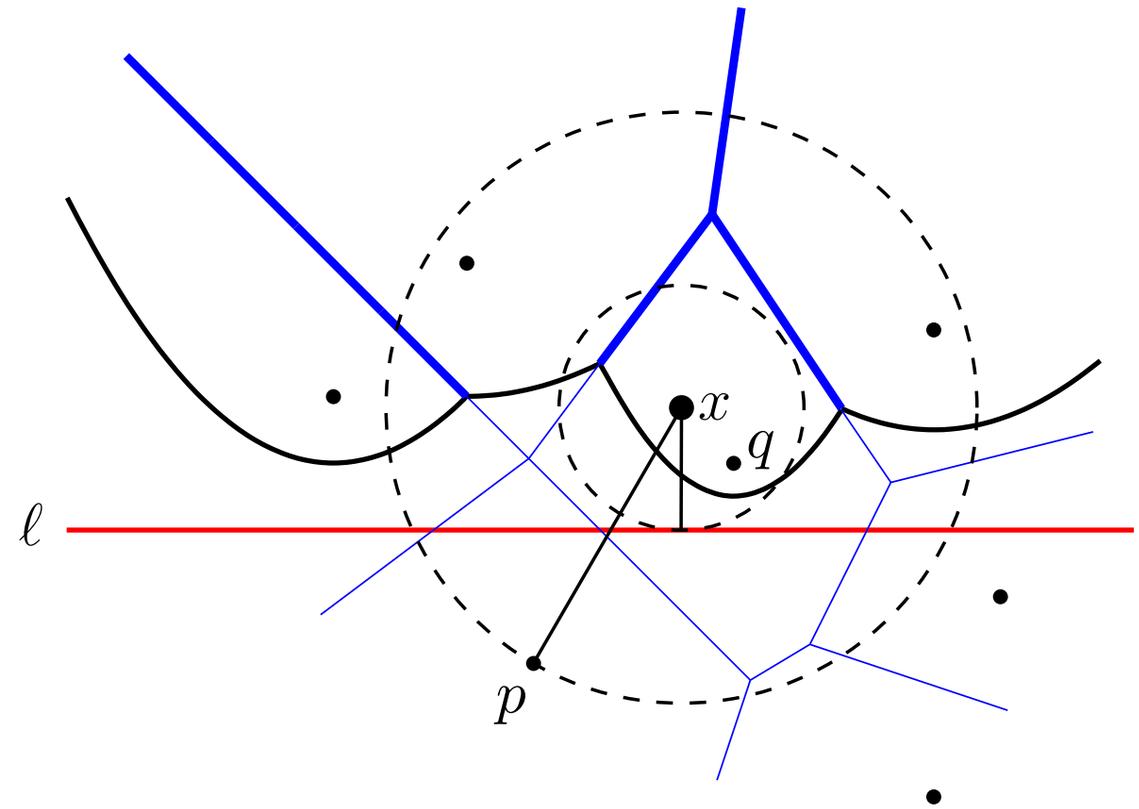

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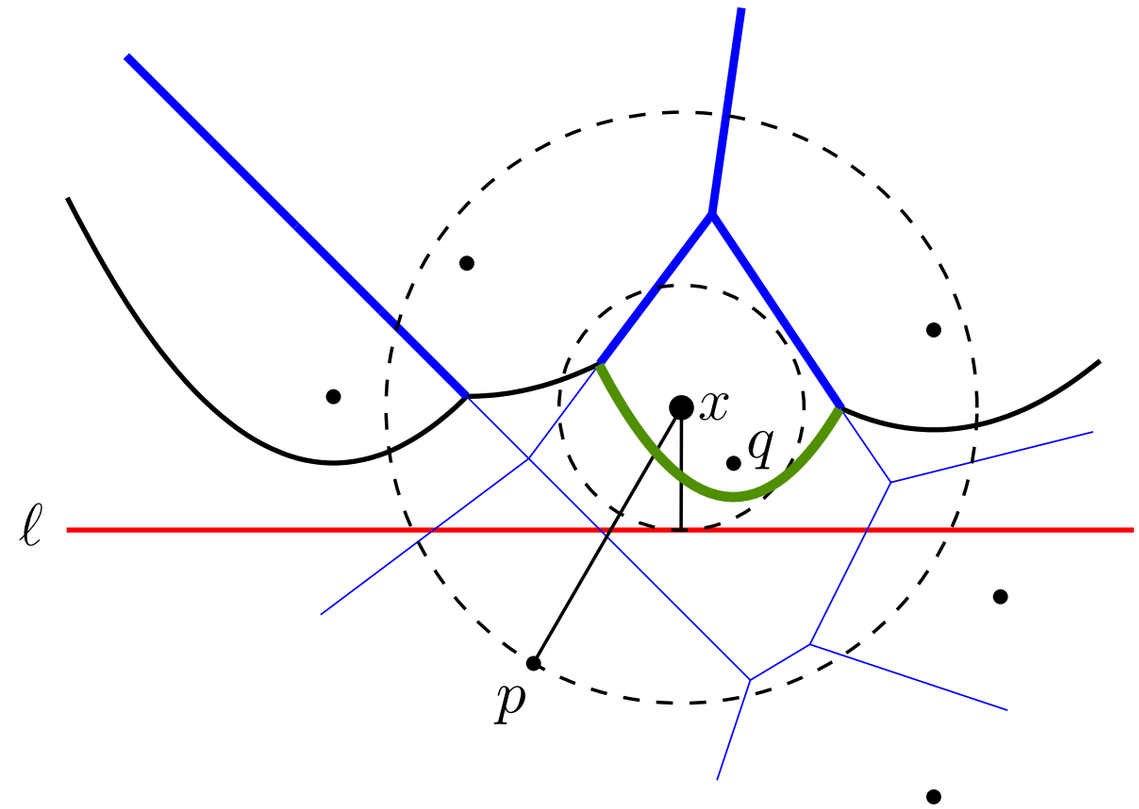

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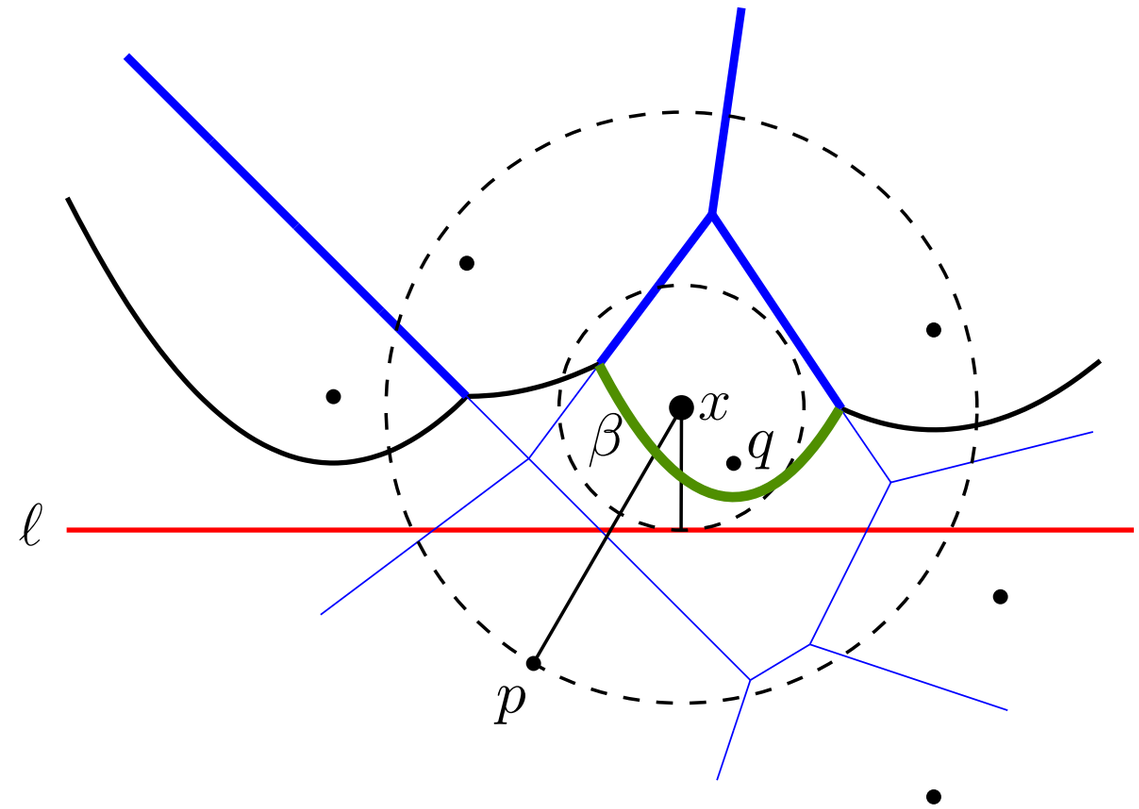

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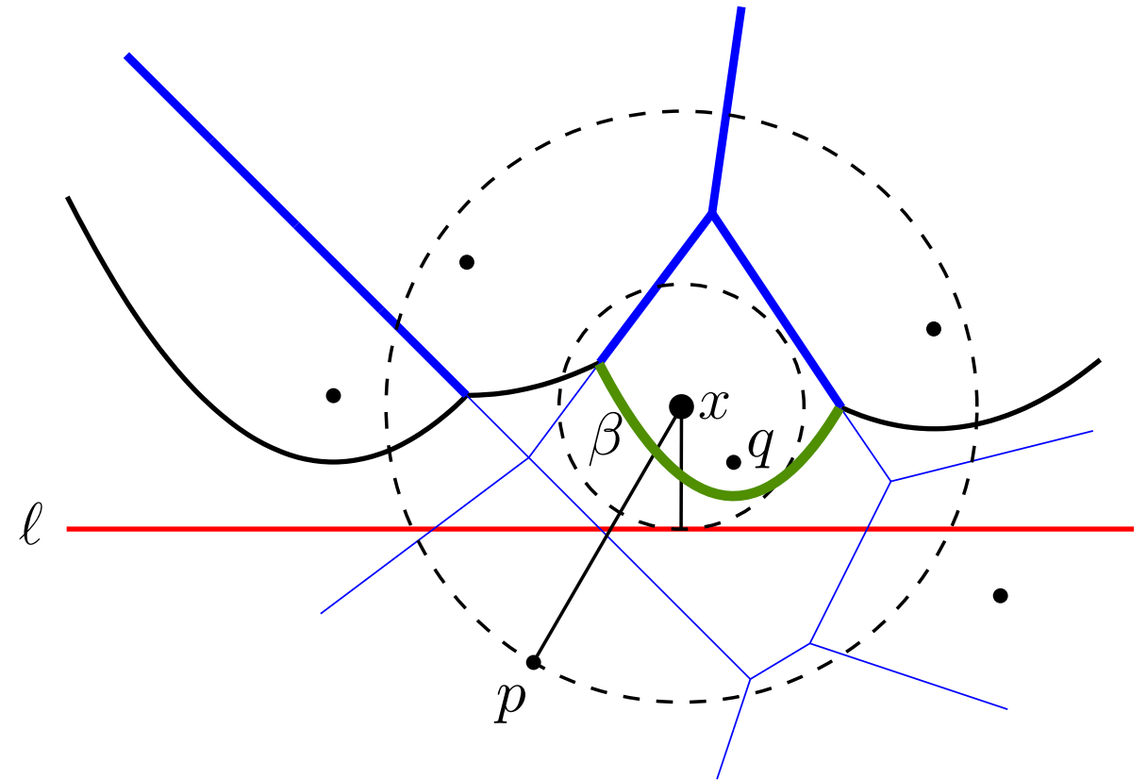

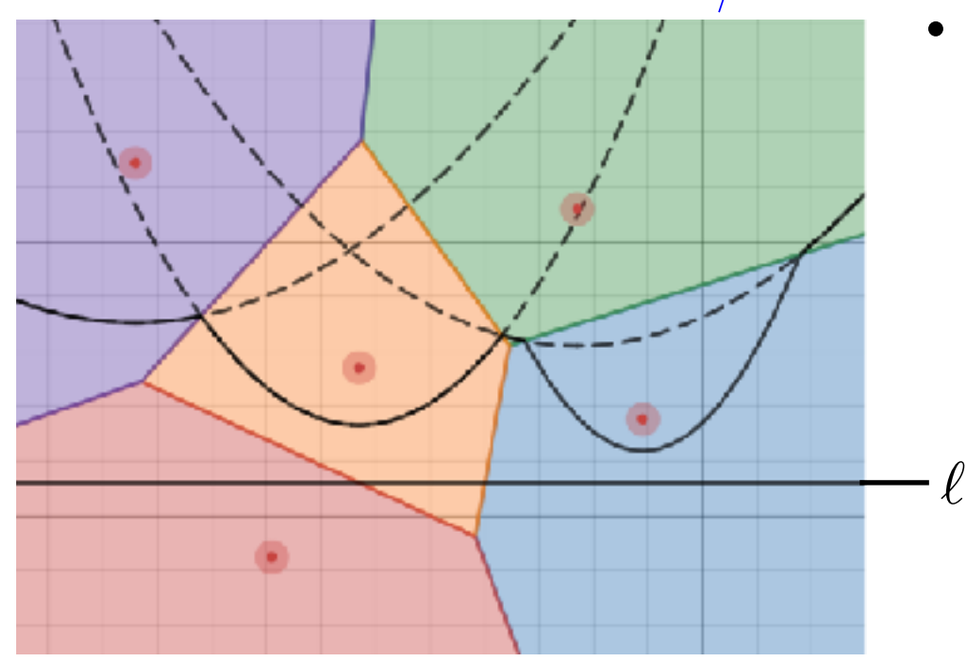

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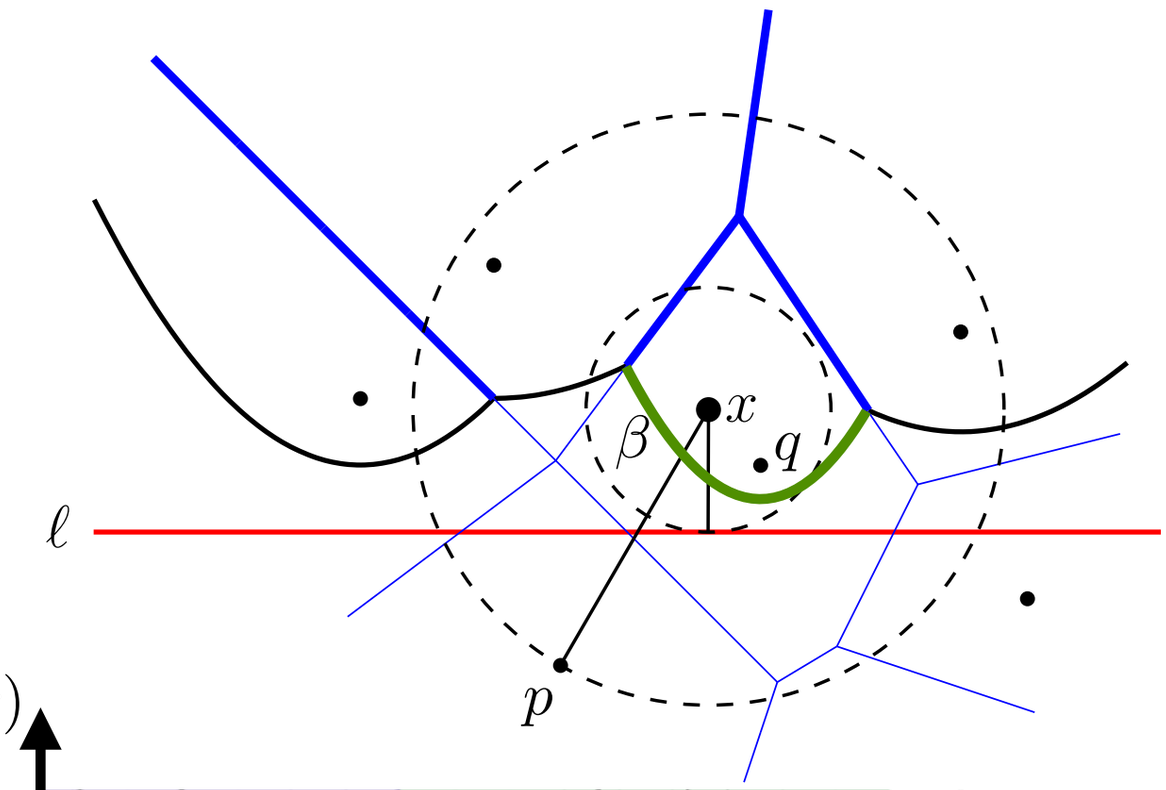

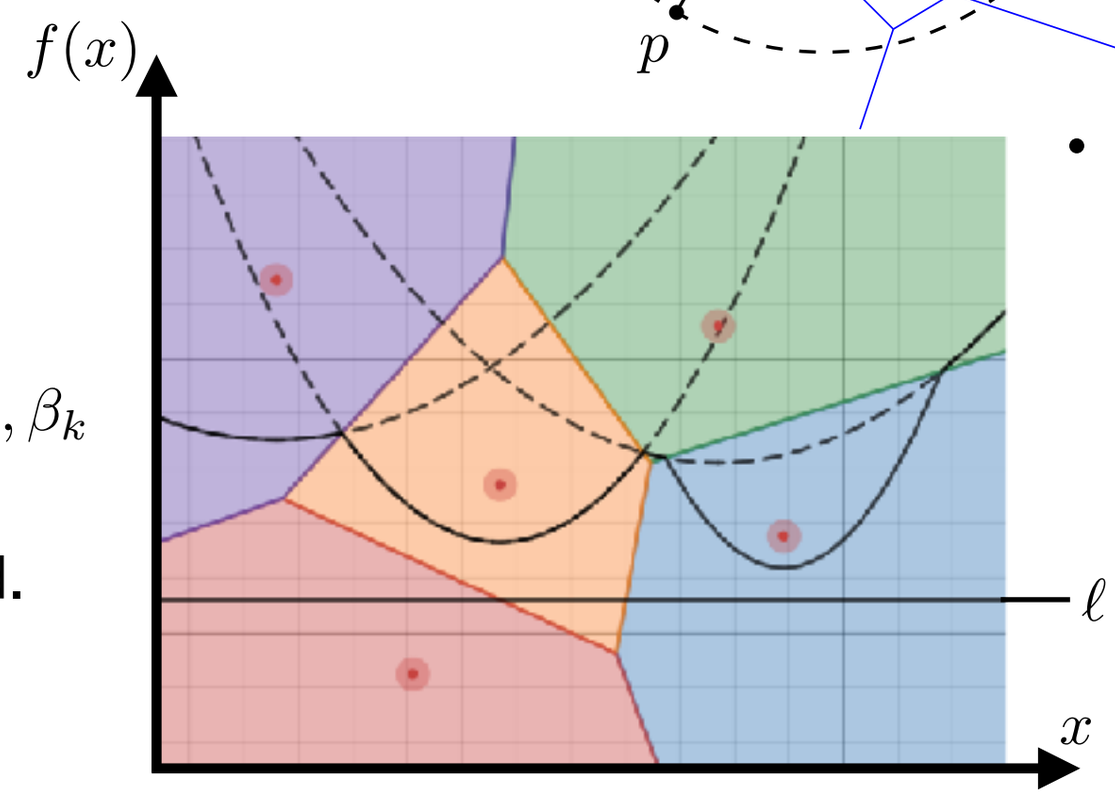

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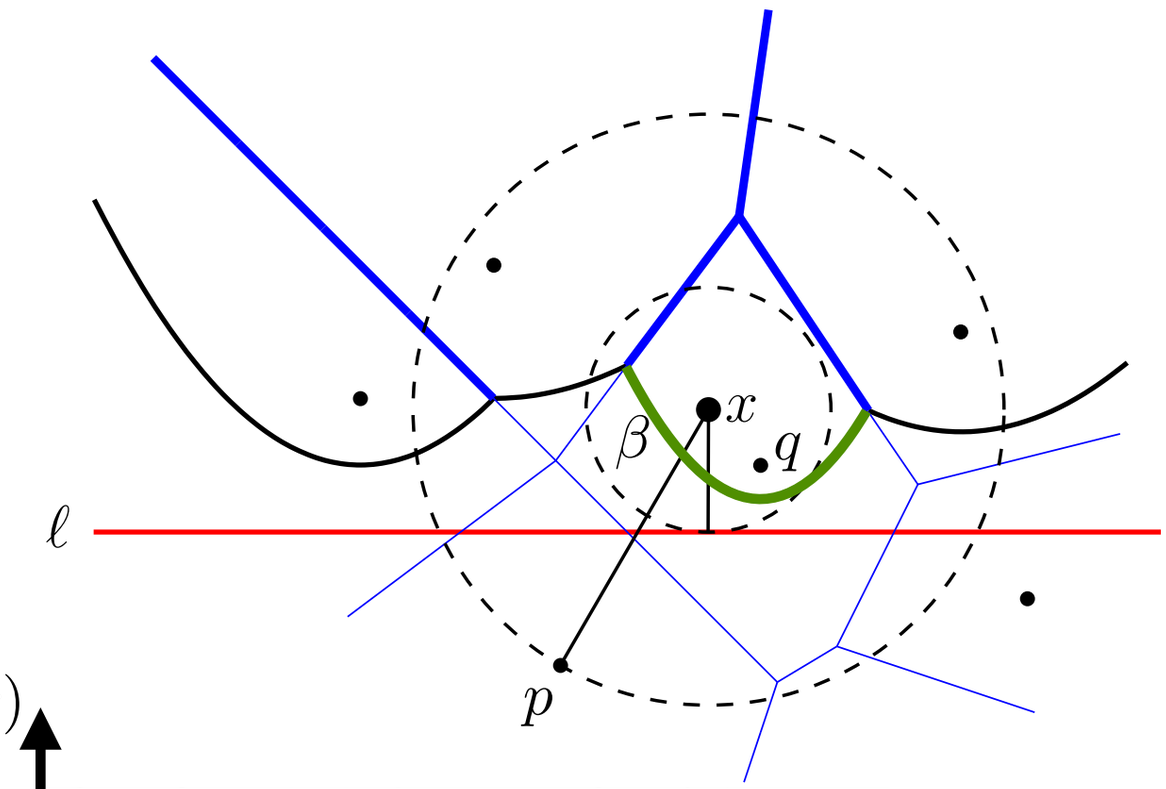

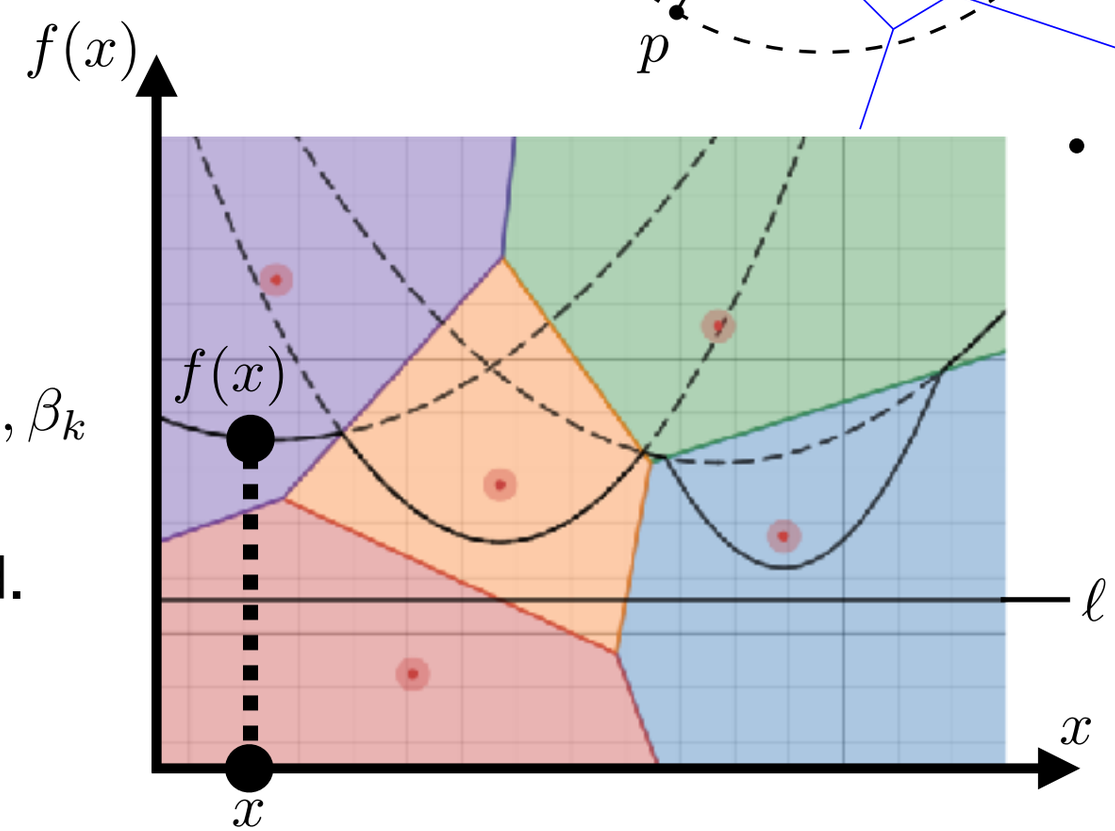

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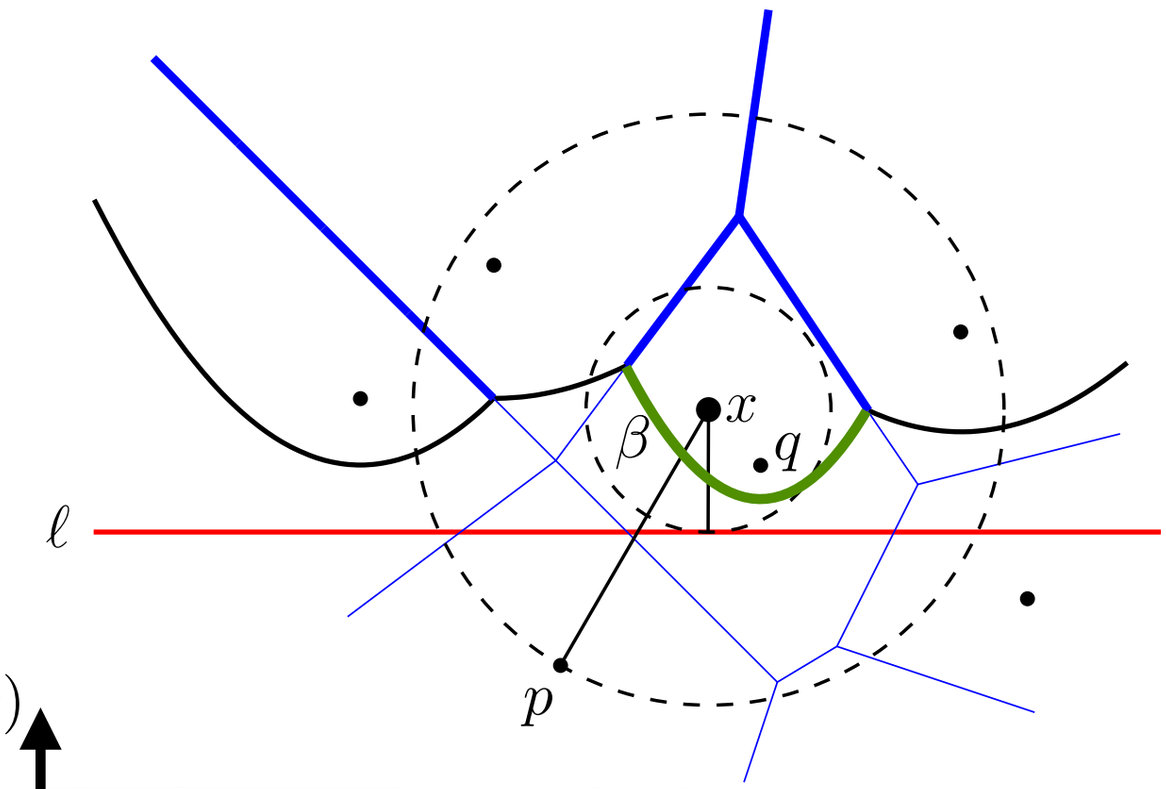

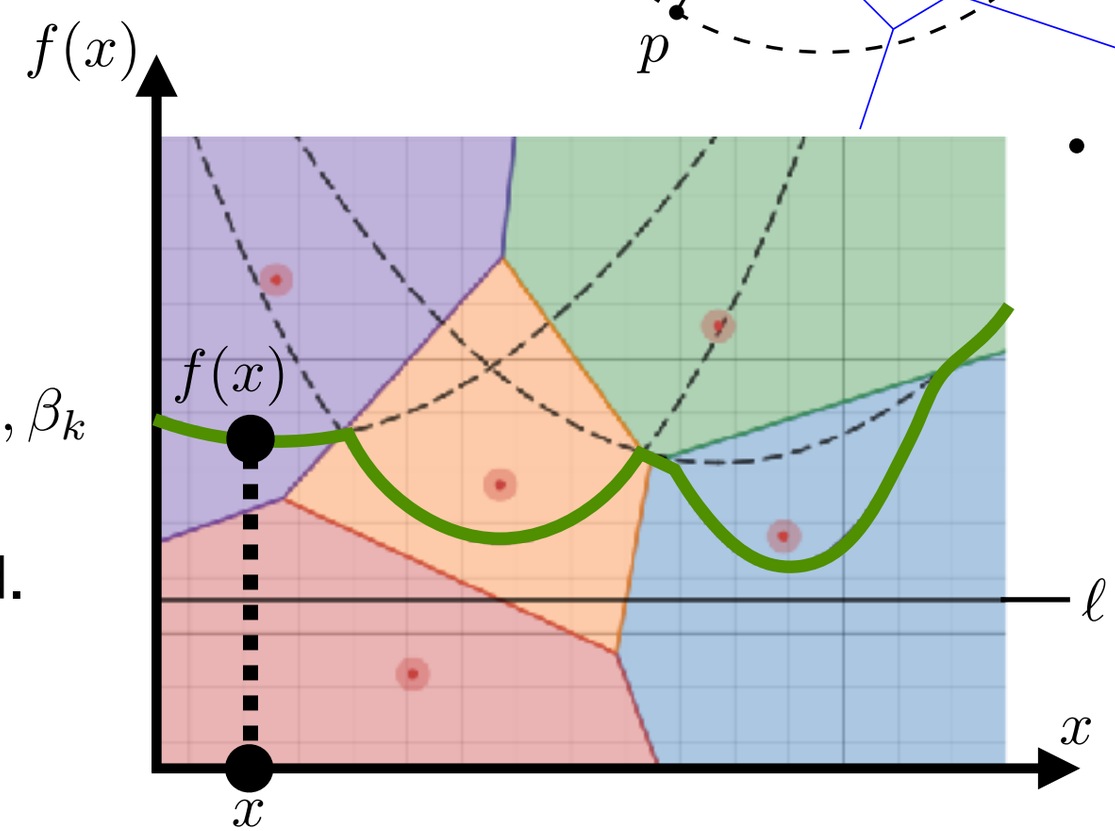

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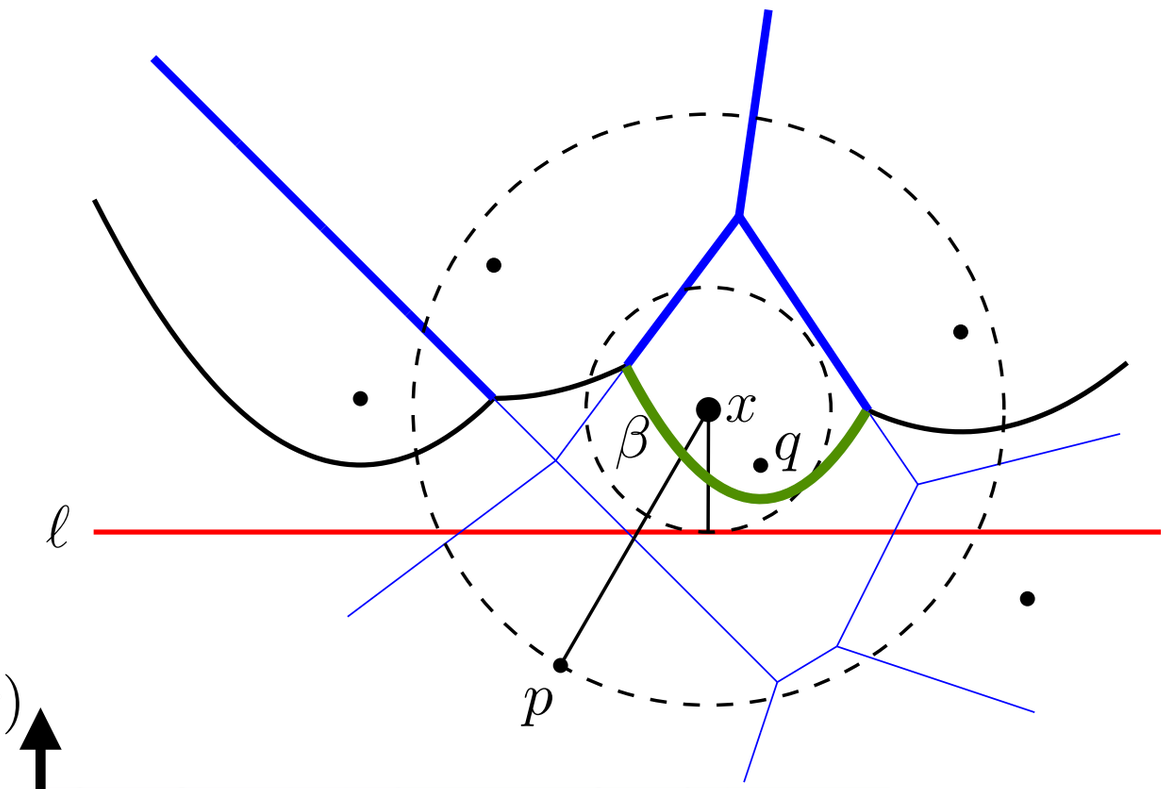

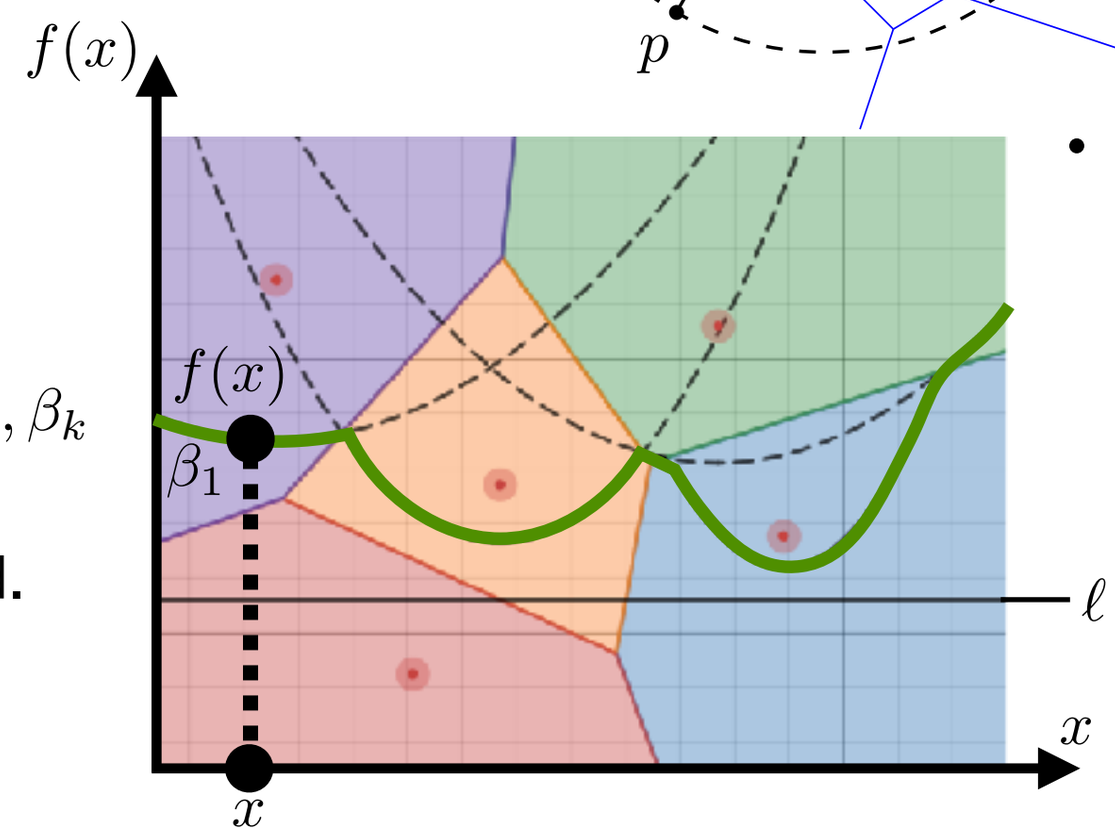

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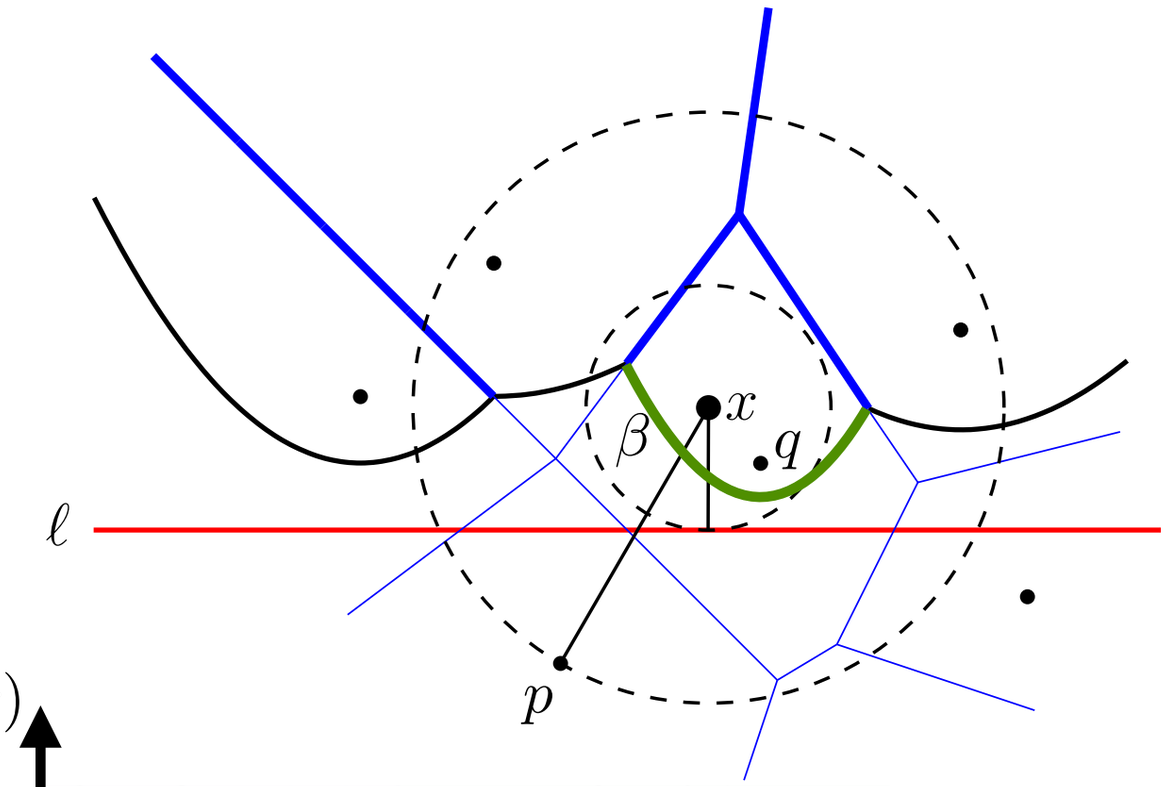

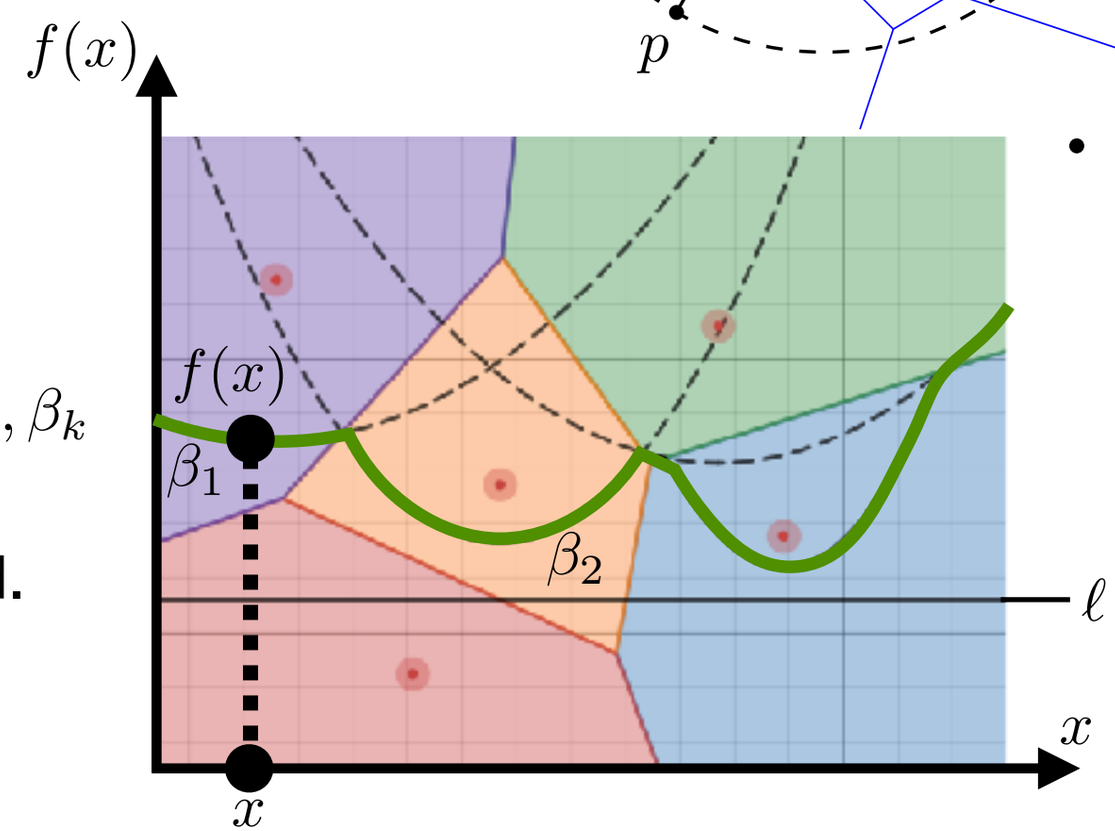

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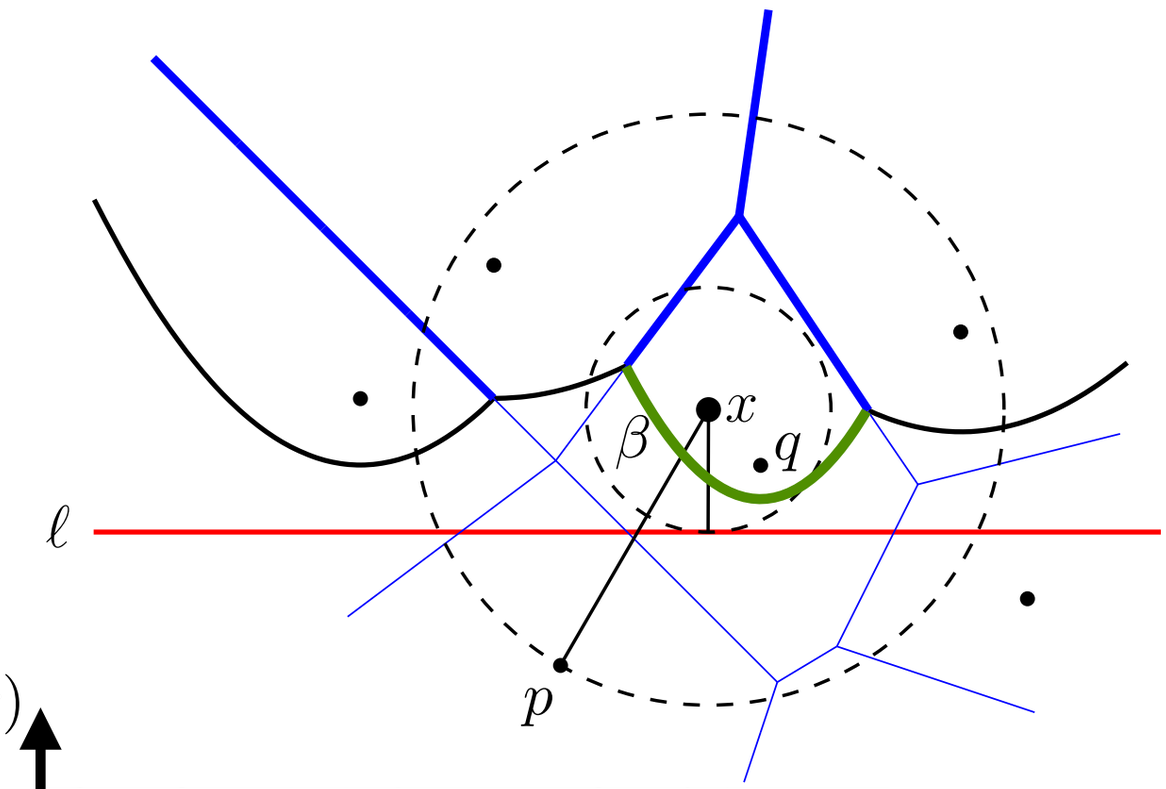

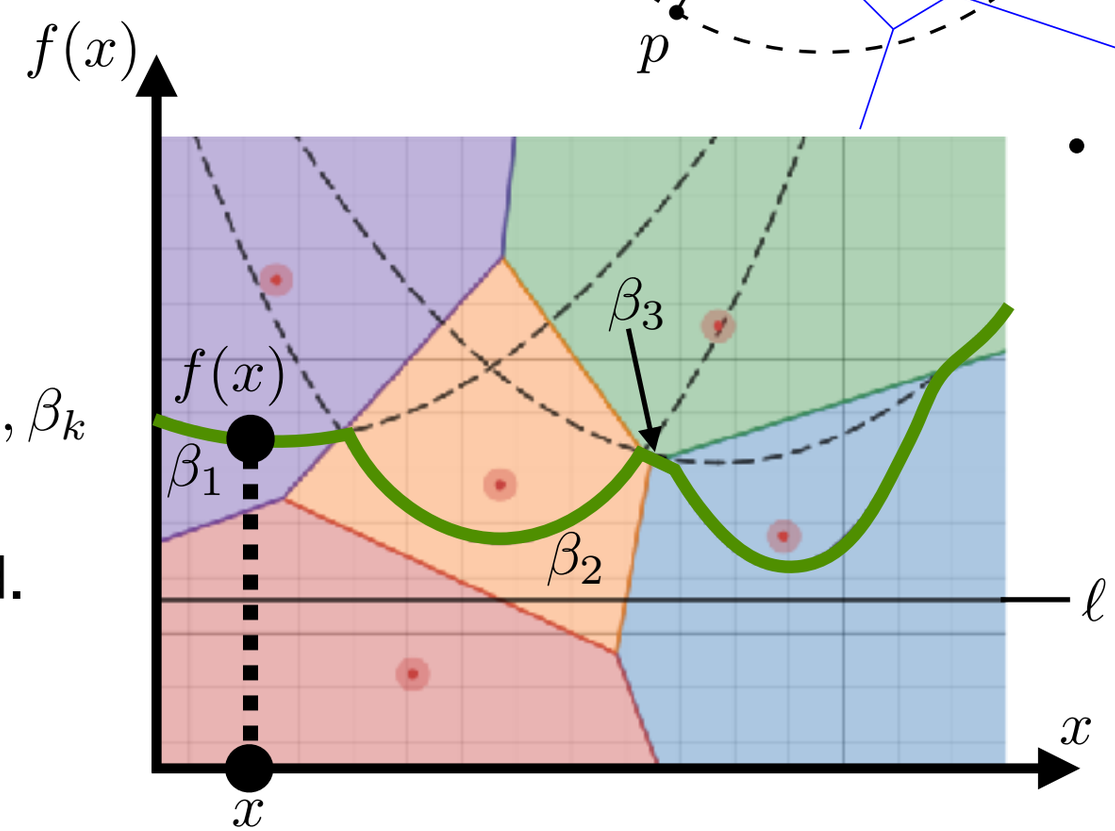

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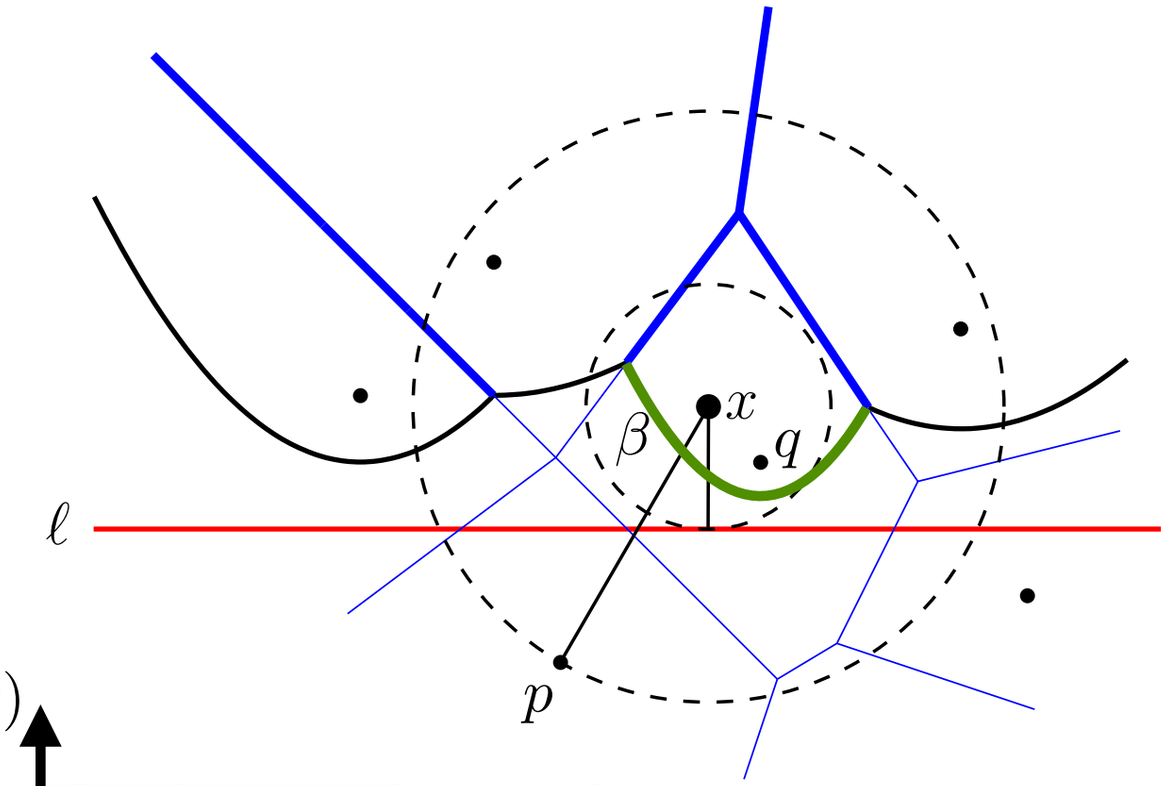

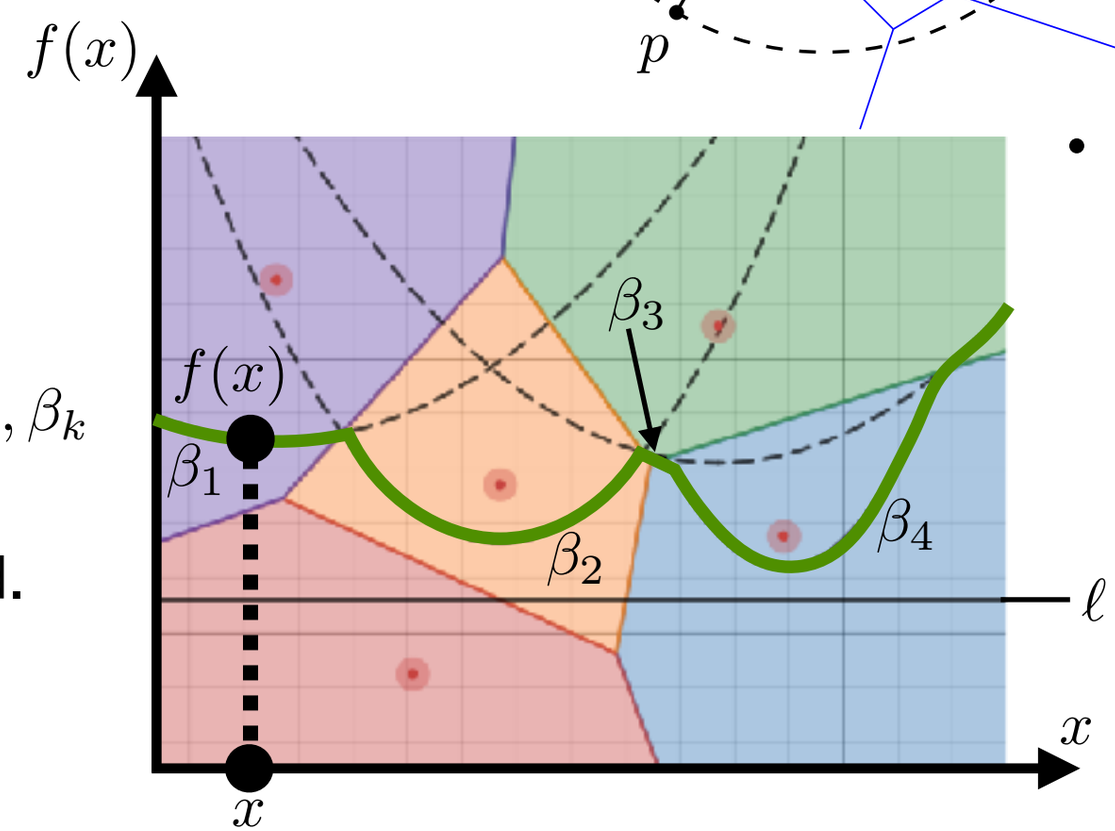

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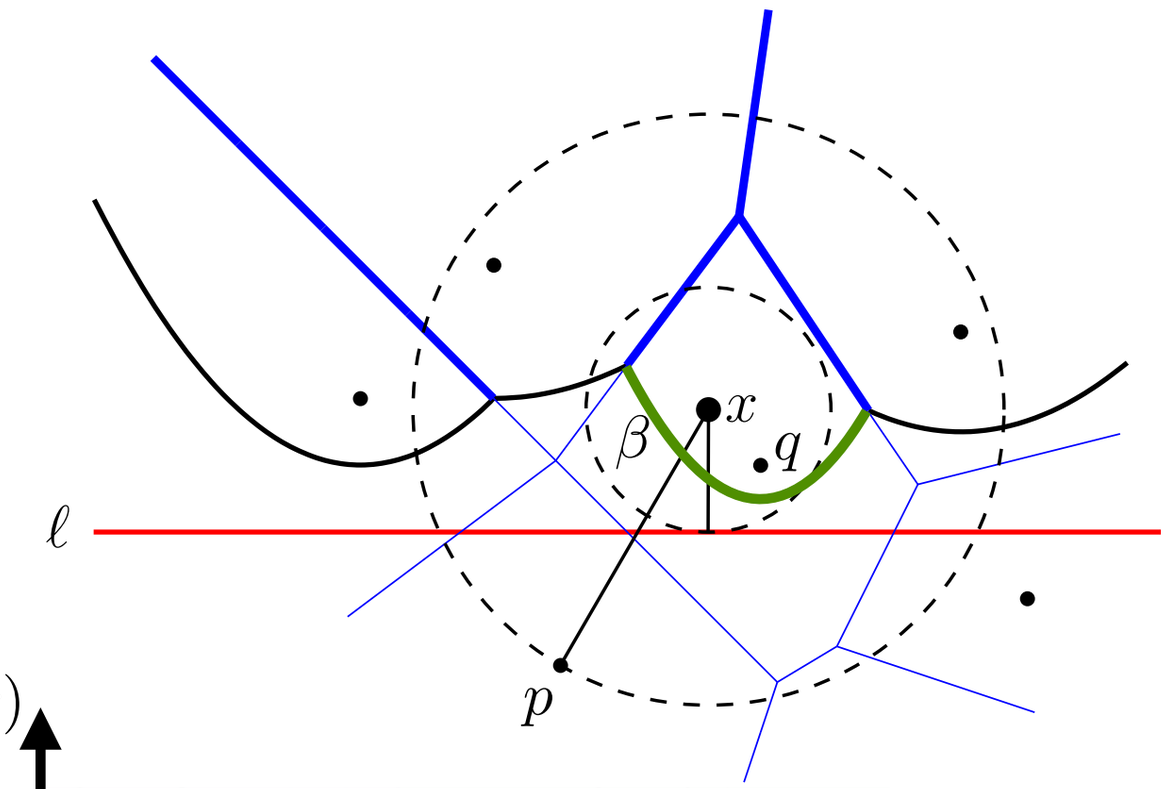

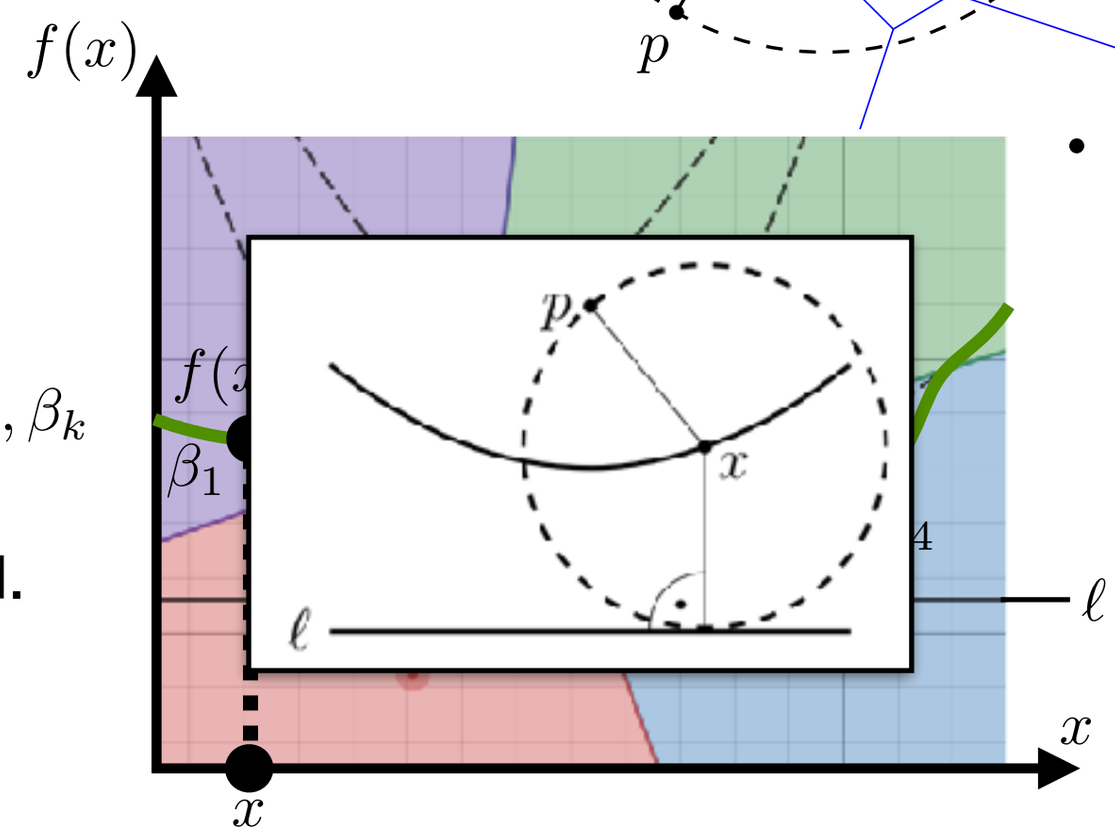

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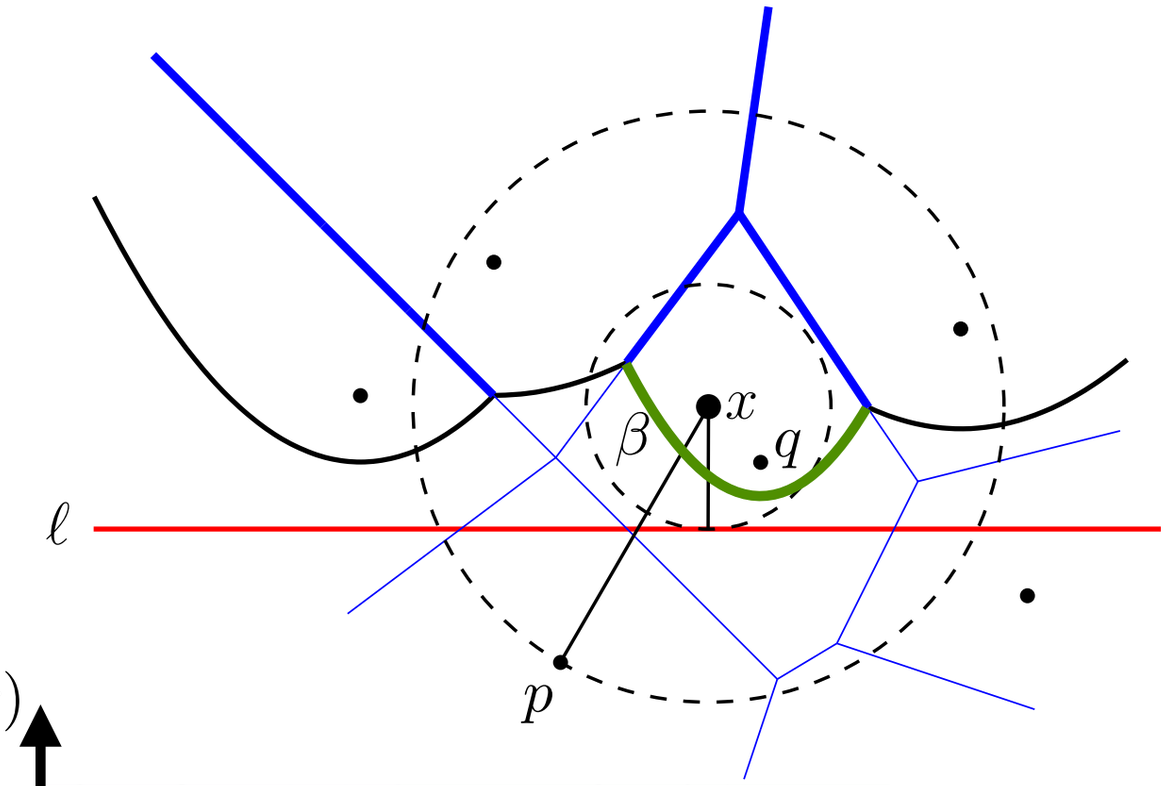
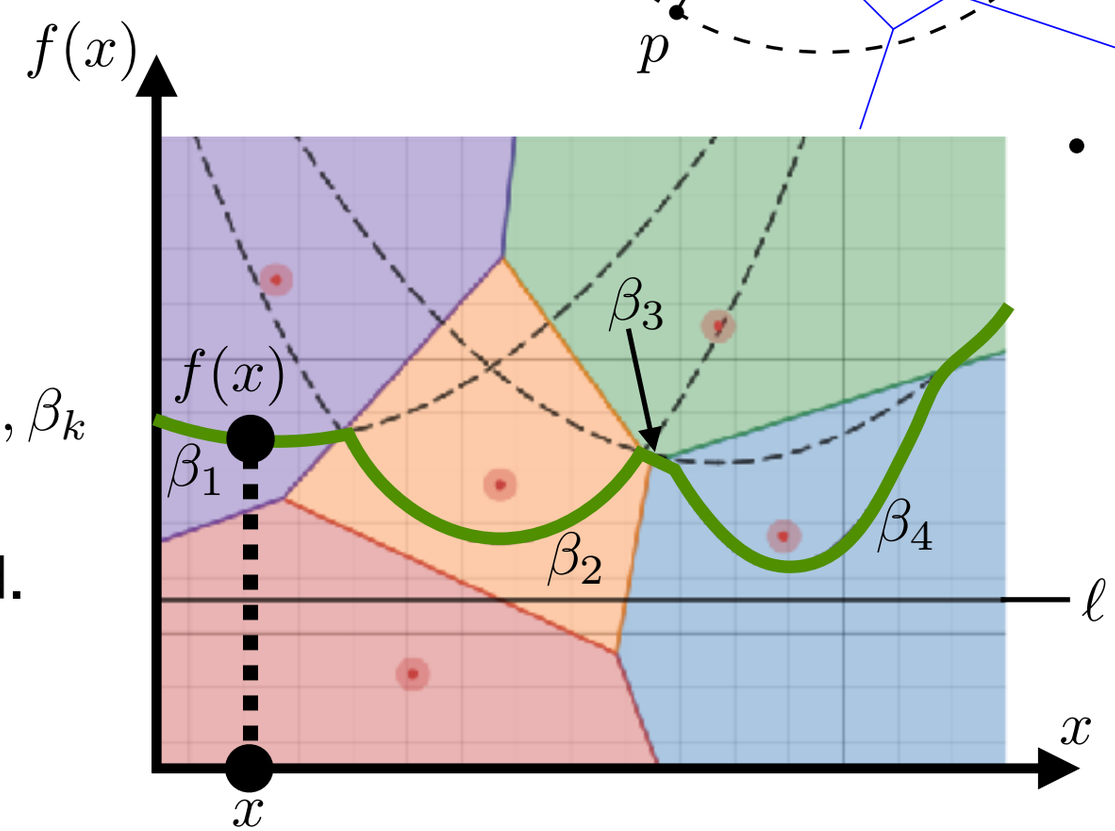
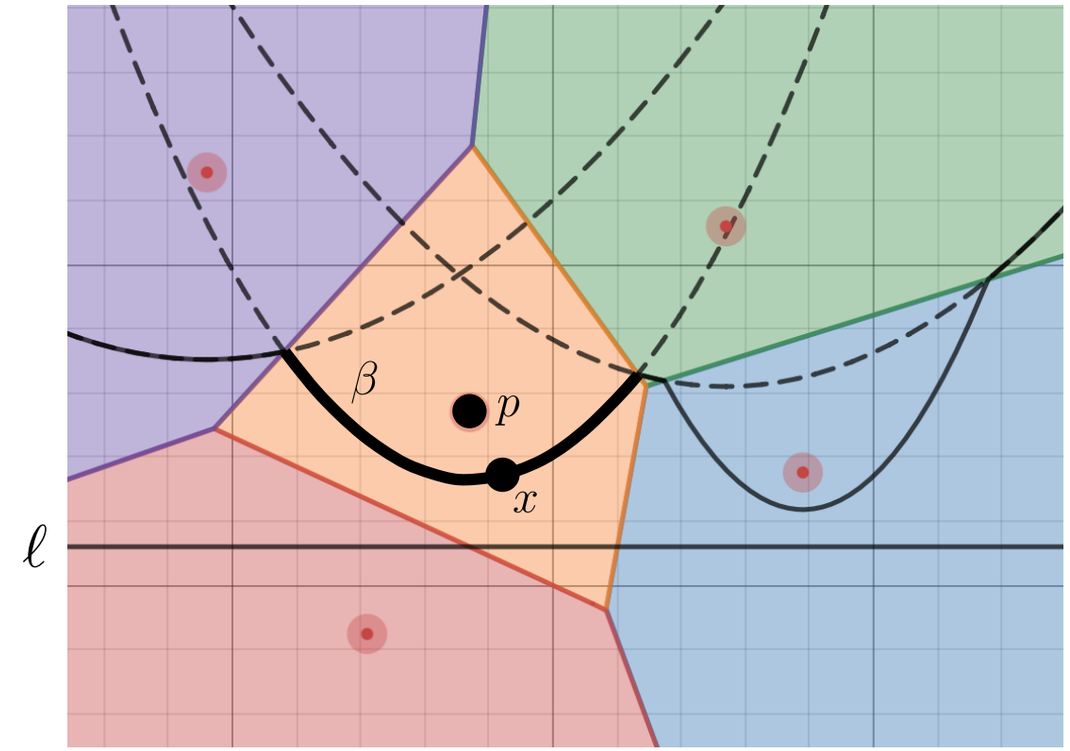




Lemma 4.15

$p \in \mathcal{P}$ defines arc β on beach line

$\Rightarrow$ p is nearest site $\forall x \in \beta$.



Proof:

- Assume: $\exists q \in P : d(x,q) < d(x,p)$
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- But $x \in \beta$ on beach line. ↯ $\square$

Corollary 4.16

Intersection points of adjacent arcs lie on Voronoi edges.

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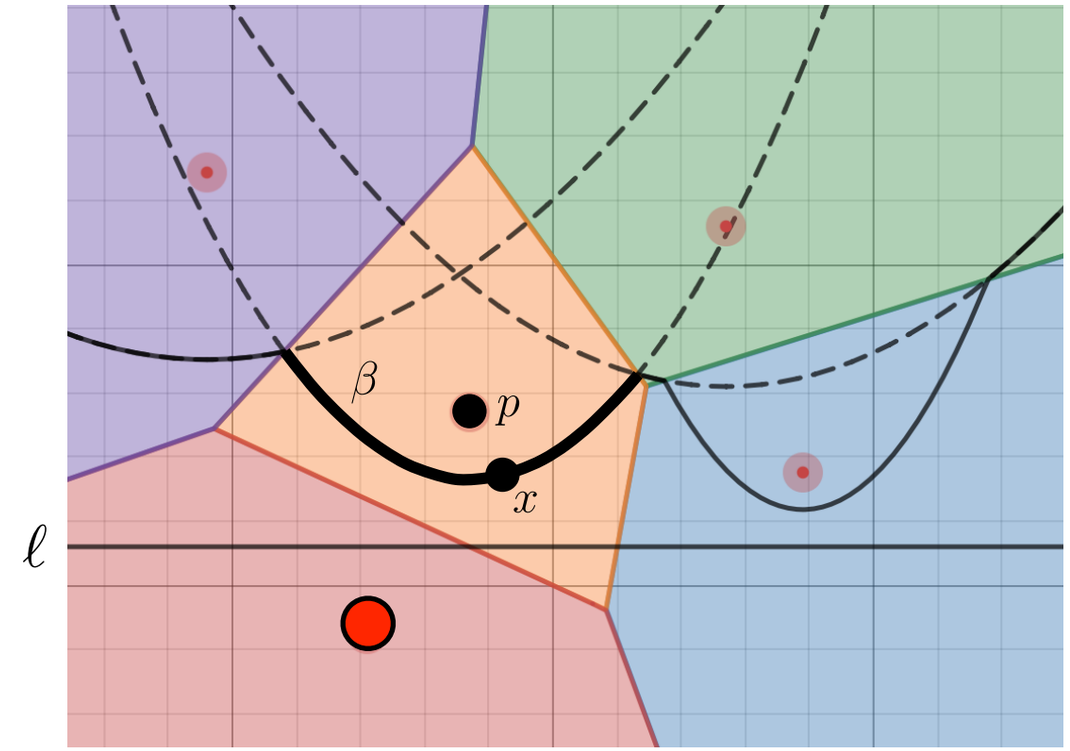

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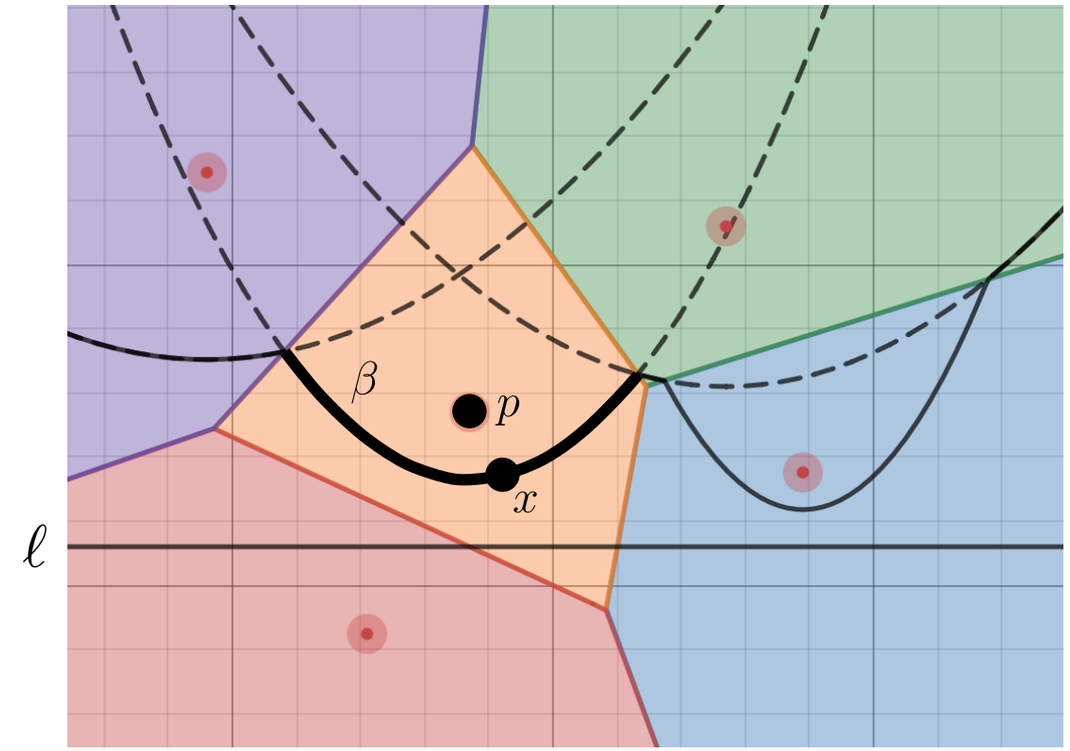

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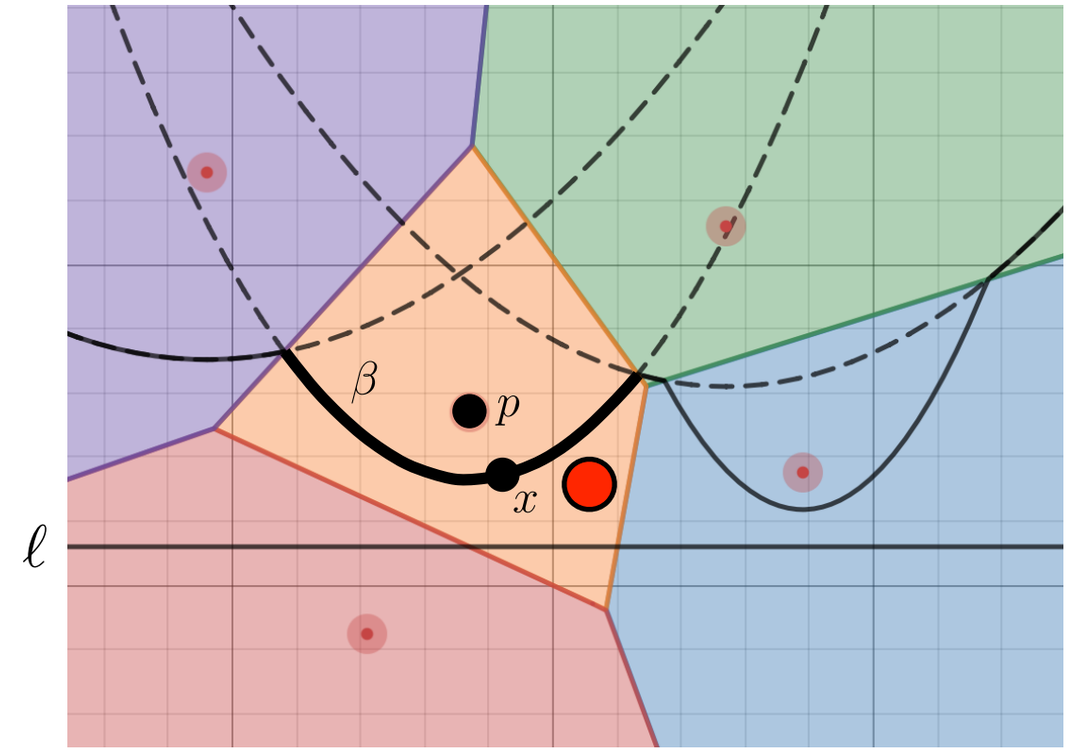

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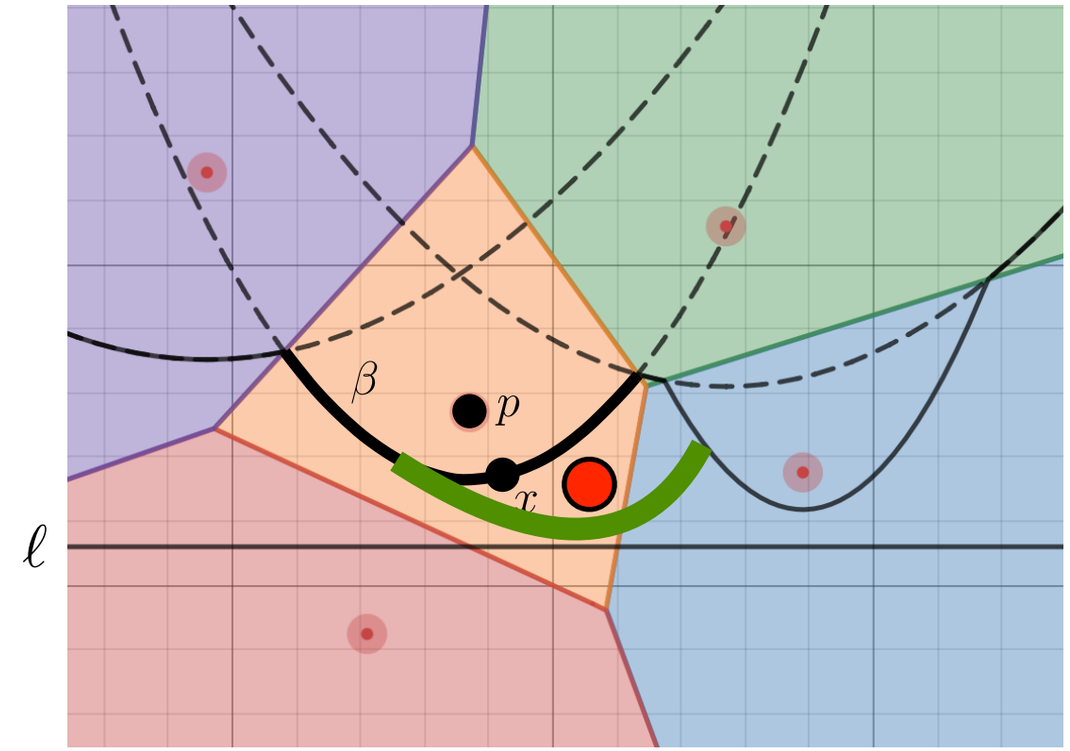

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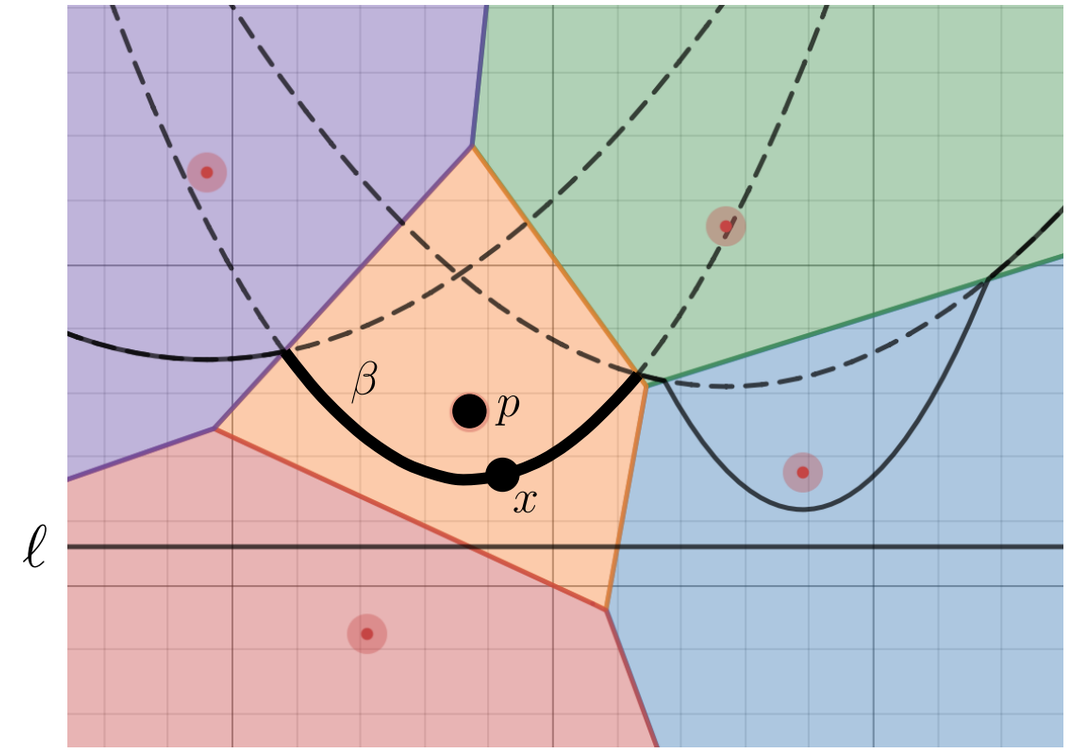

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Approach:

- Plane sweep
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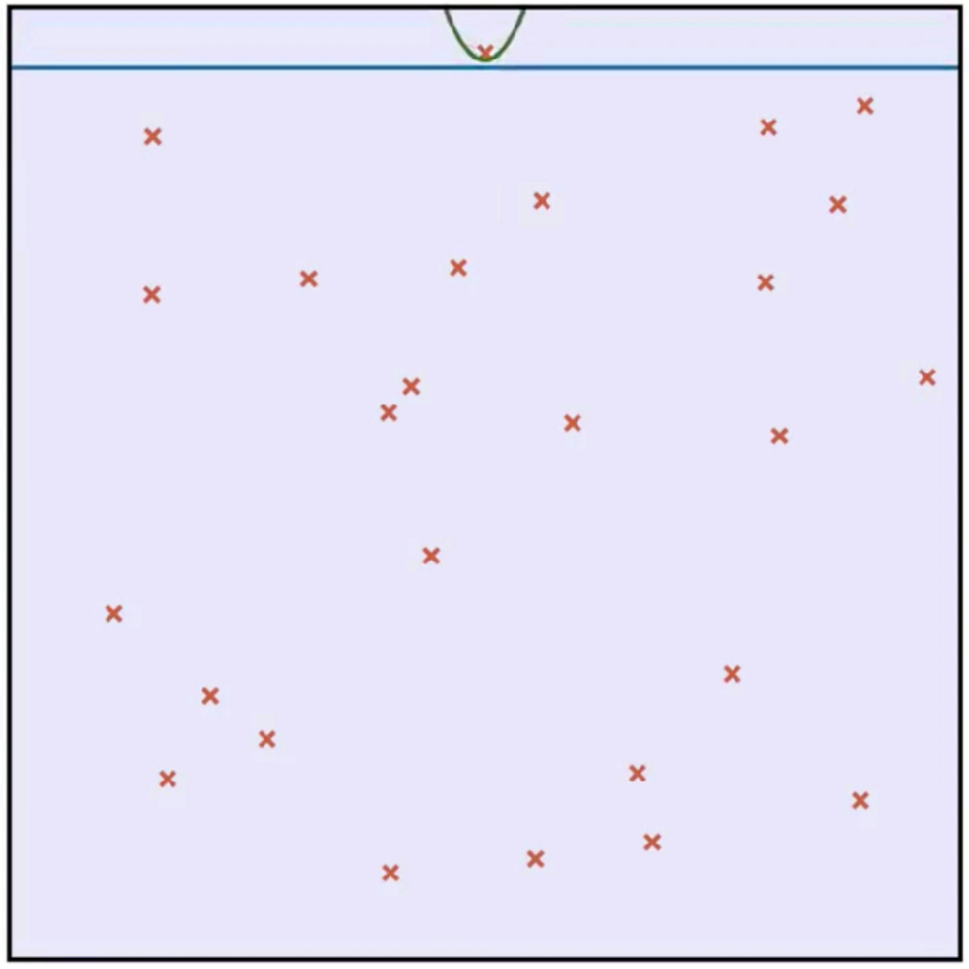

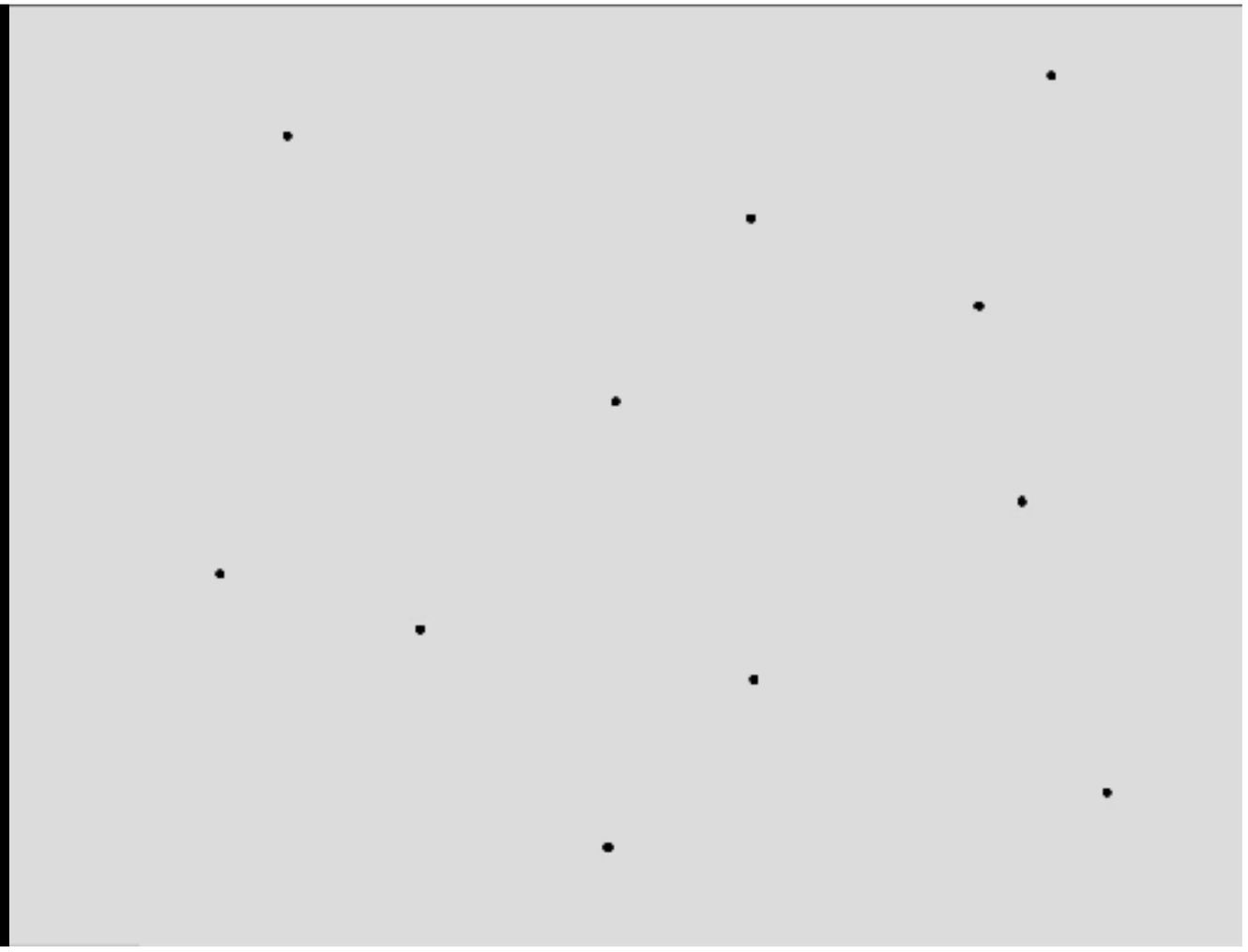

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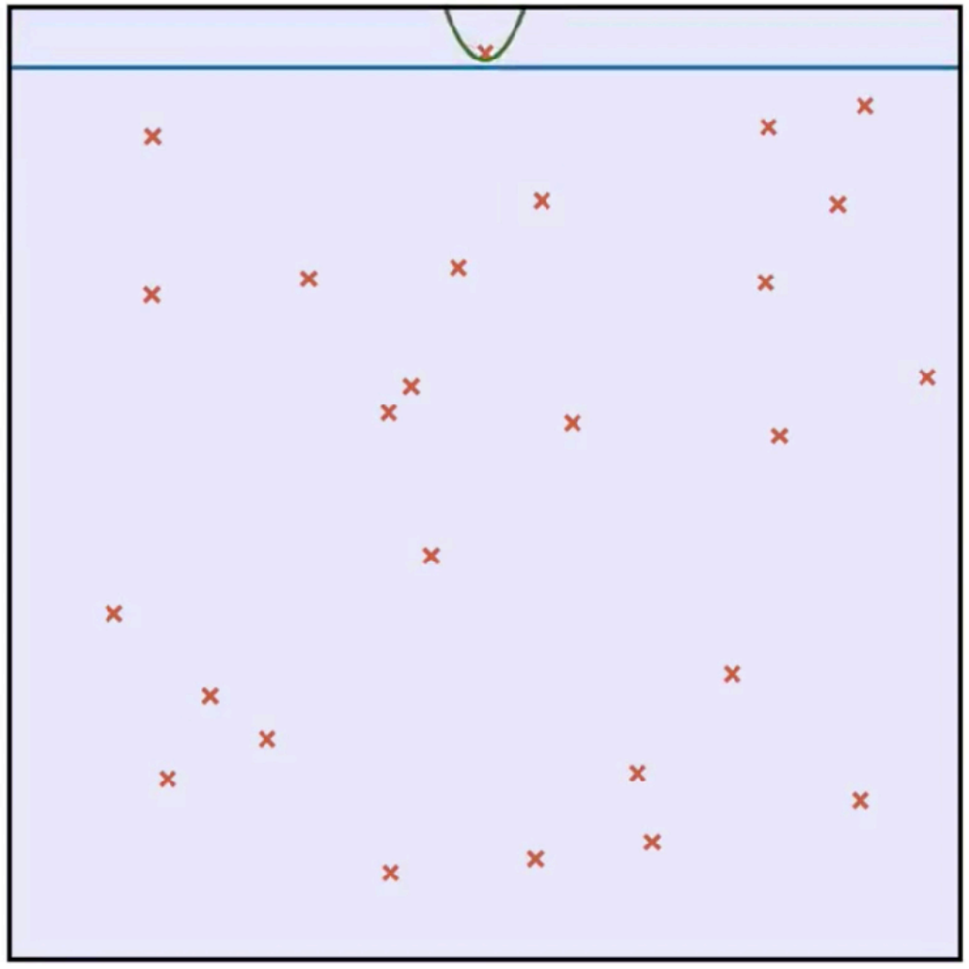

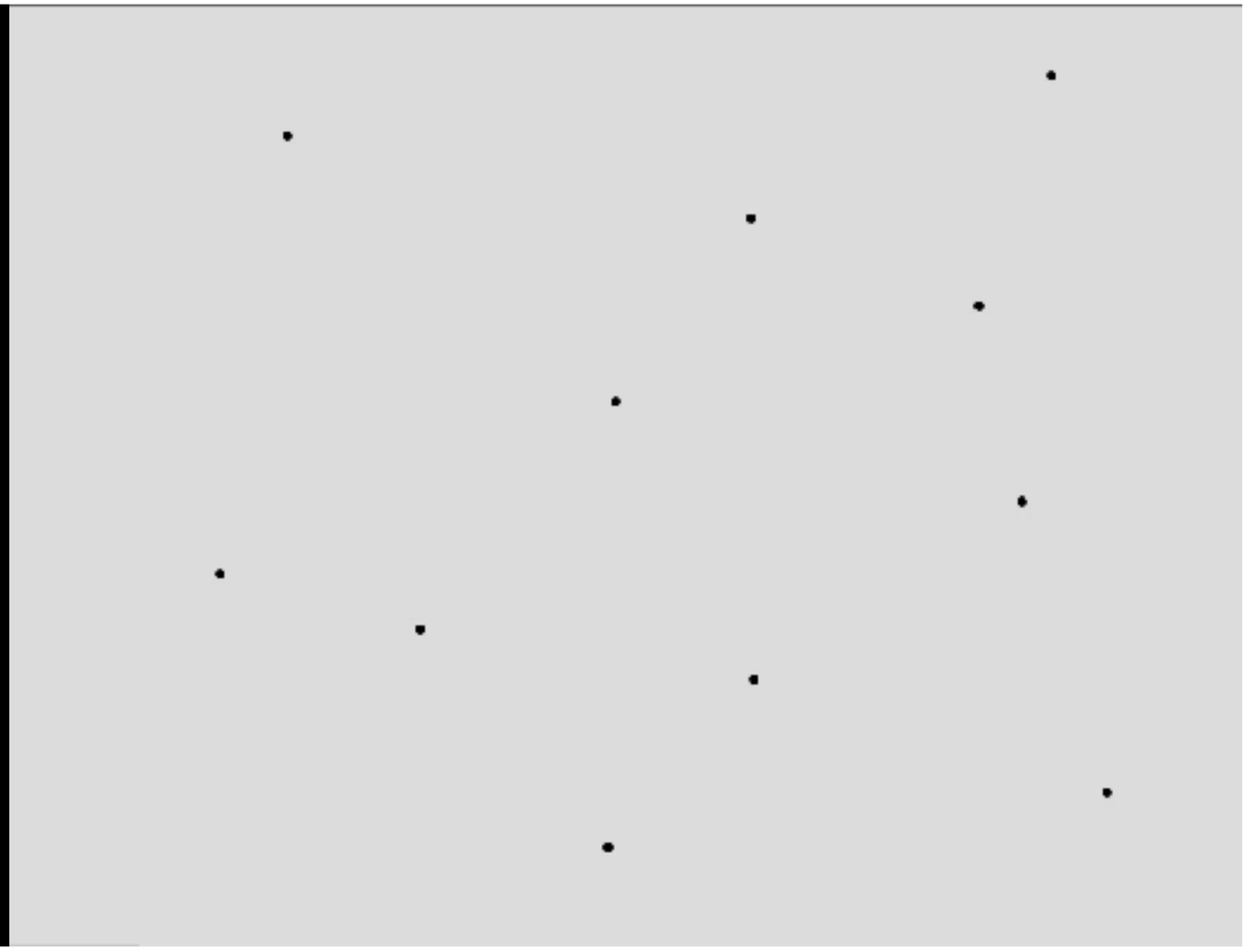

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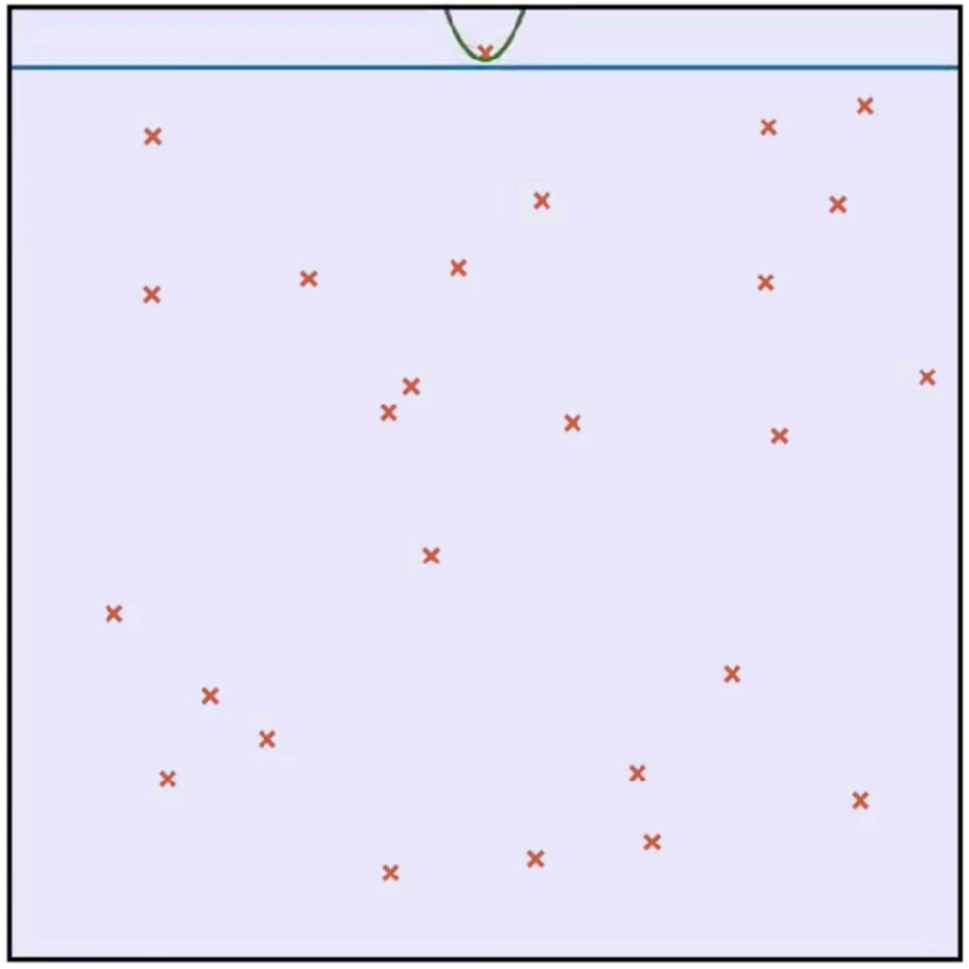

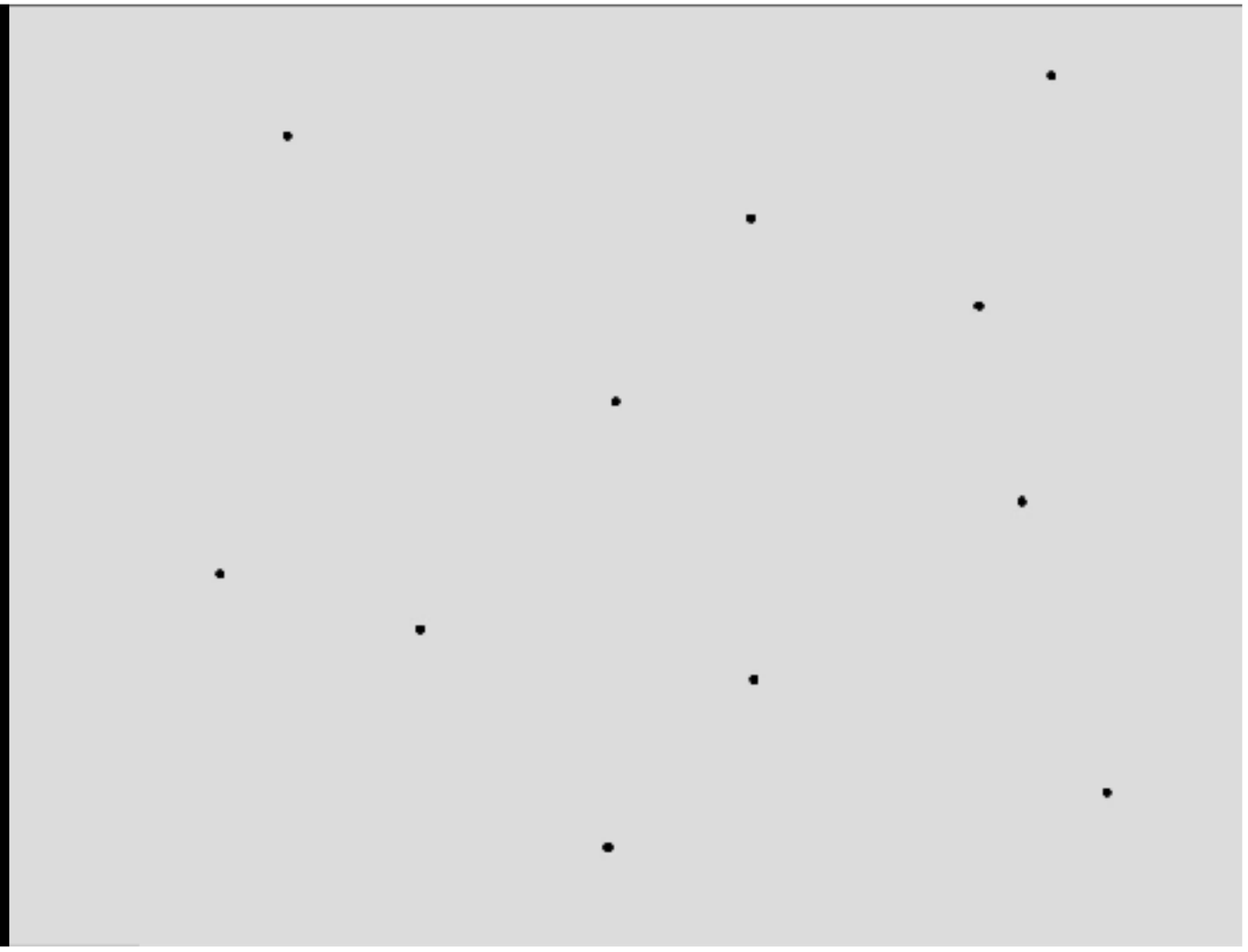

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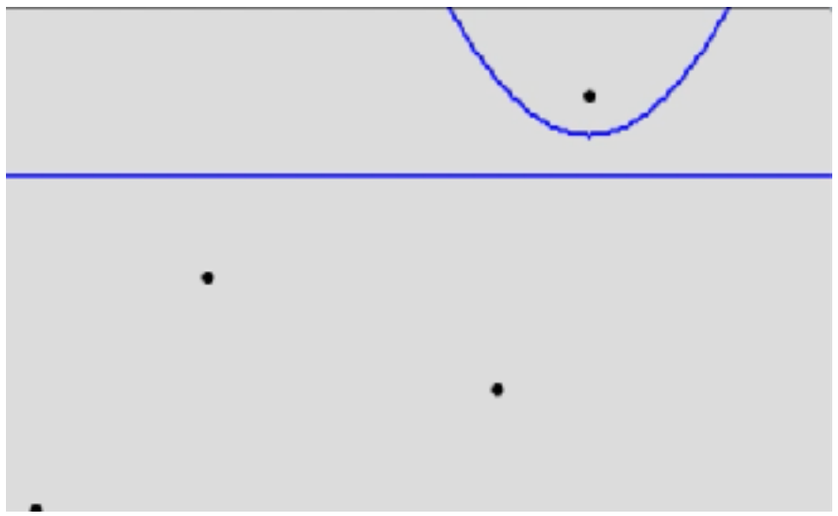

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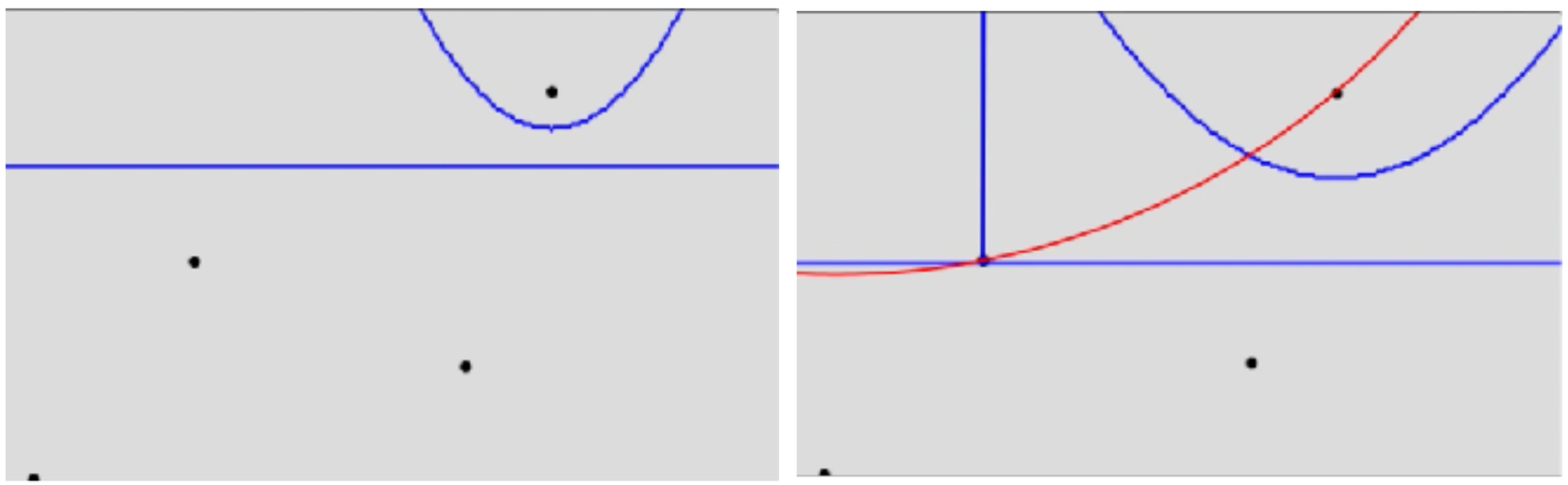

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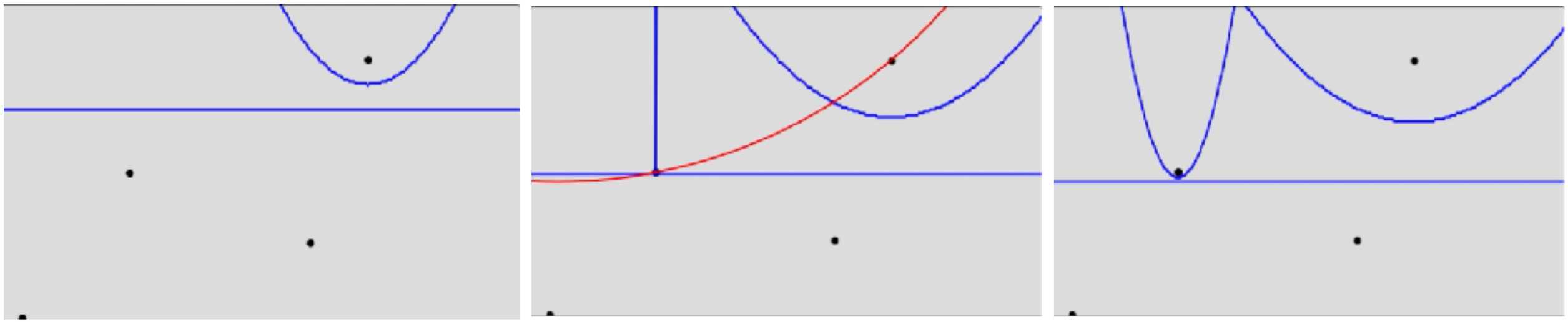

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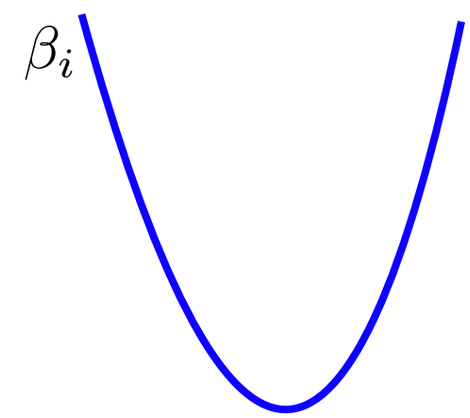

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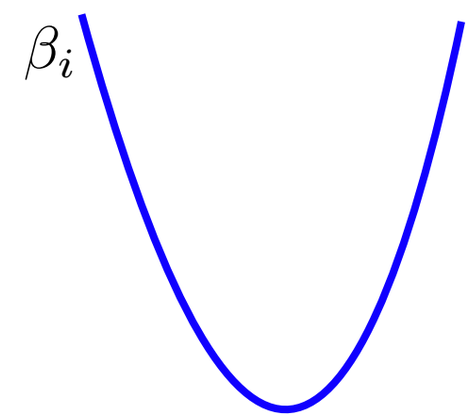

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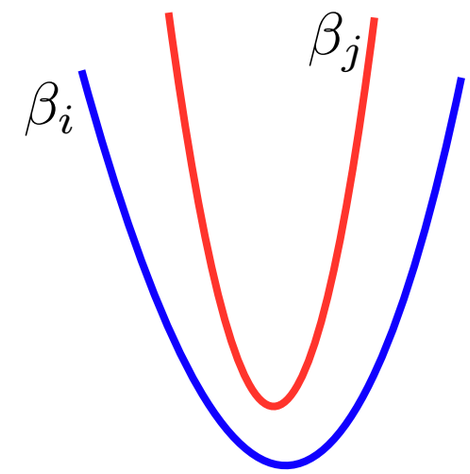

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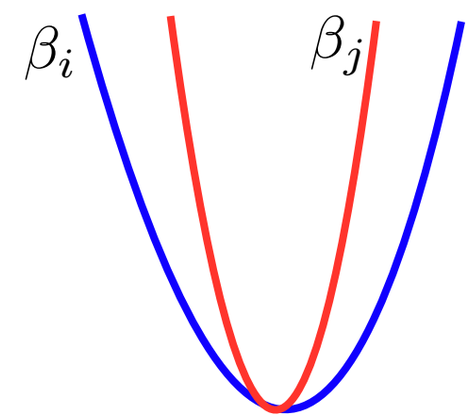

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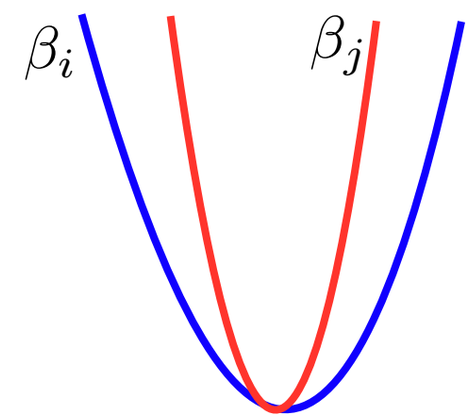

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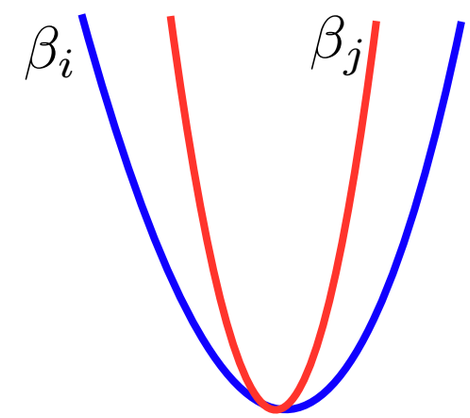

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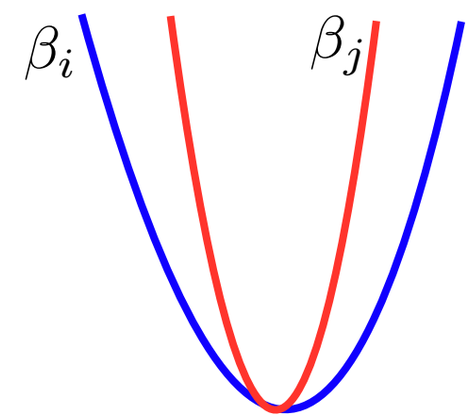

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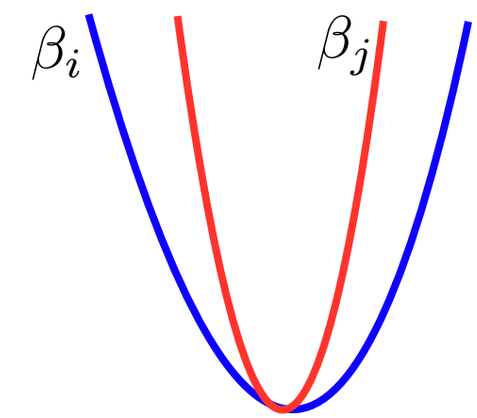

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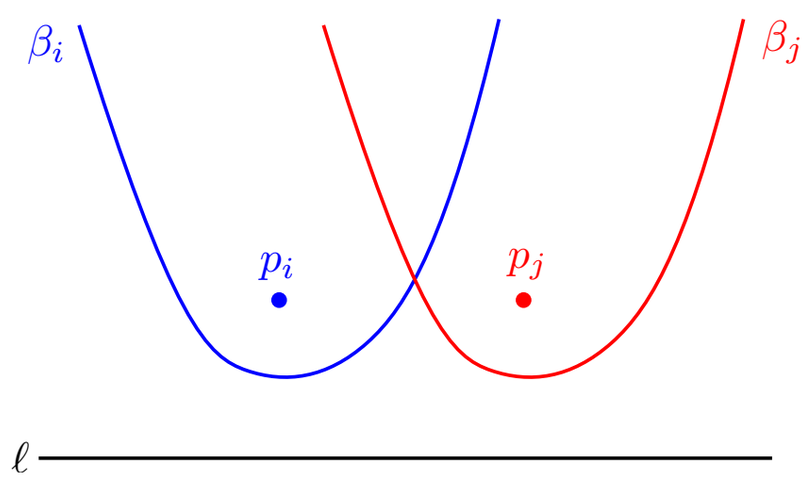

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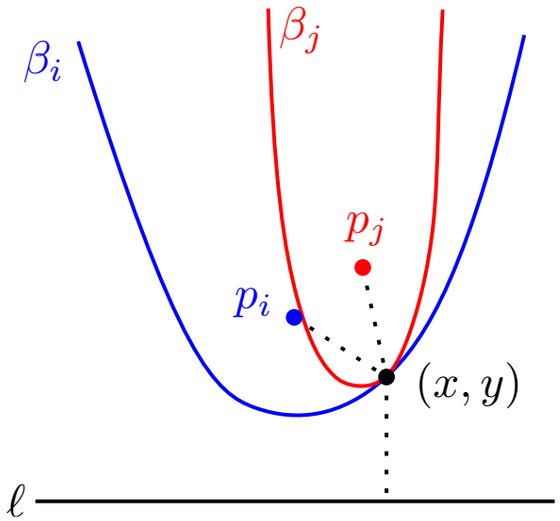

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- Equation for β_j:

$$(\ell.y - y)^2 = (p_j.x - x)^2 + (p_j.x - y)^2$$

$$\Leftrightarrow \ell.y^2 - 2 \cdot \ell.y \cdot y + y^2$$

$$= p_j.x^2 - 2 \cdot x \cdot p_j.x + x^2 + p_j.y^2 - 2 \cdot y \cdot p_j.y + y^2$$

$$\Leftrightarrow y \cdot 2\,(p_j.y - \ell.y)$$

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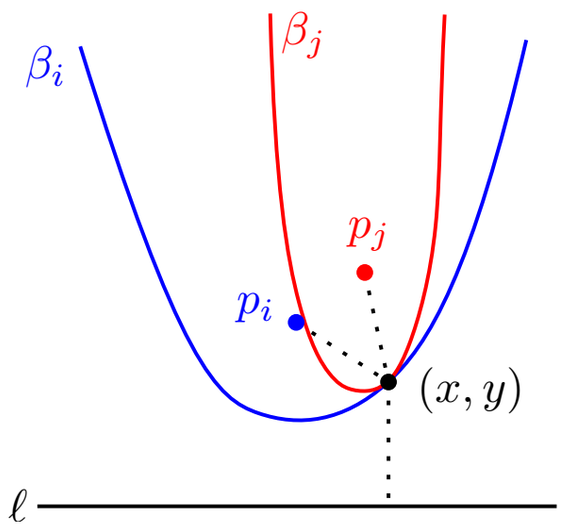

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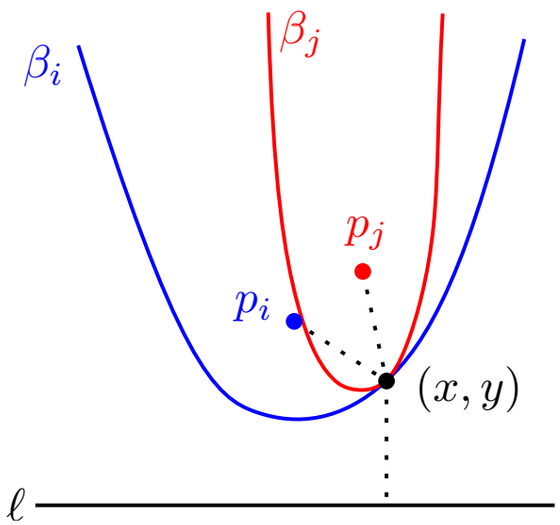

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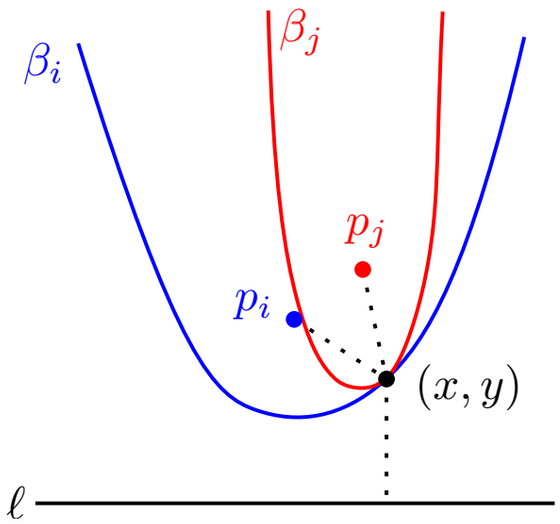

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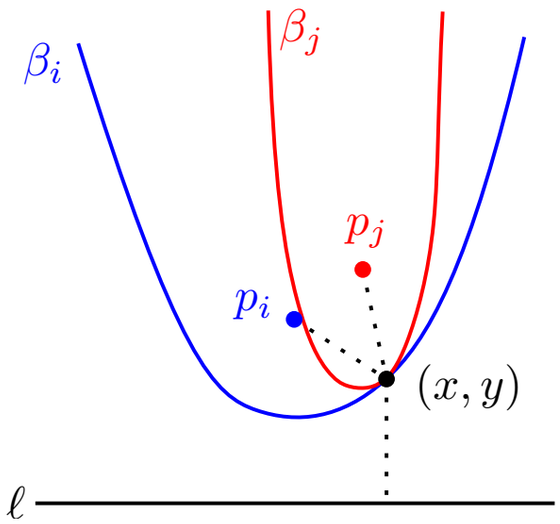

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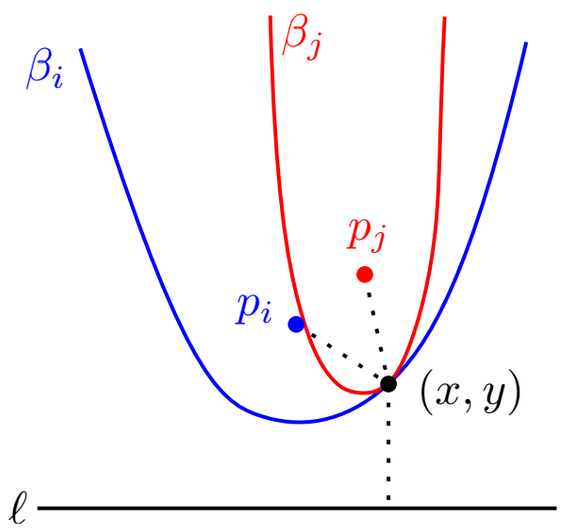

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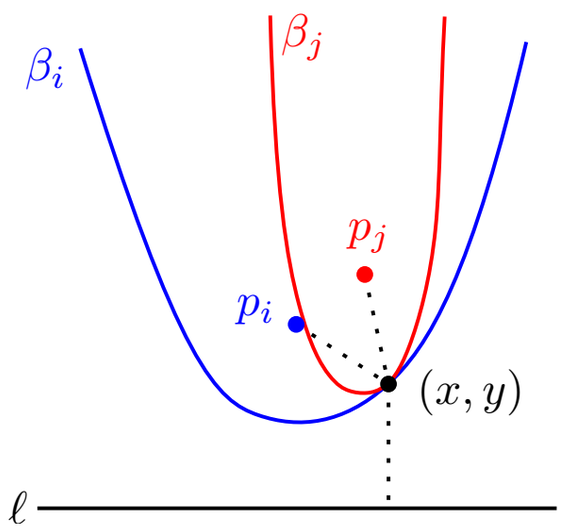

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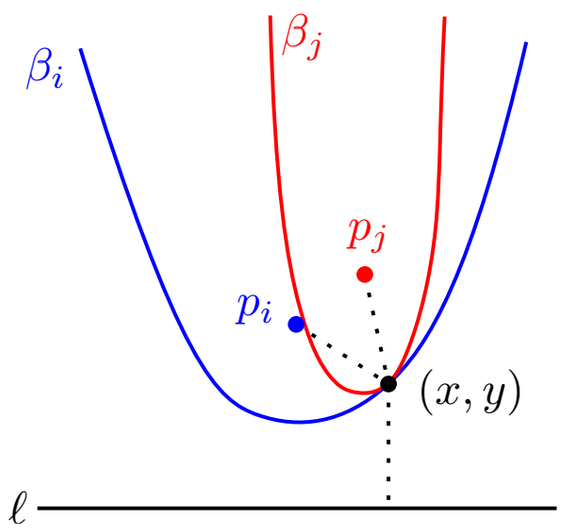

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  $\Rightarrow$ Two intersection points ↯

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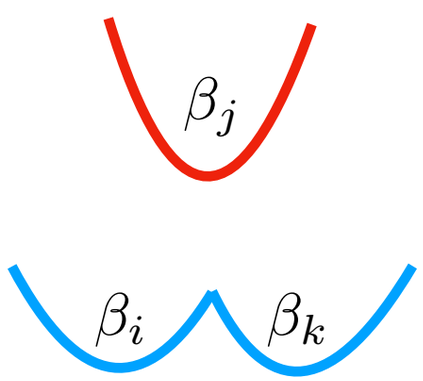

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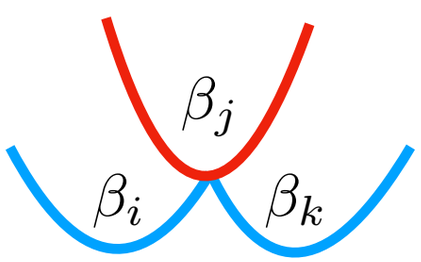

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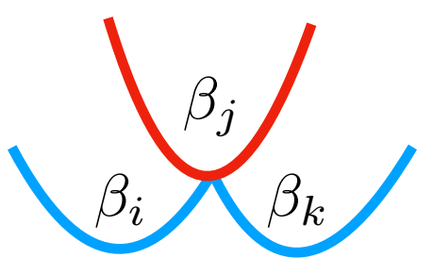

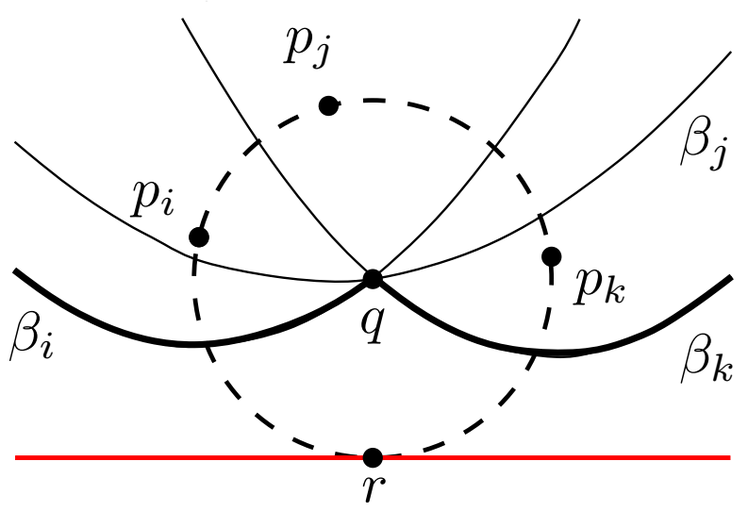

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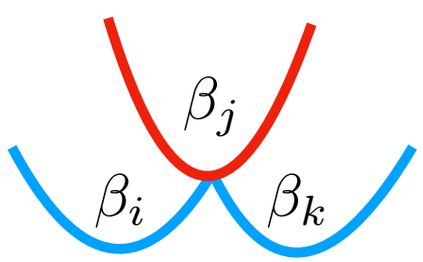

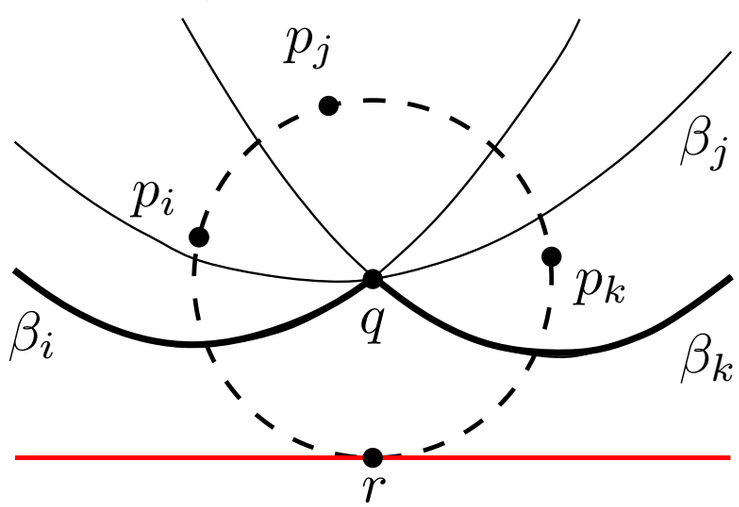

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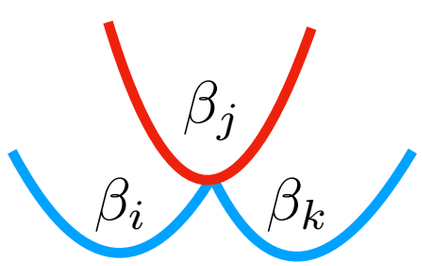

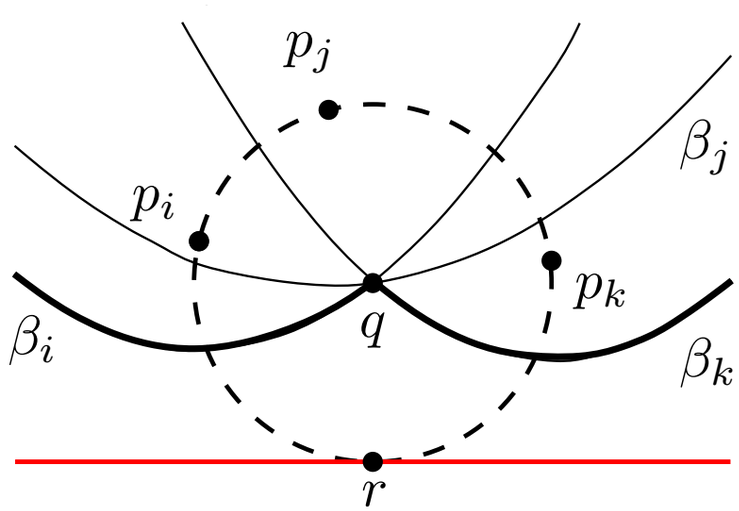

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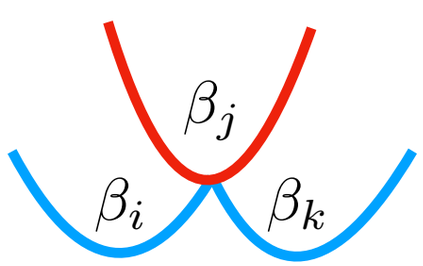

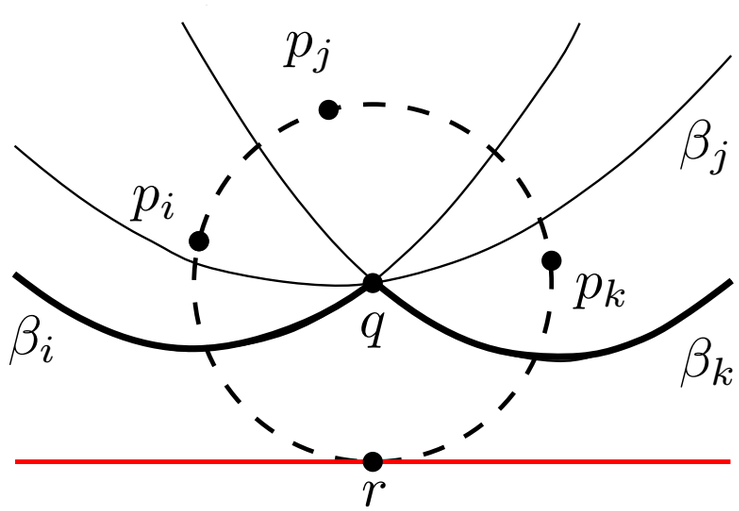

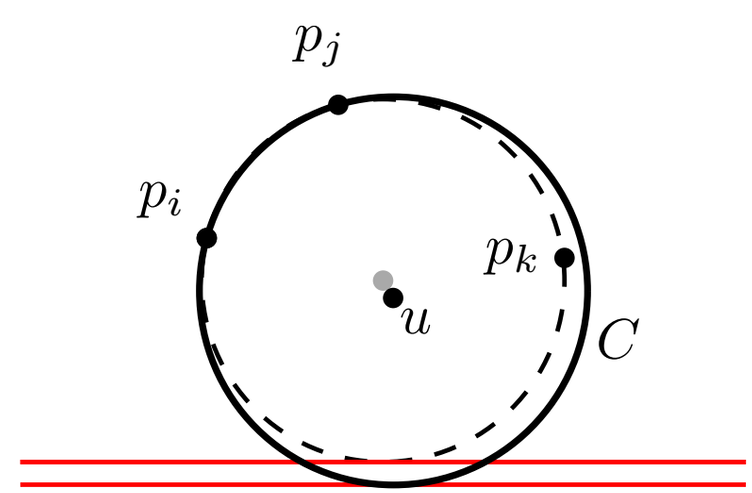

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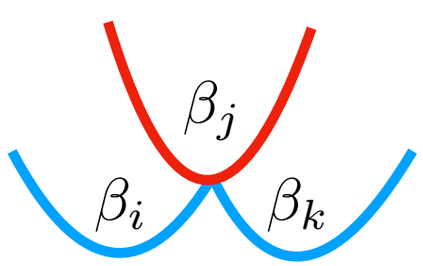

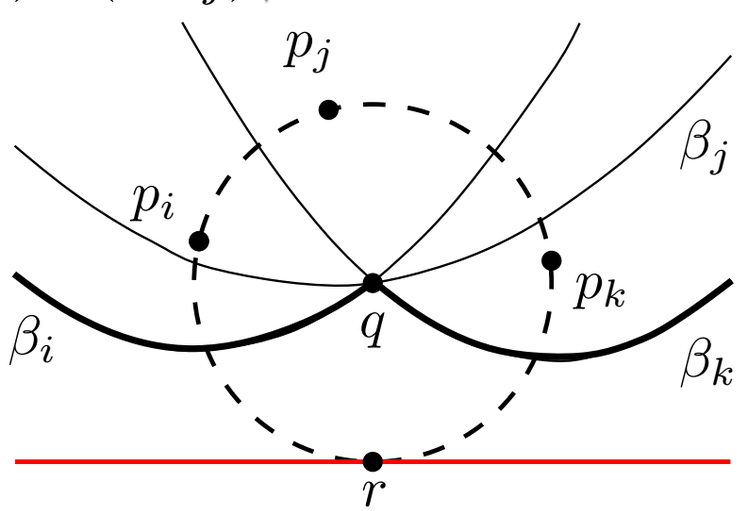

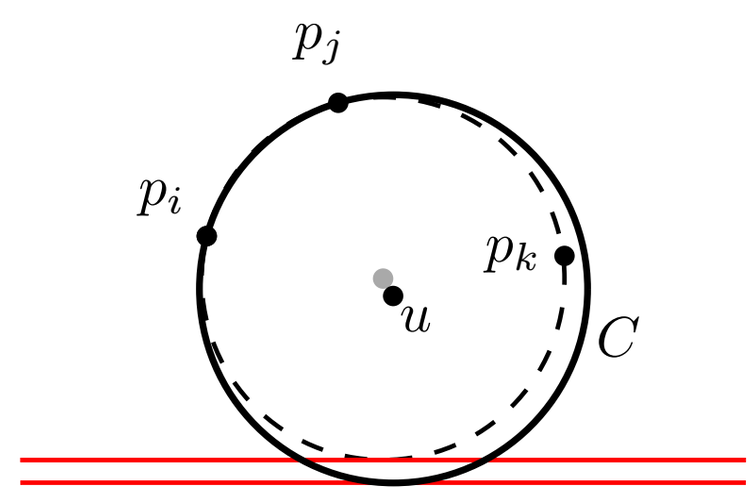

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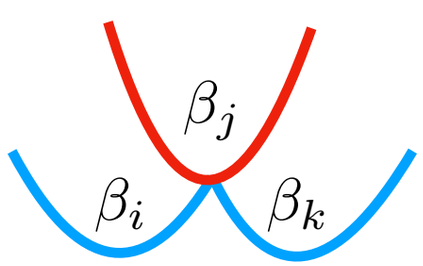

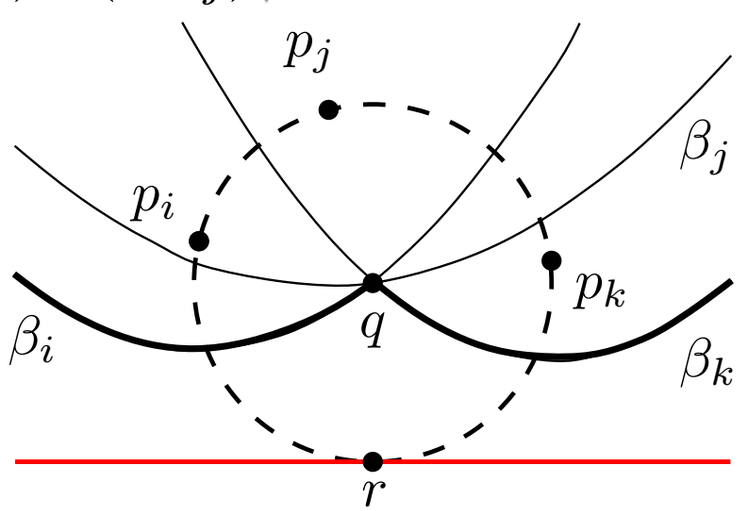

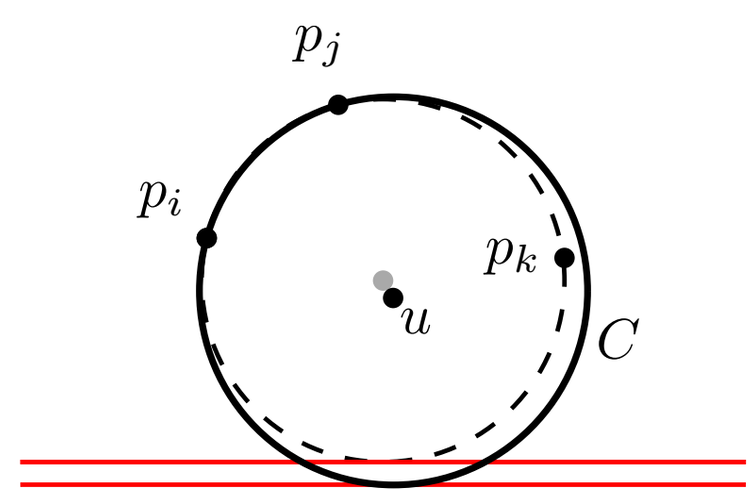

Complexity of the Beach Line - V

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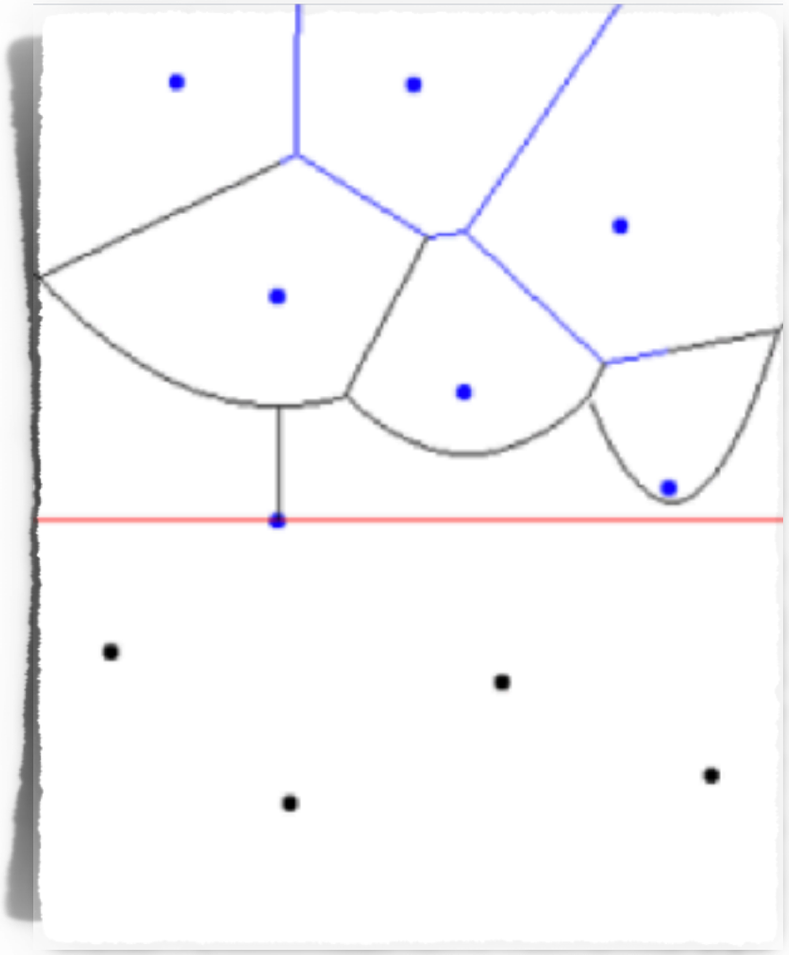

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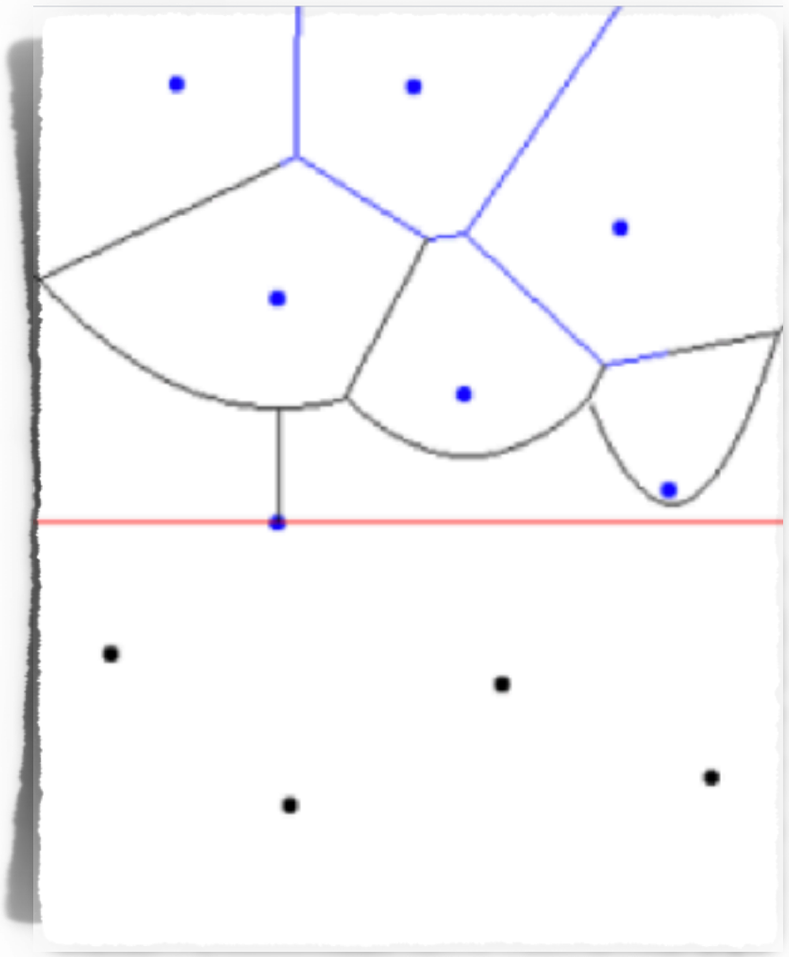

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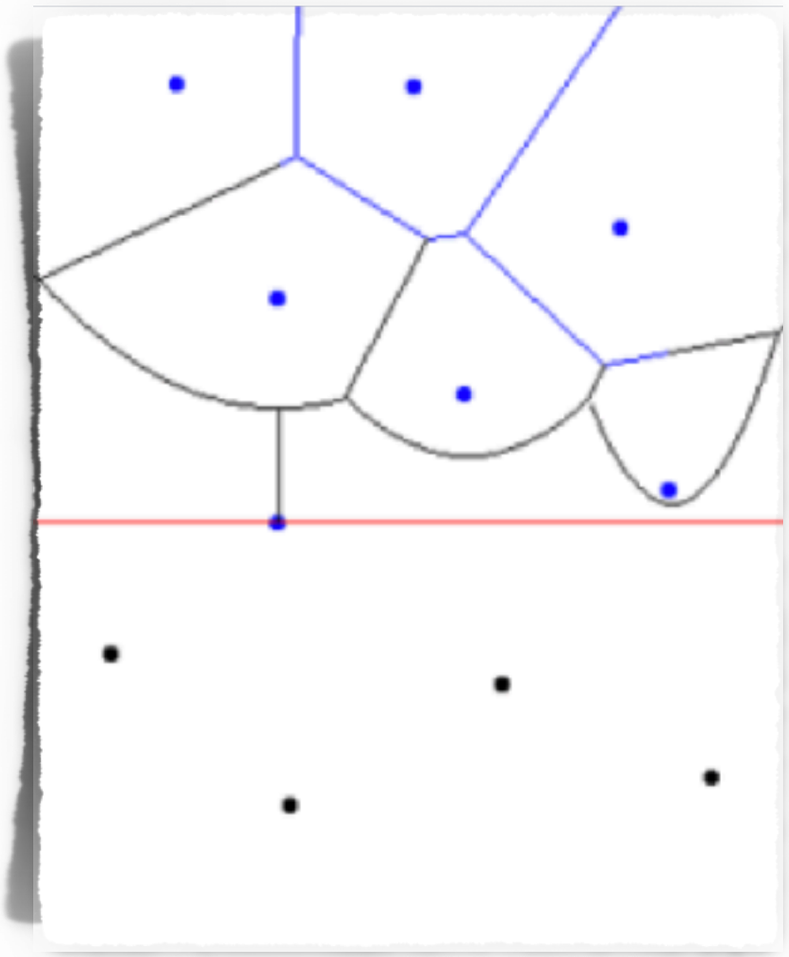

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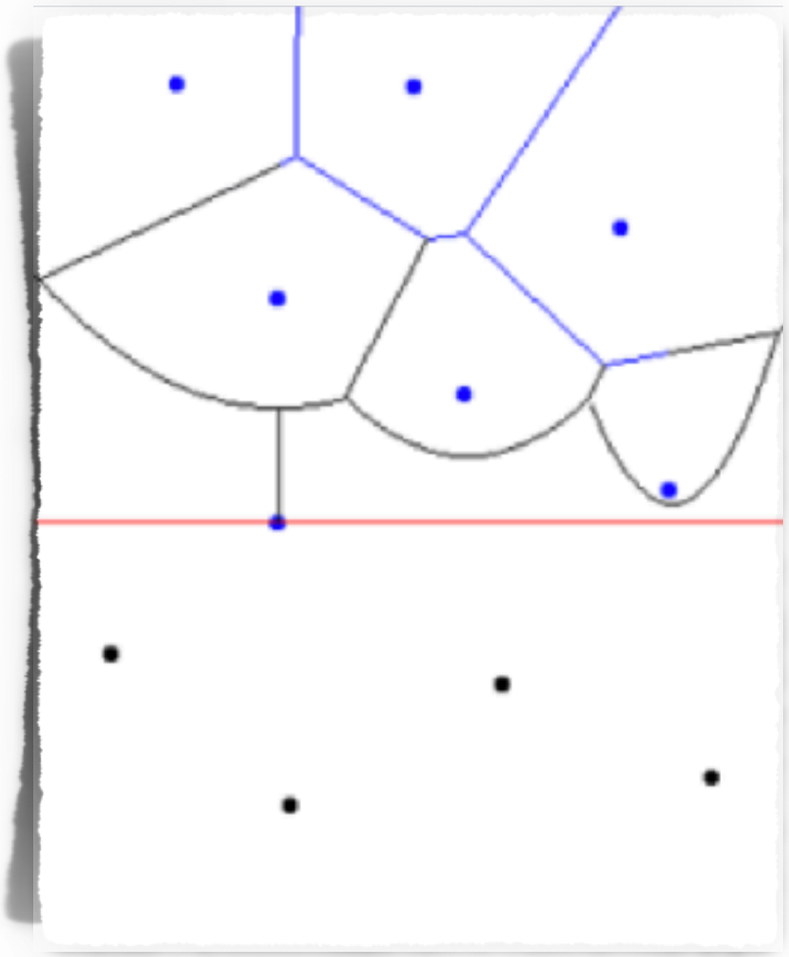

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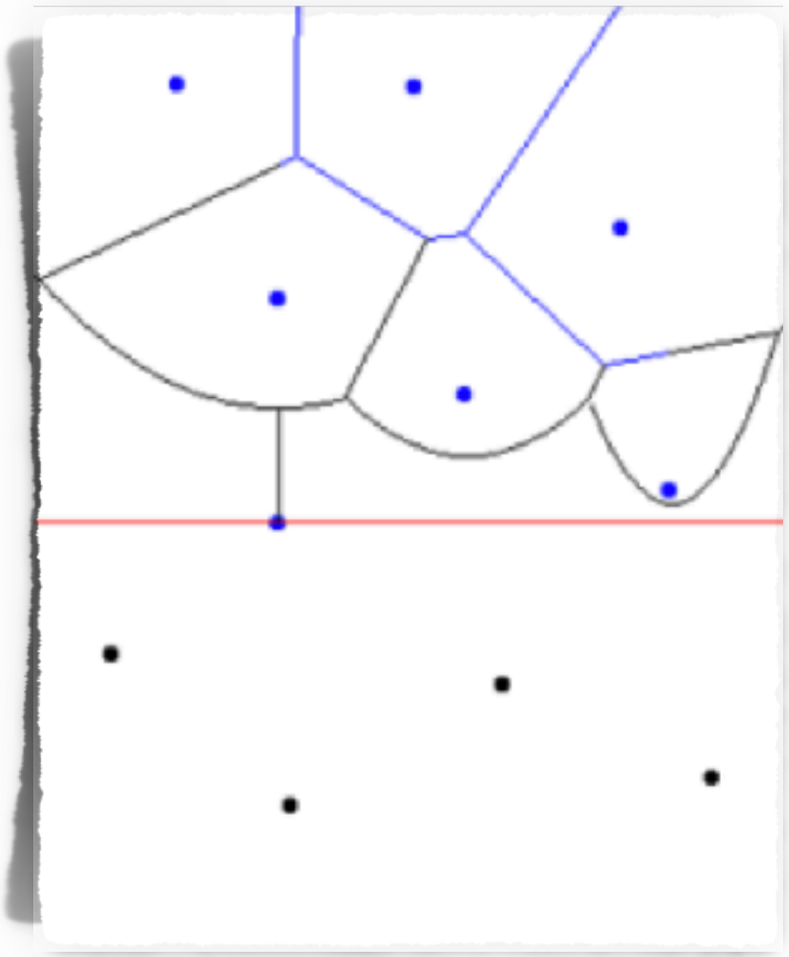

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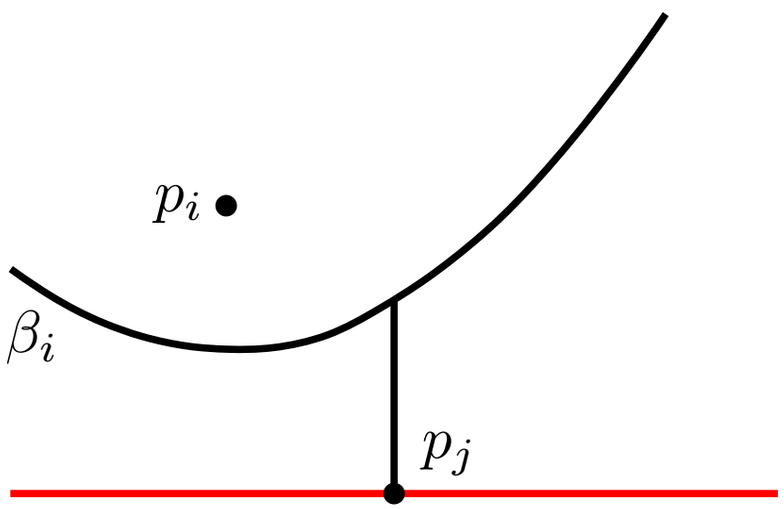

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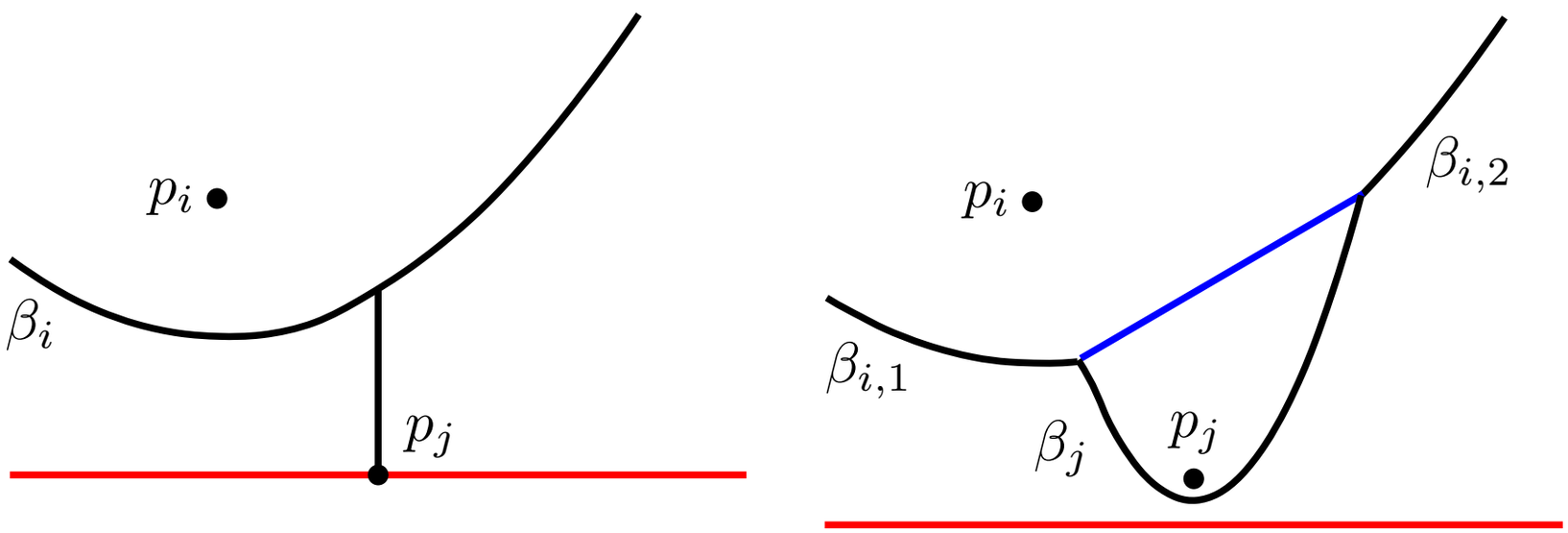

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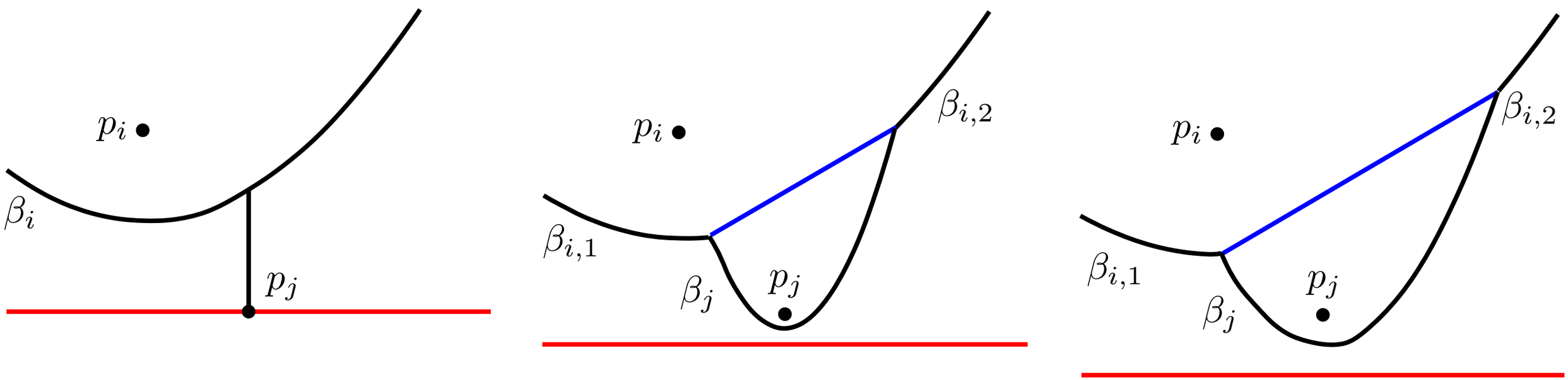

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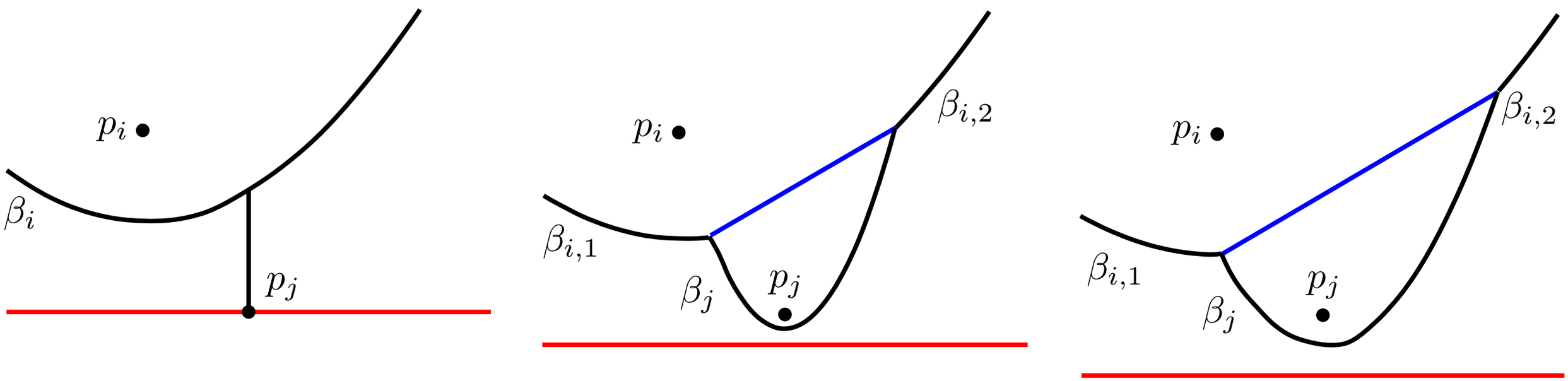

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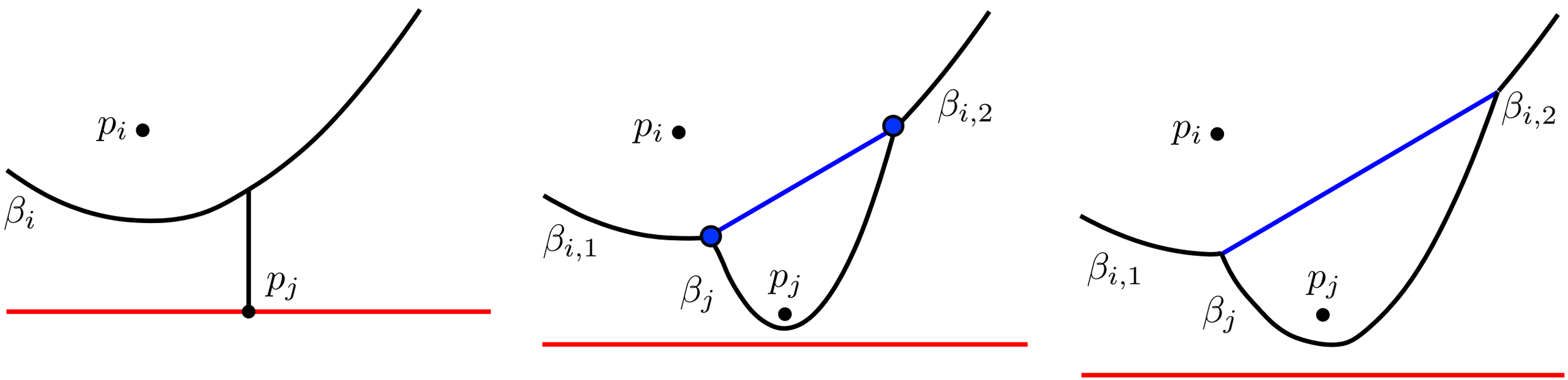

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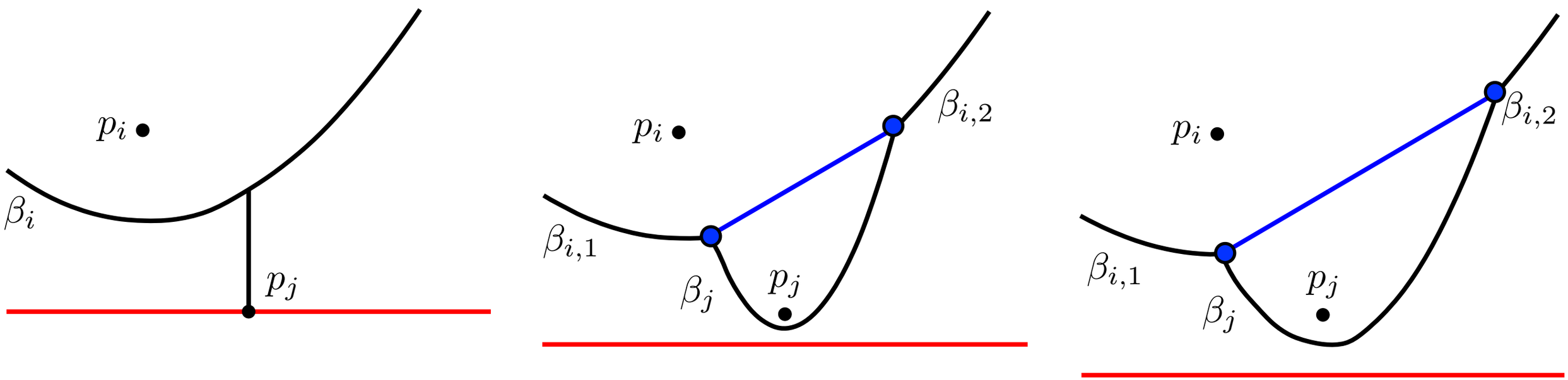

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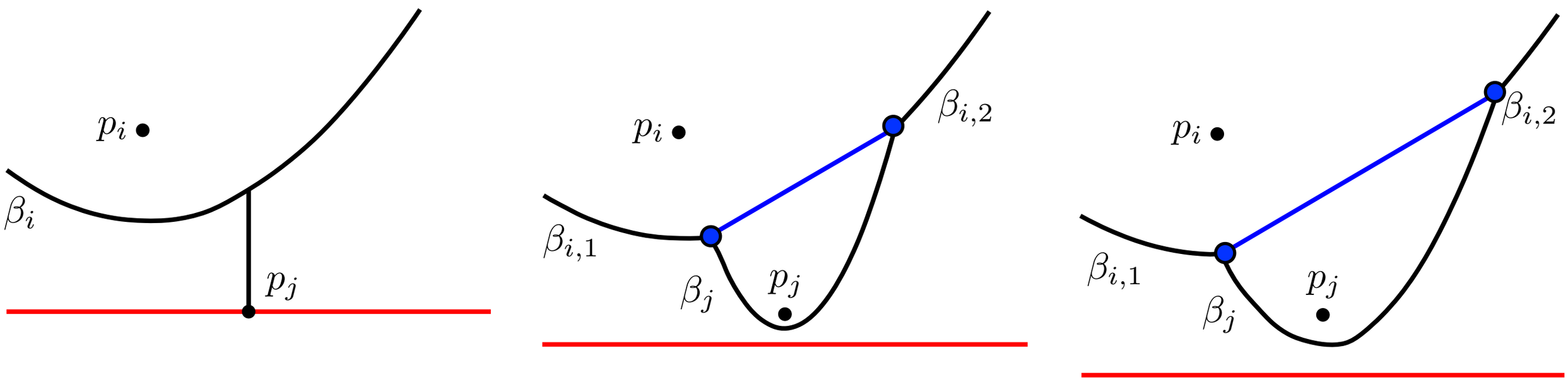

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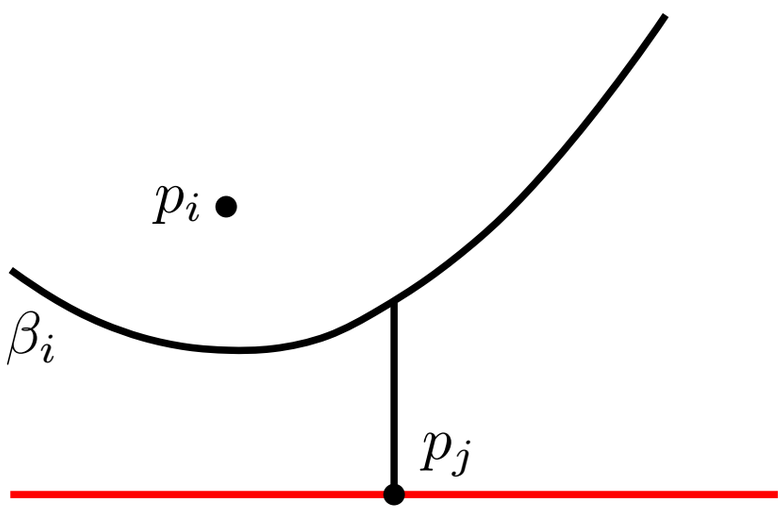

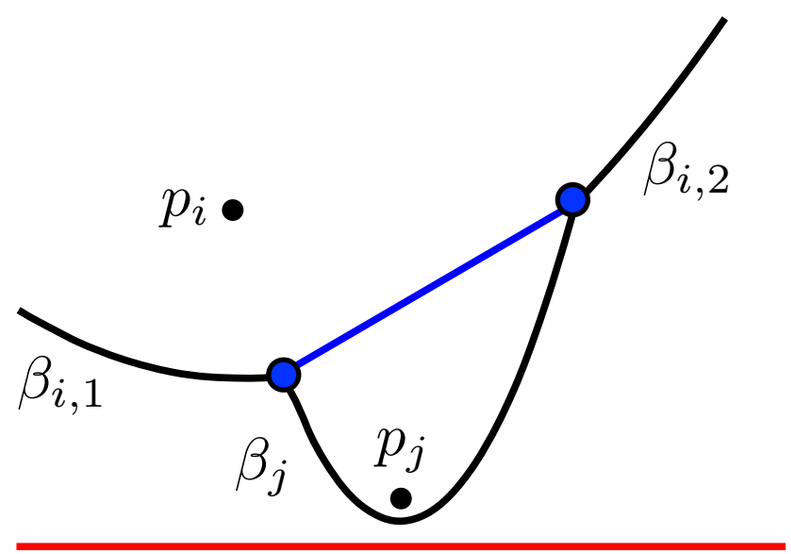

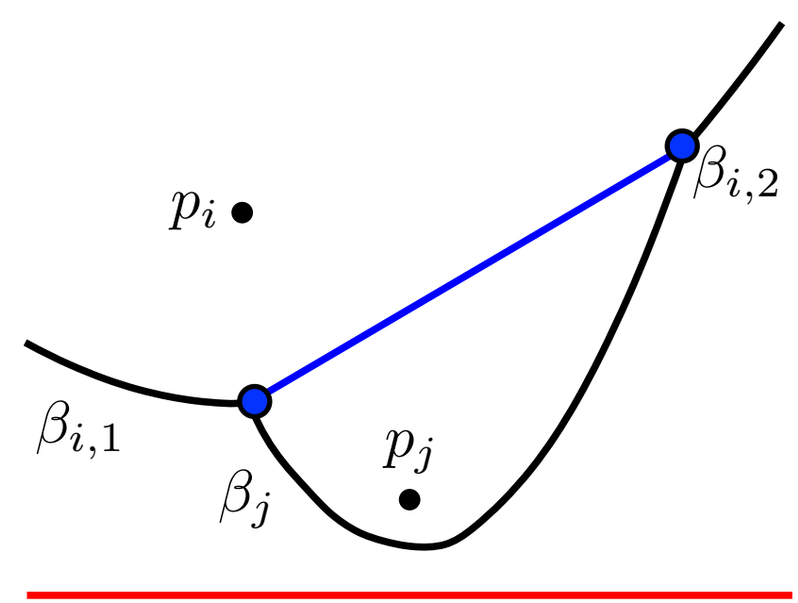

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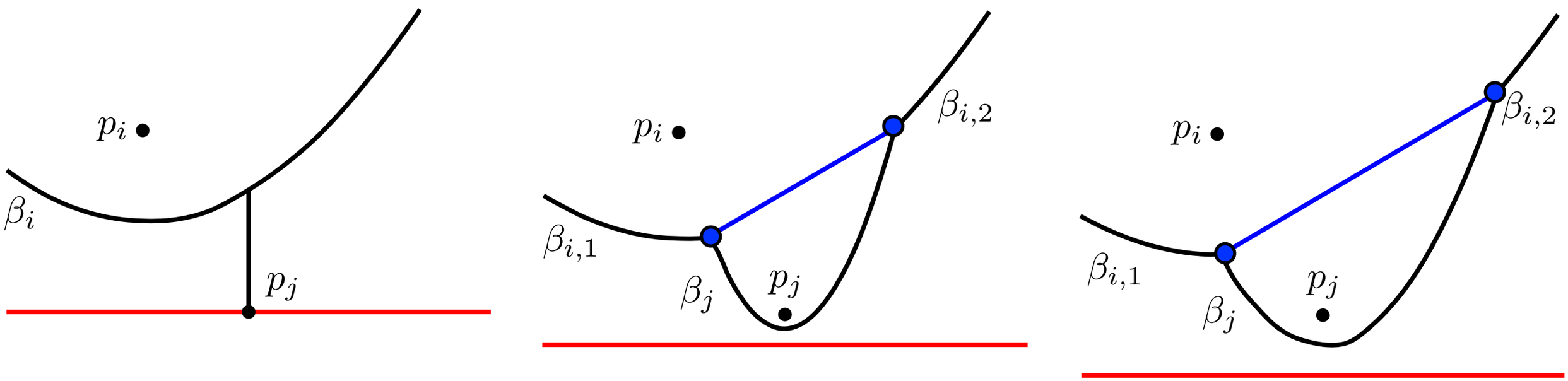

- Edge i.g. not connected to already computed part of $Vor(\mathcal{P})$

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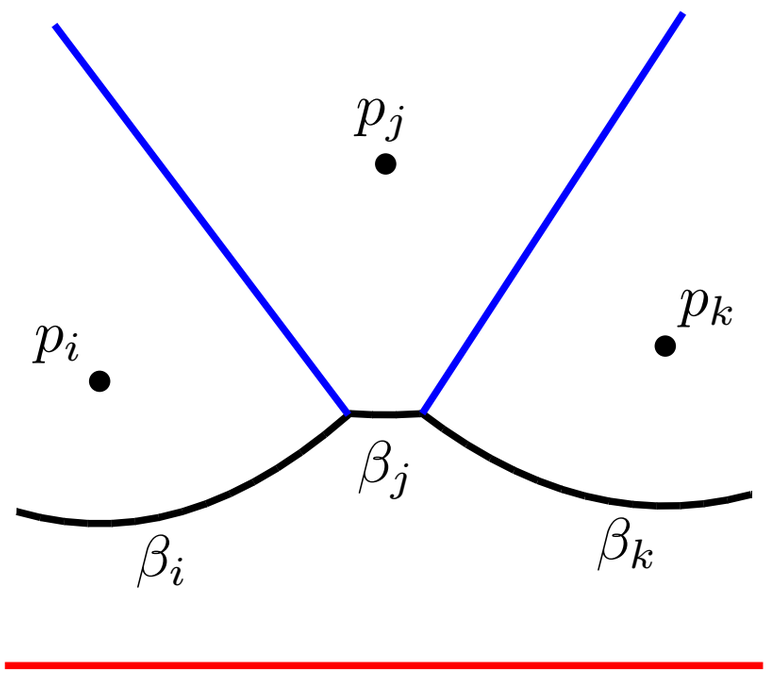

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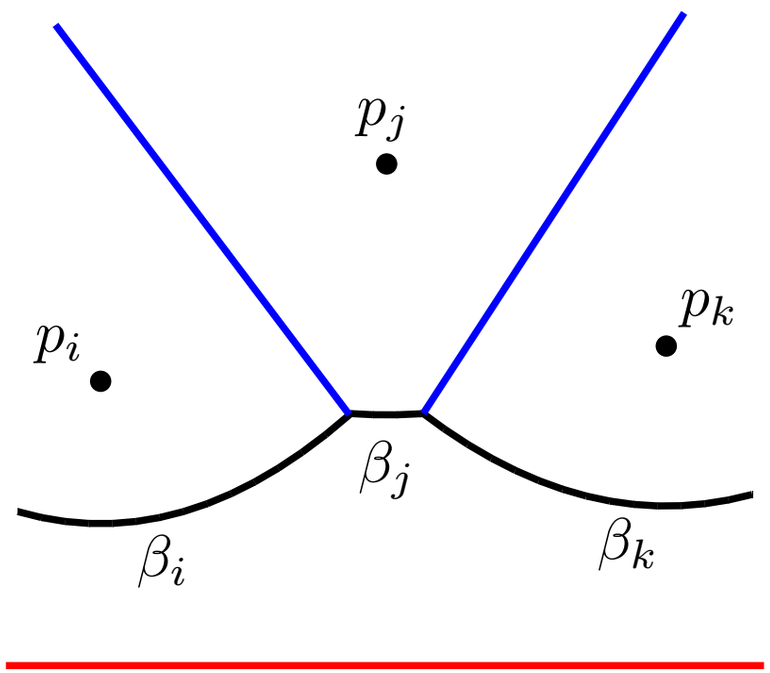

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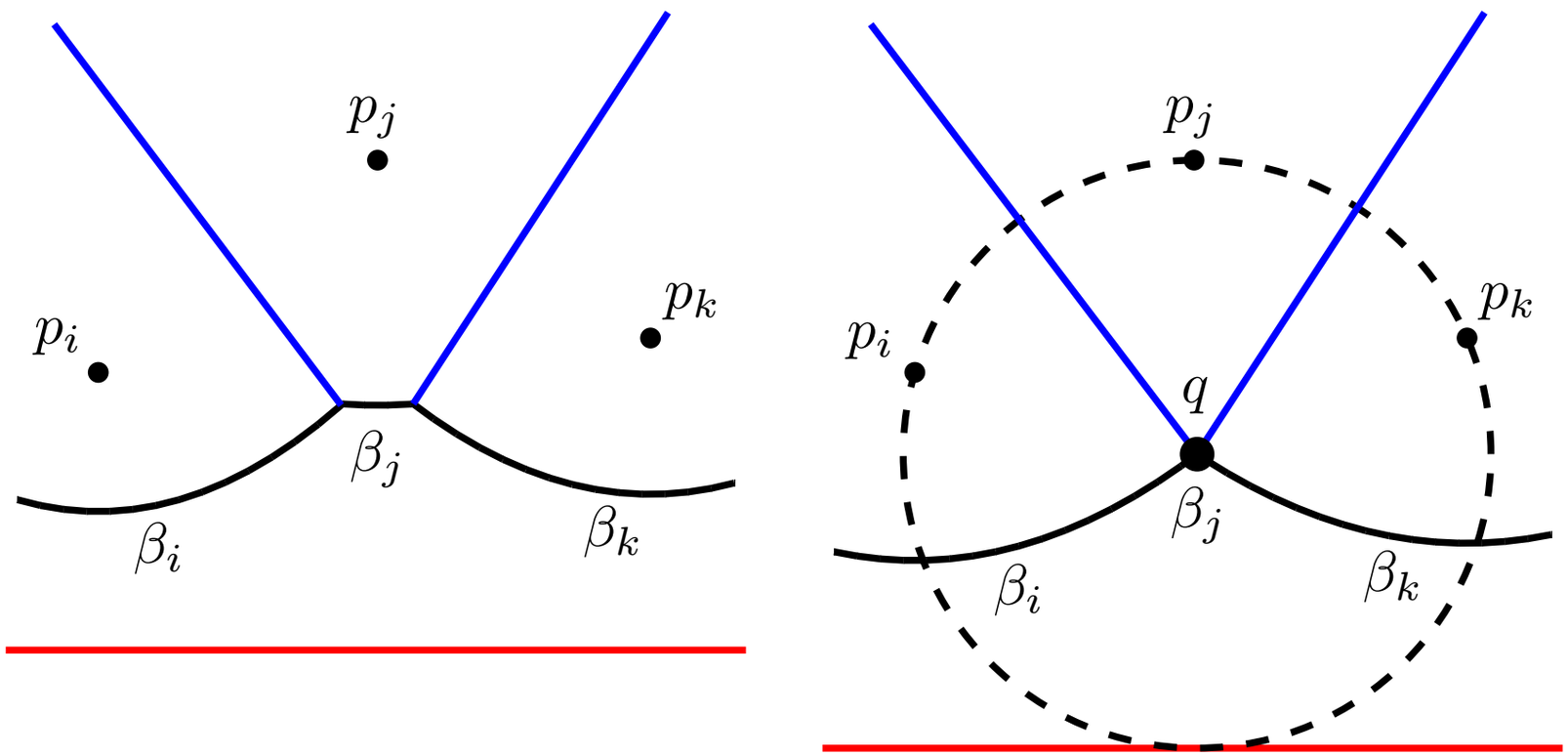

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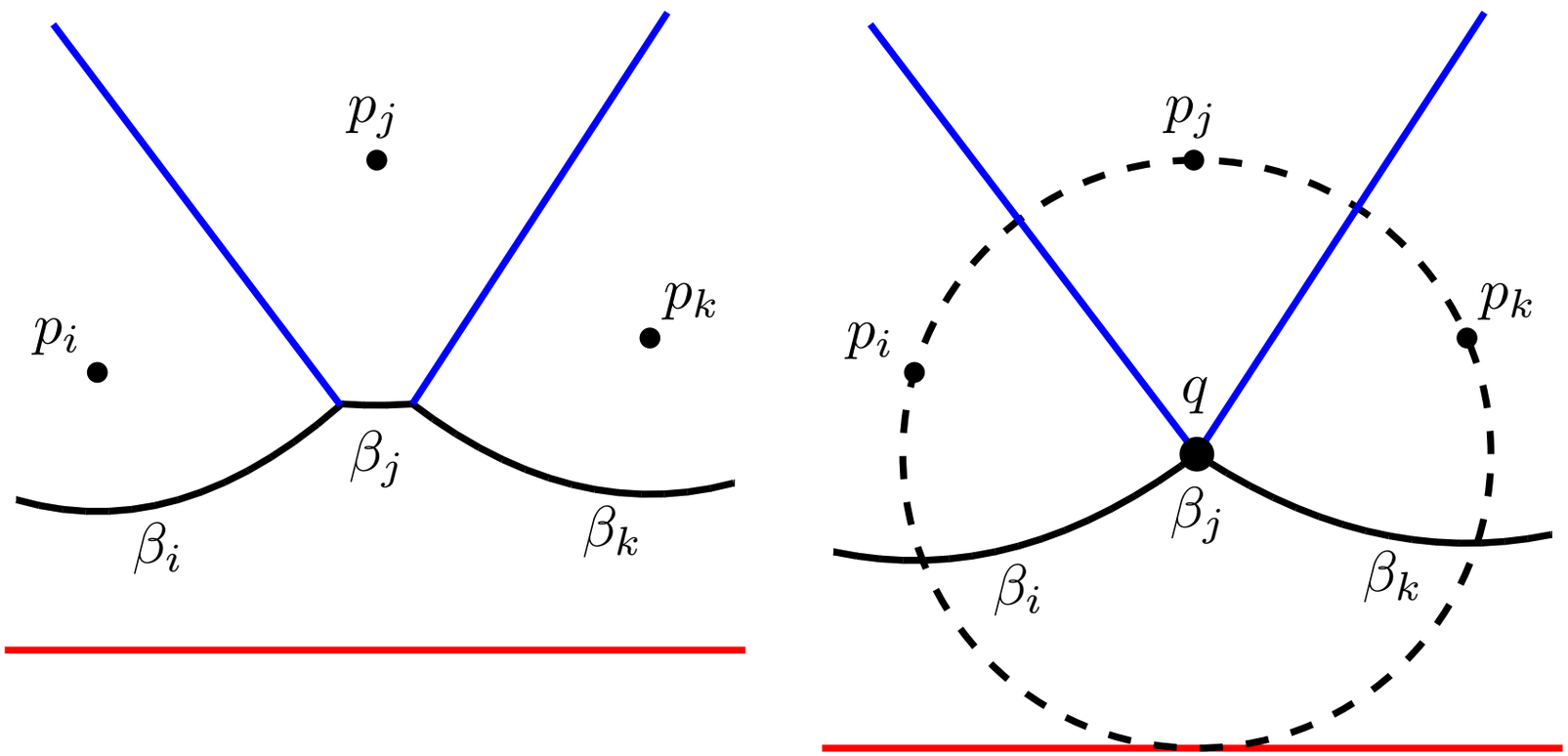

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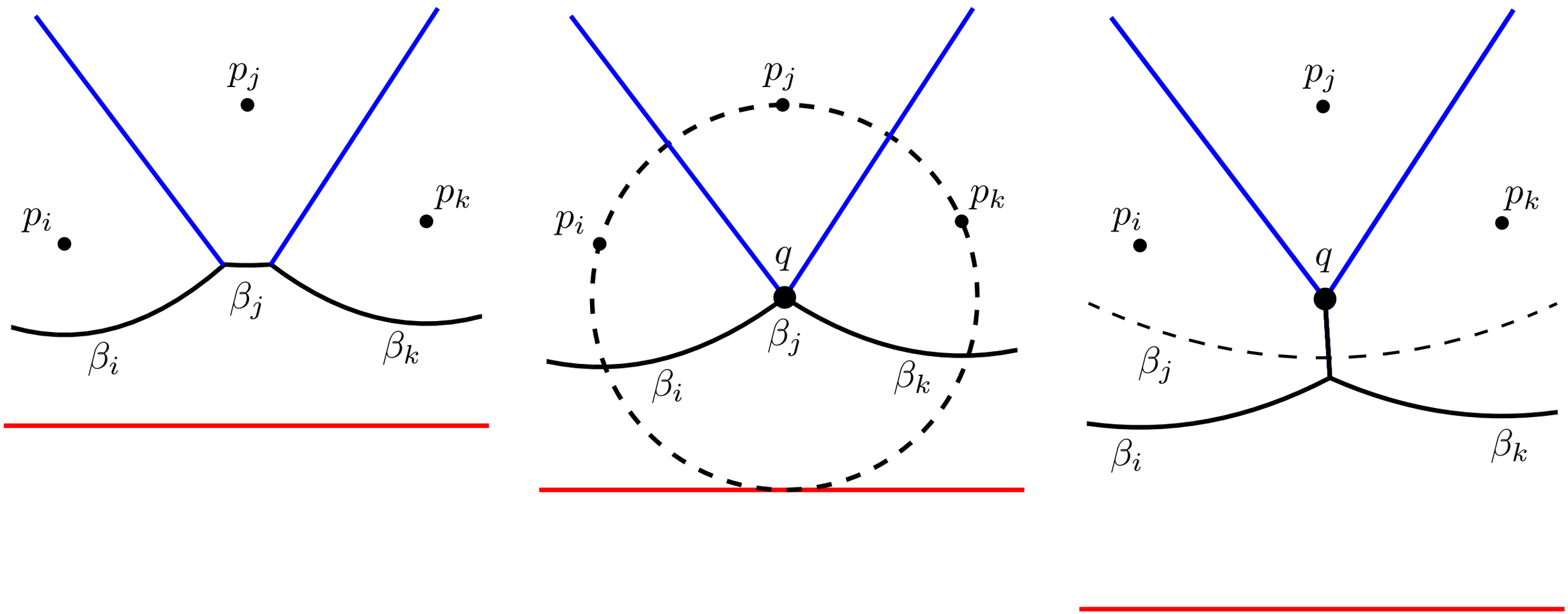

Circle Events - II

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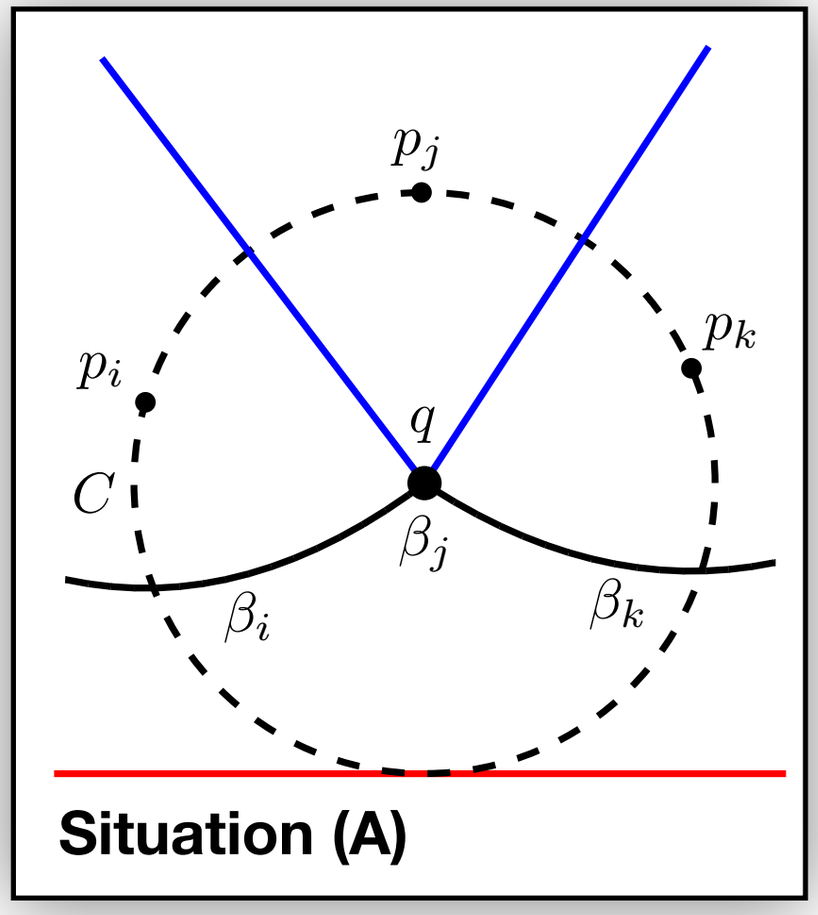

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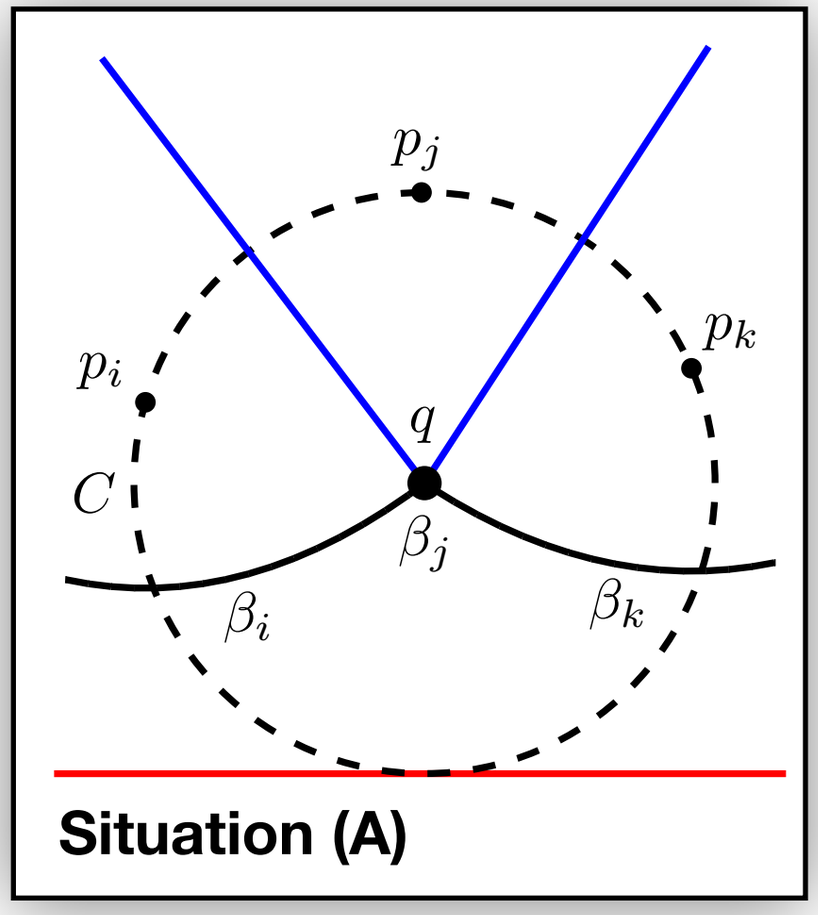

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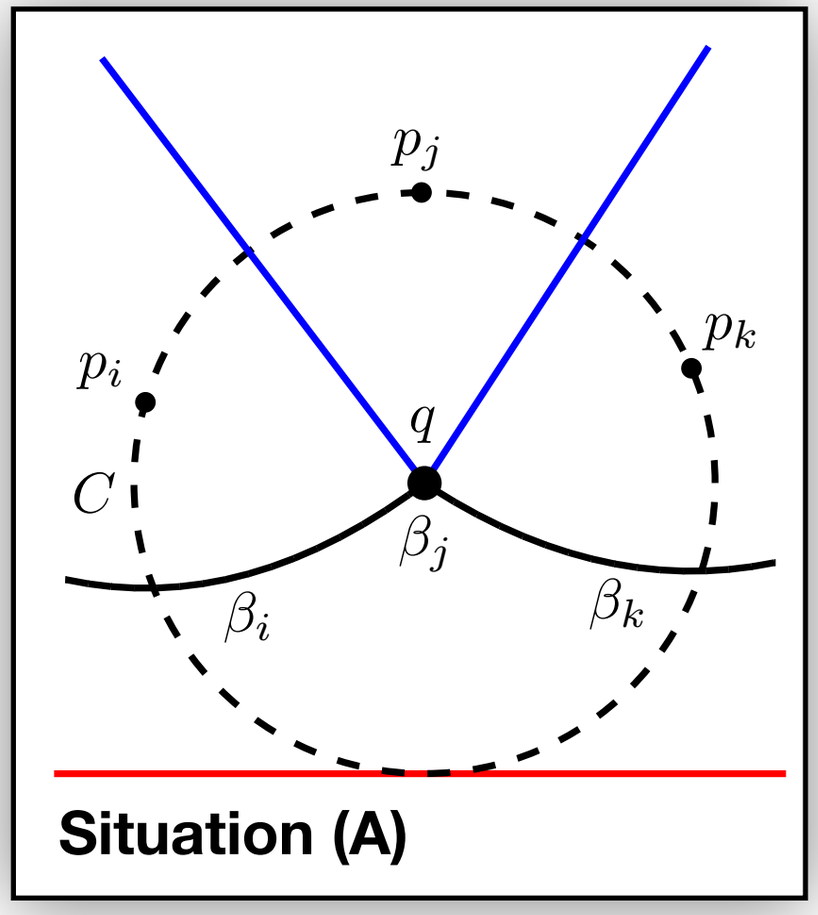

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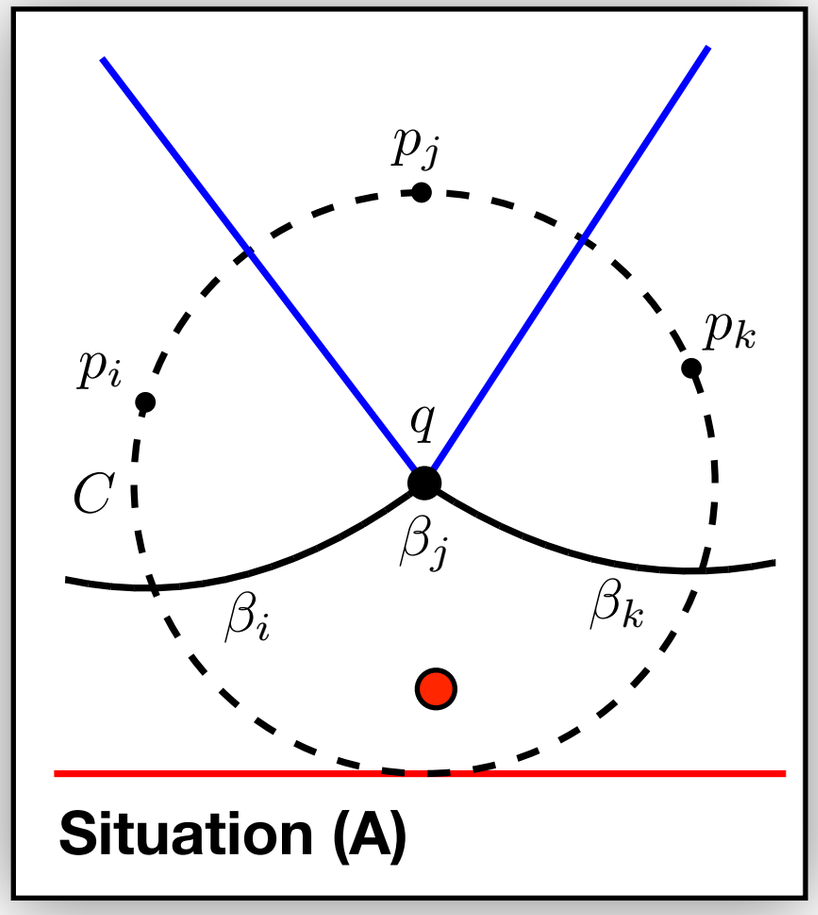

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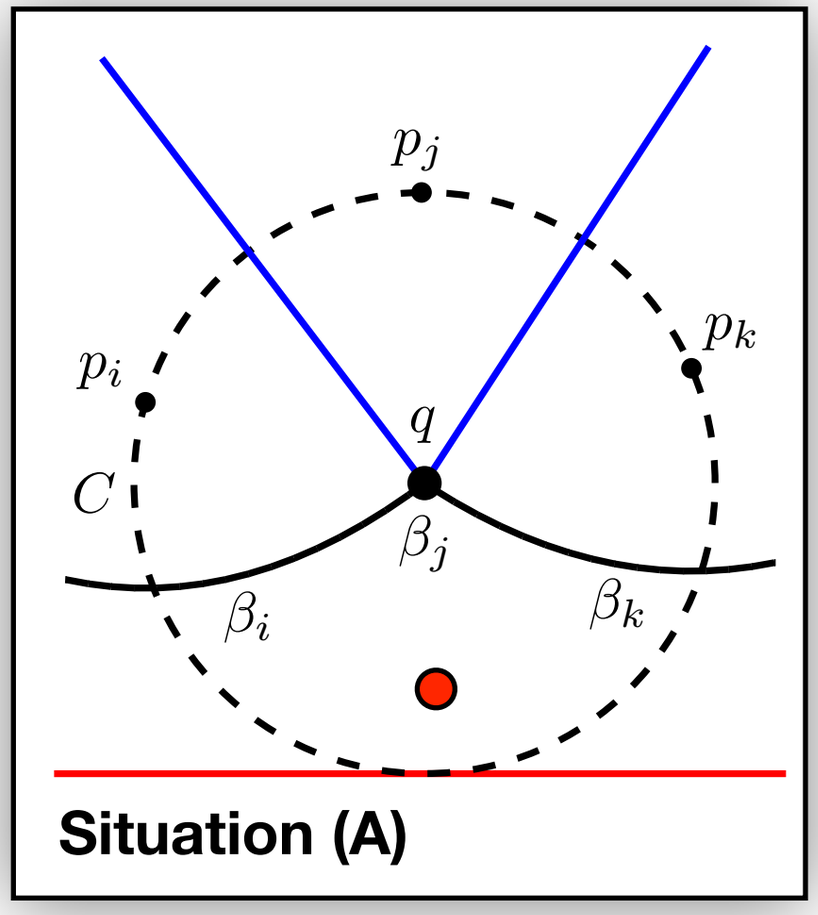

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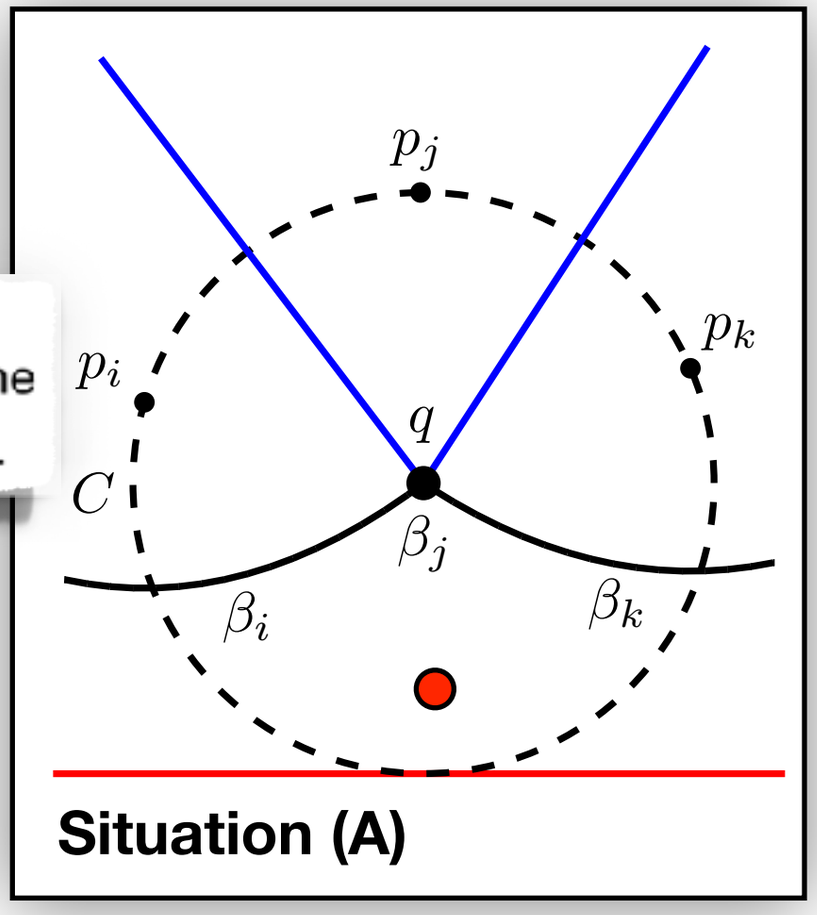

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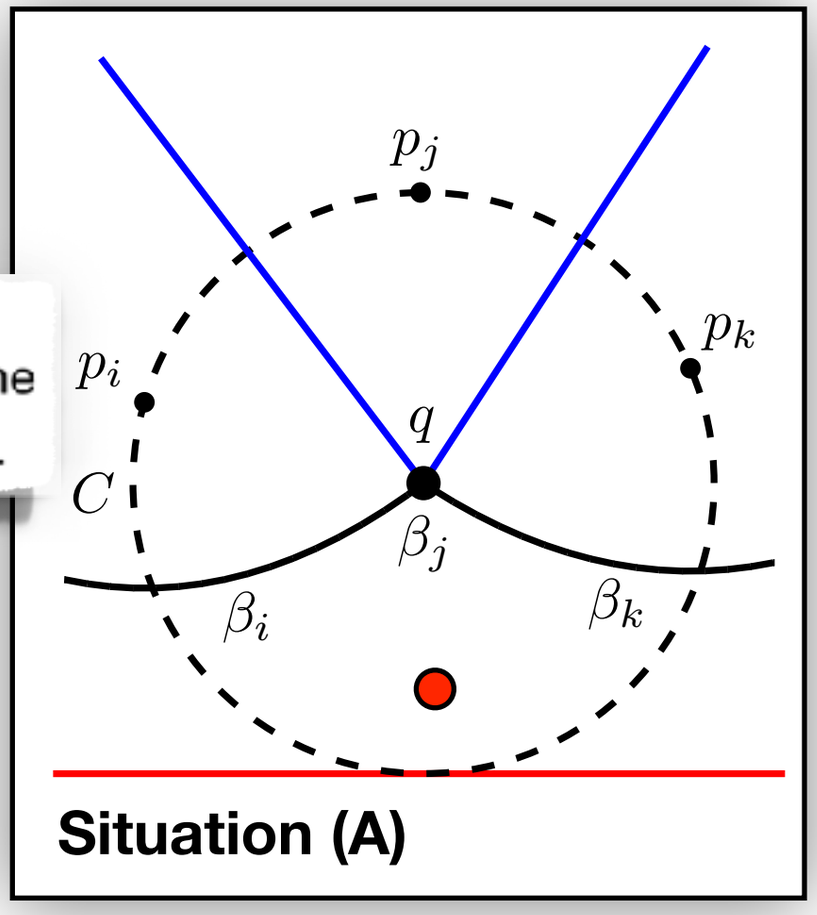

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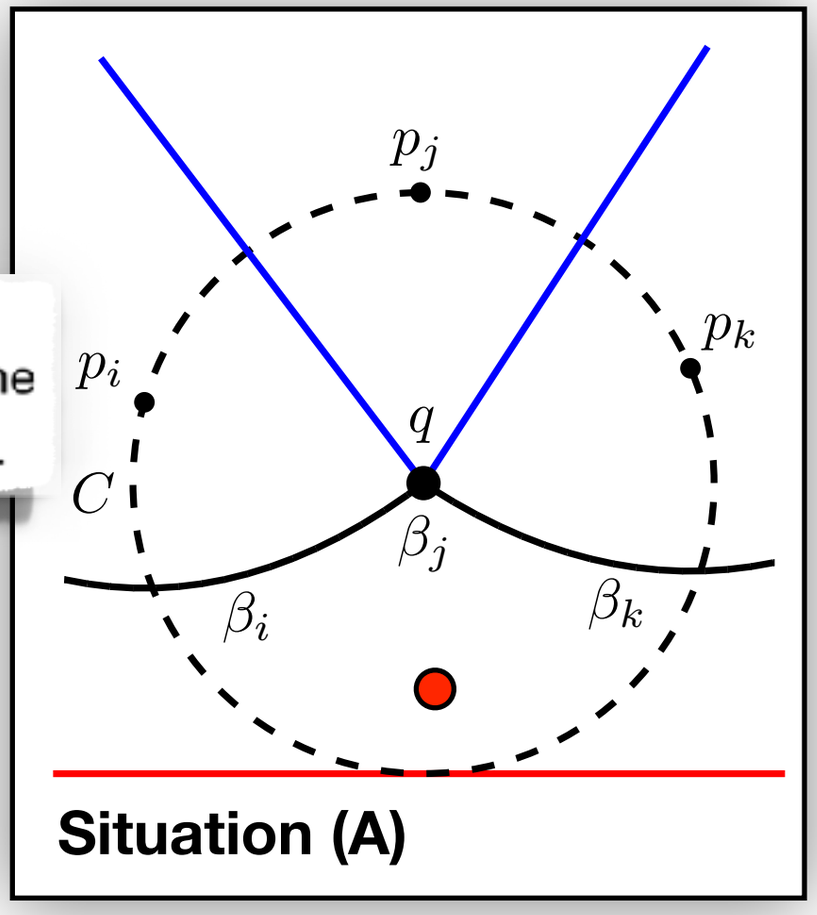

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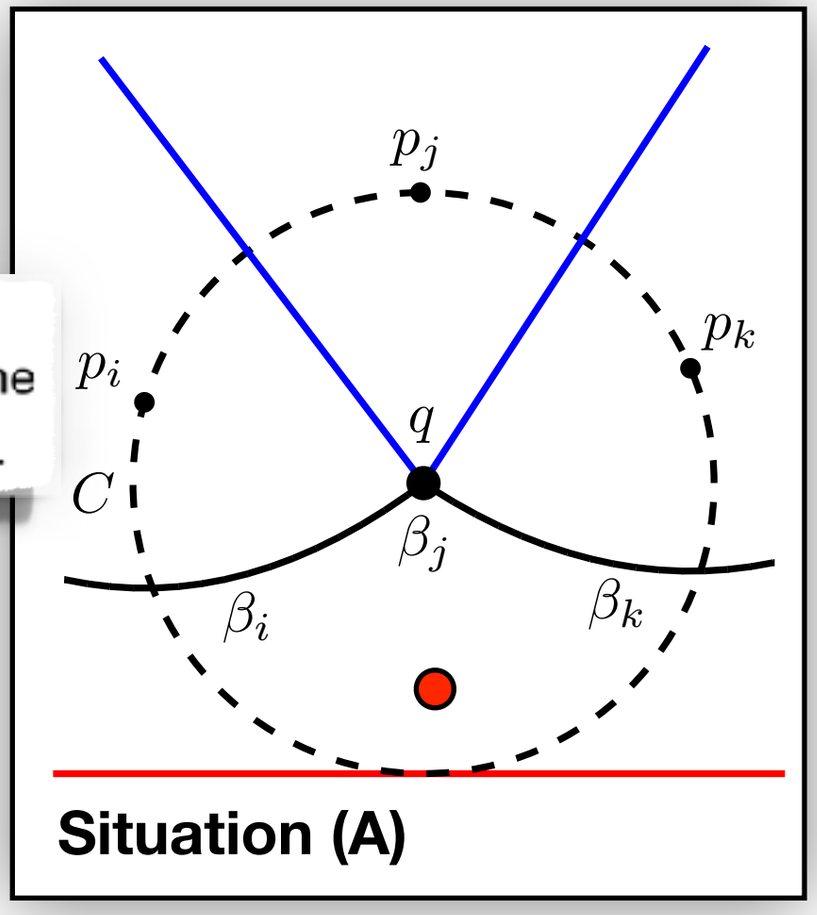

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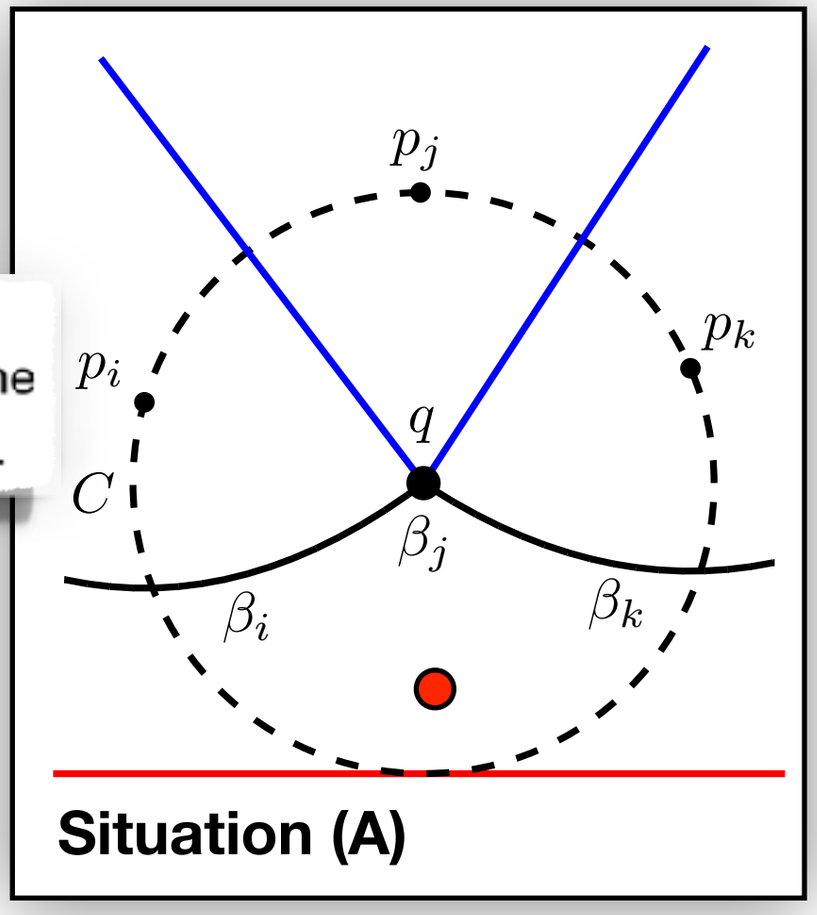

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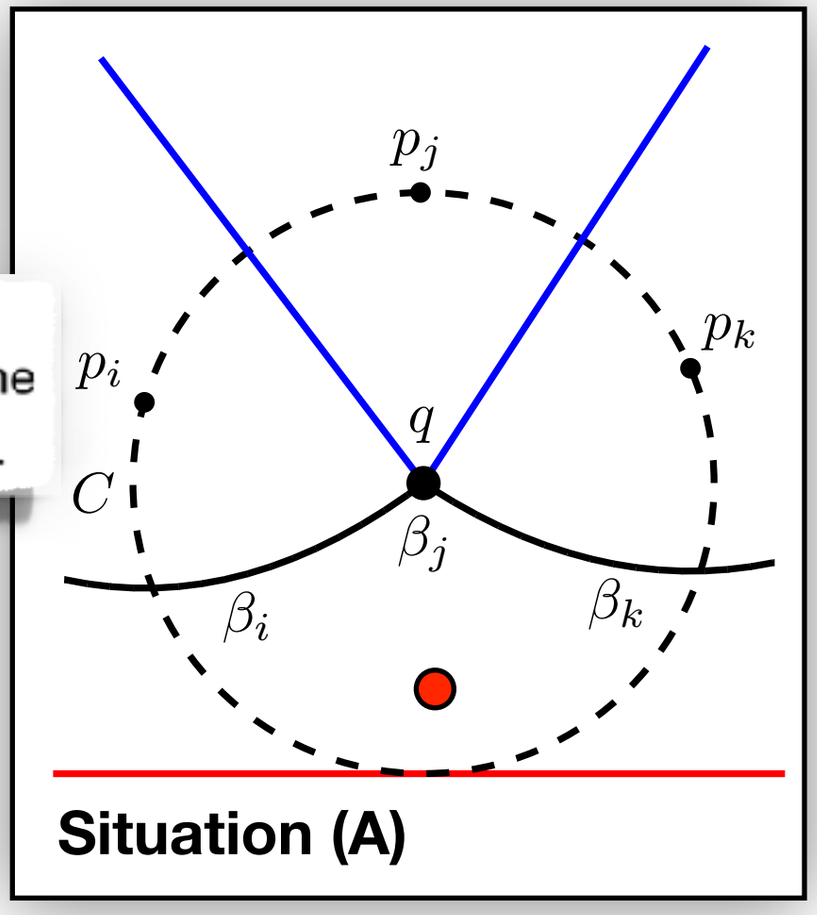

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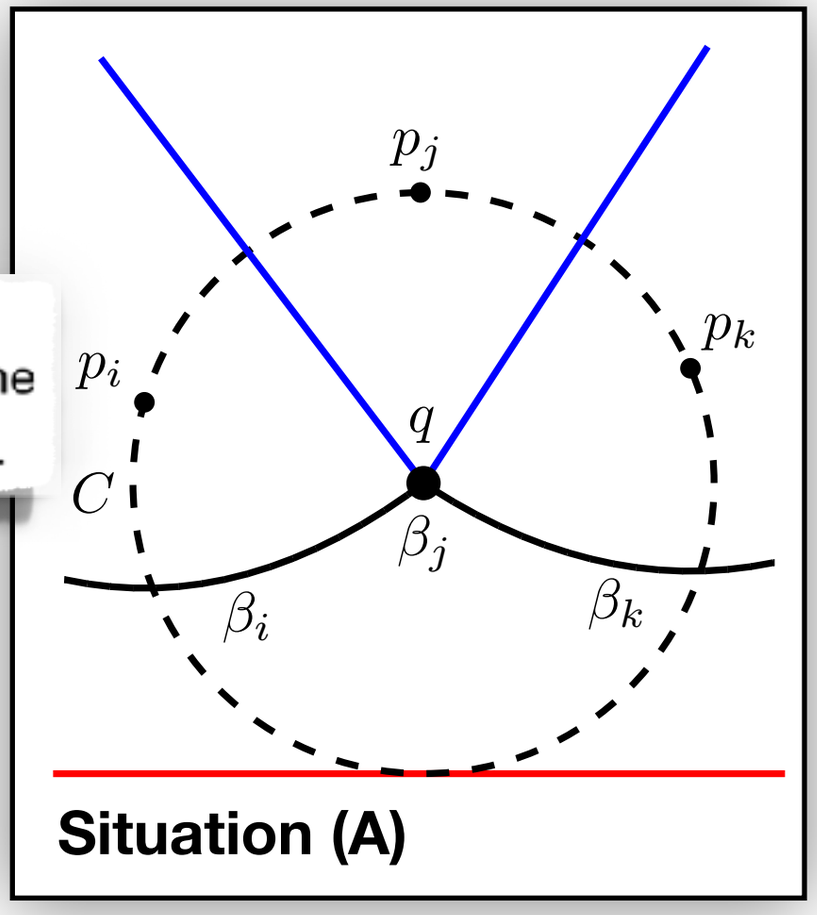

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Properties of the Voronoi Diagram

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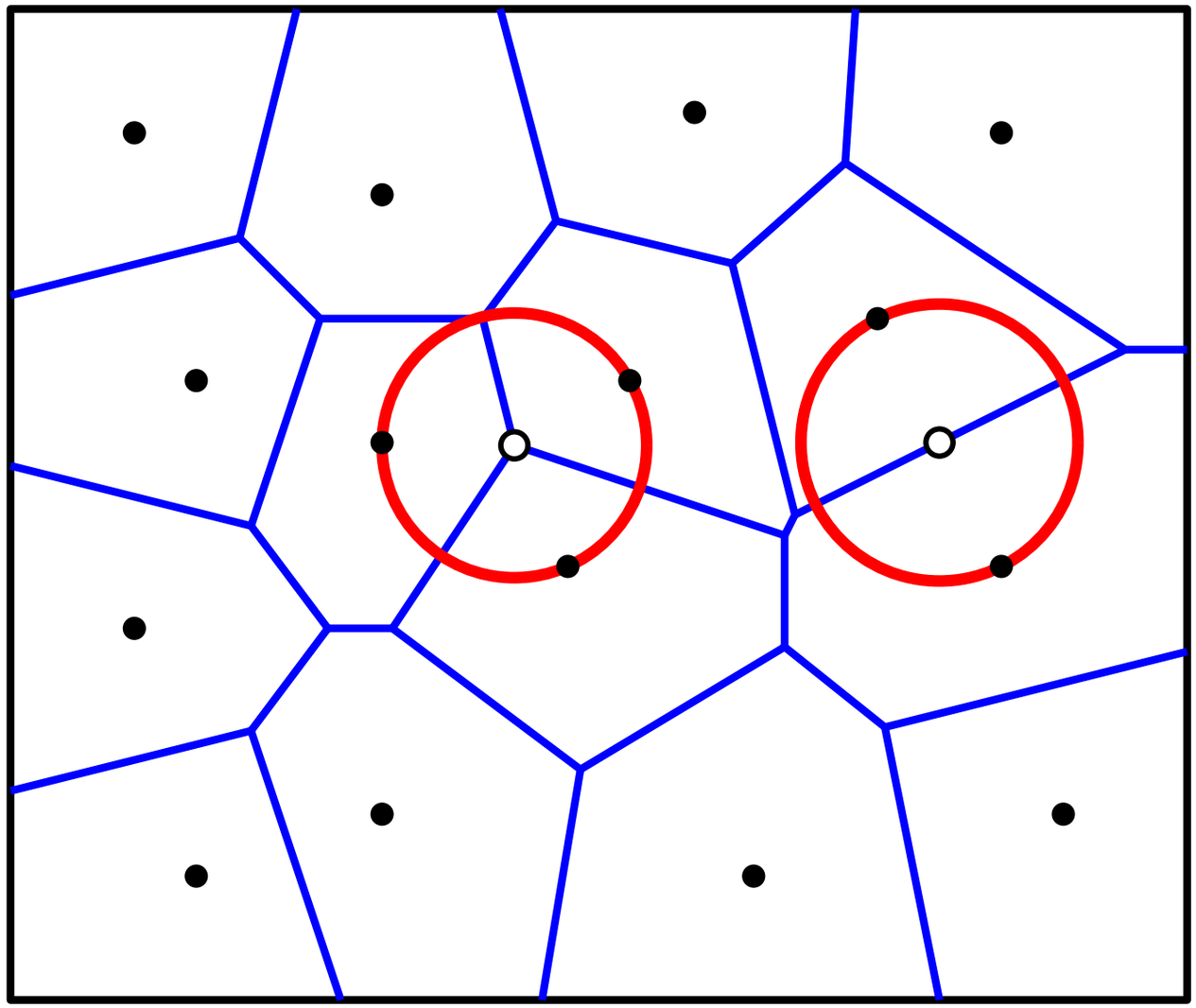

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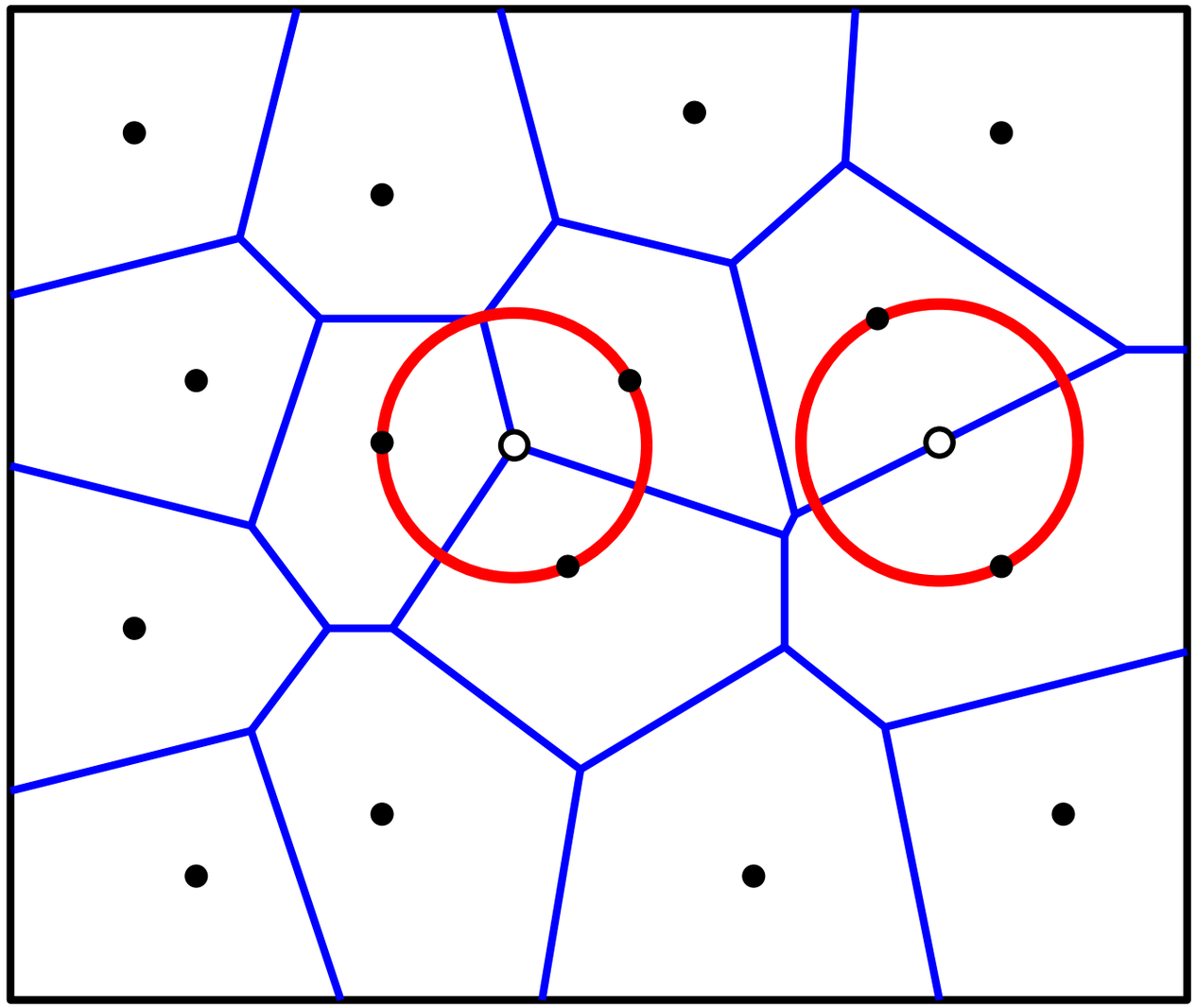

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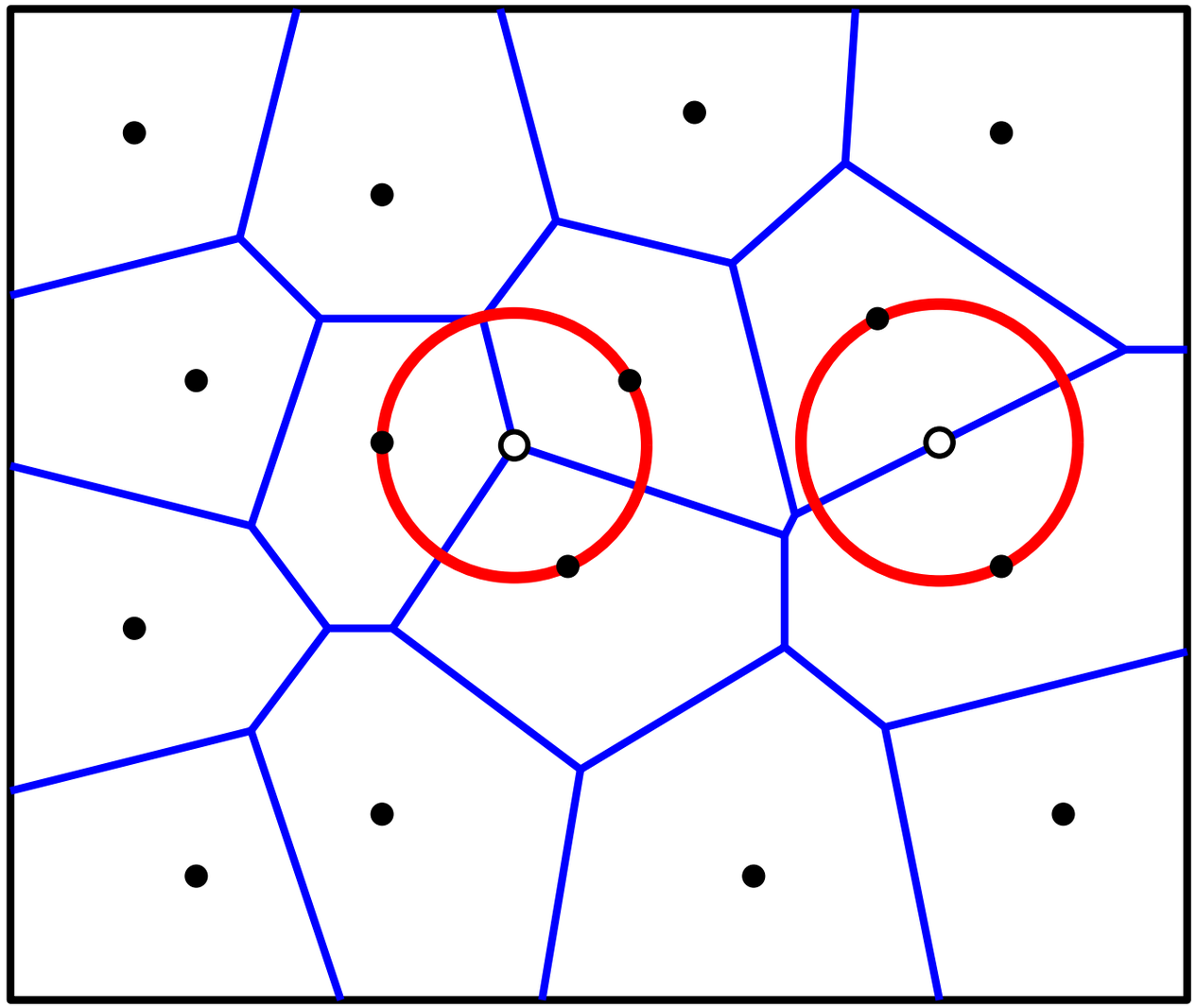

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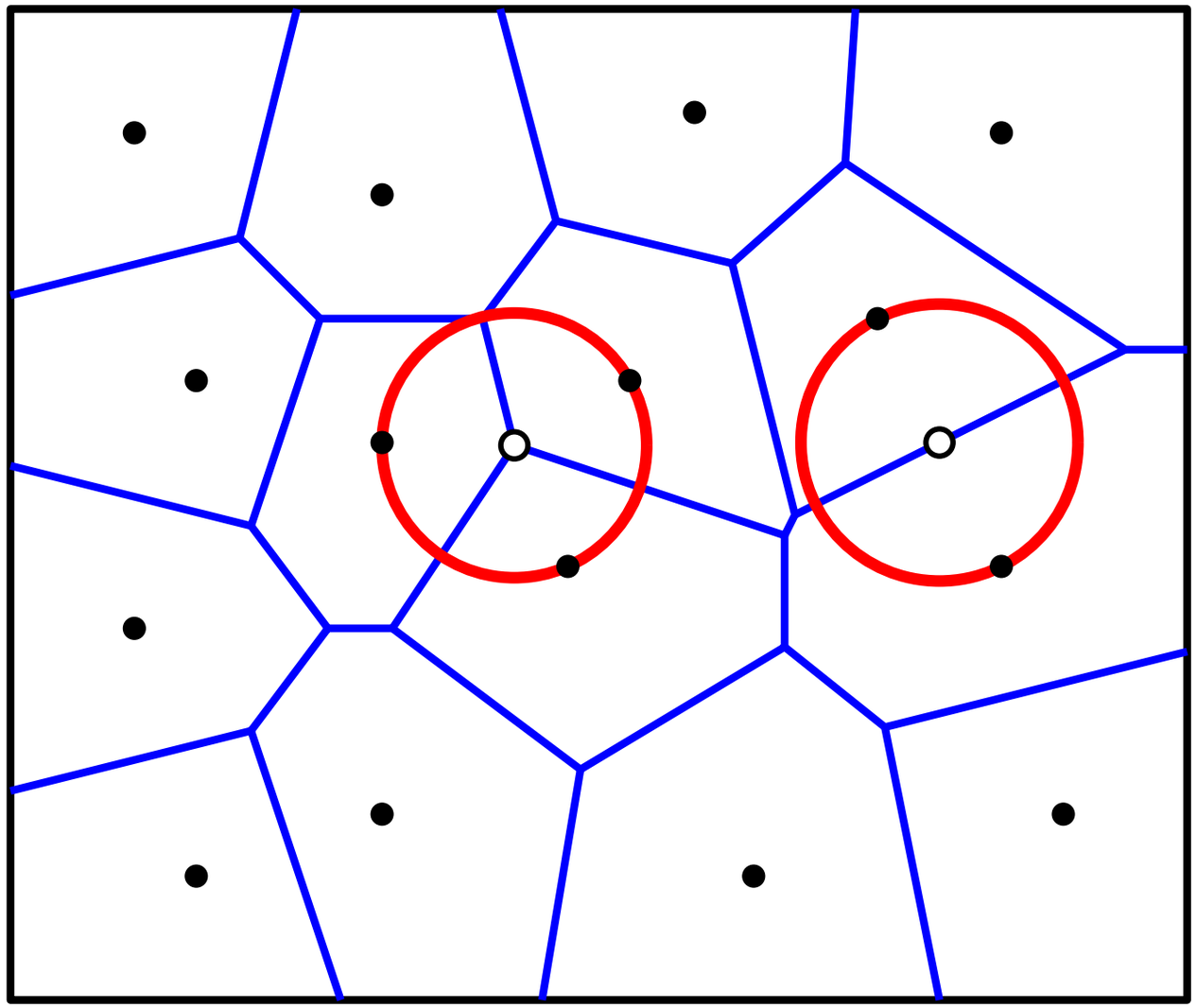

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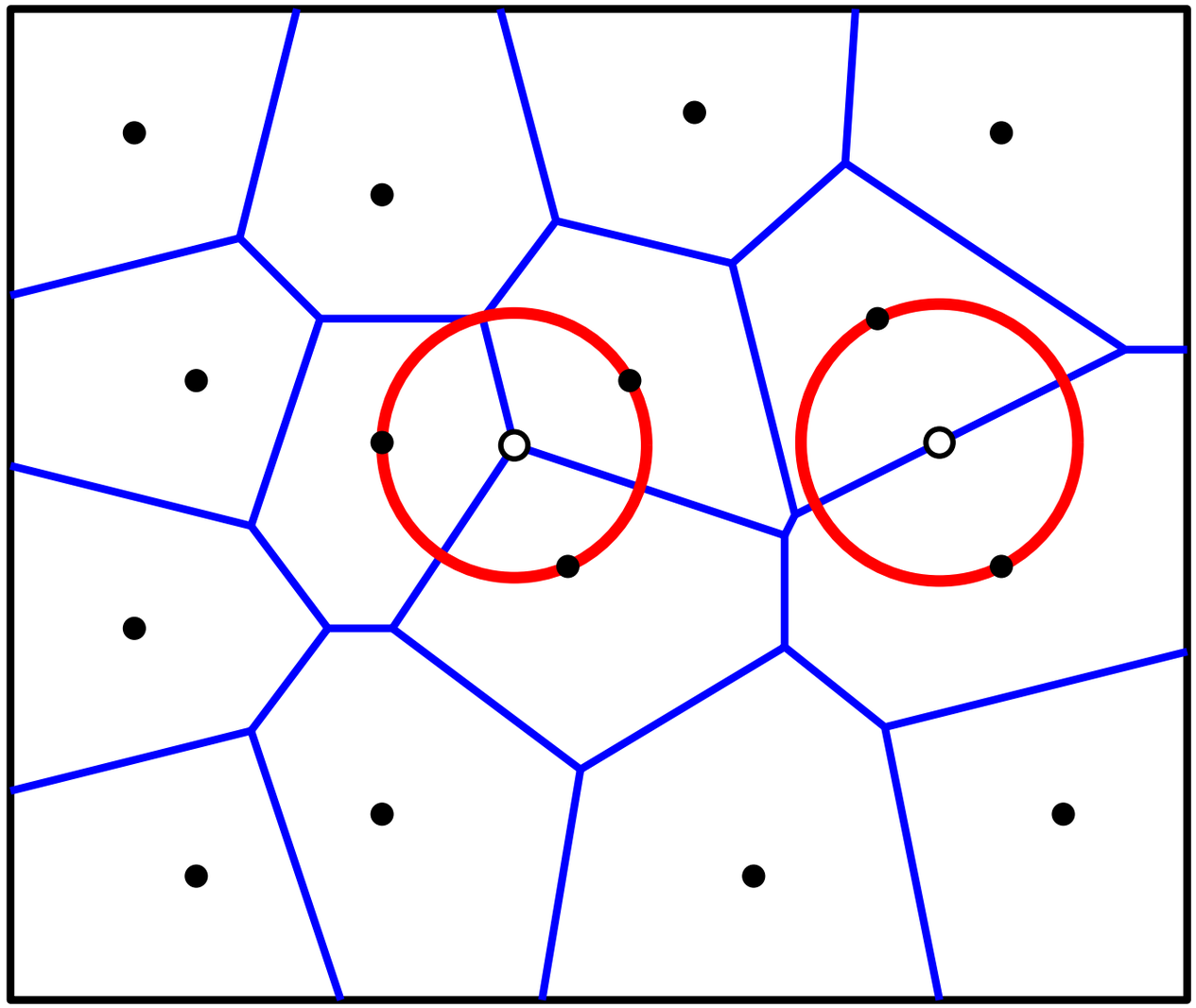

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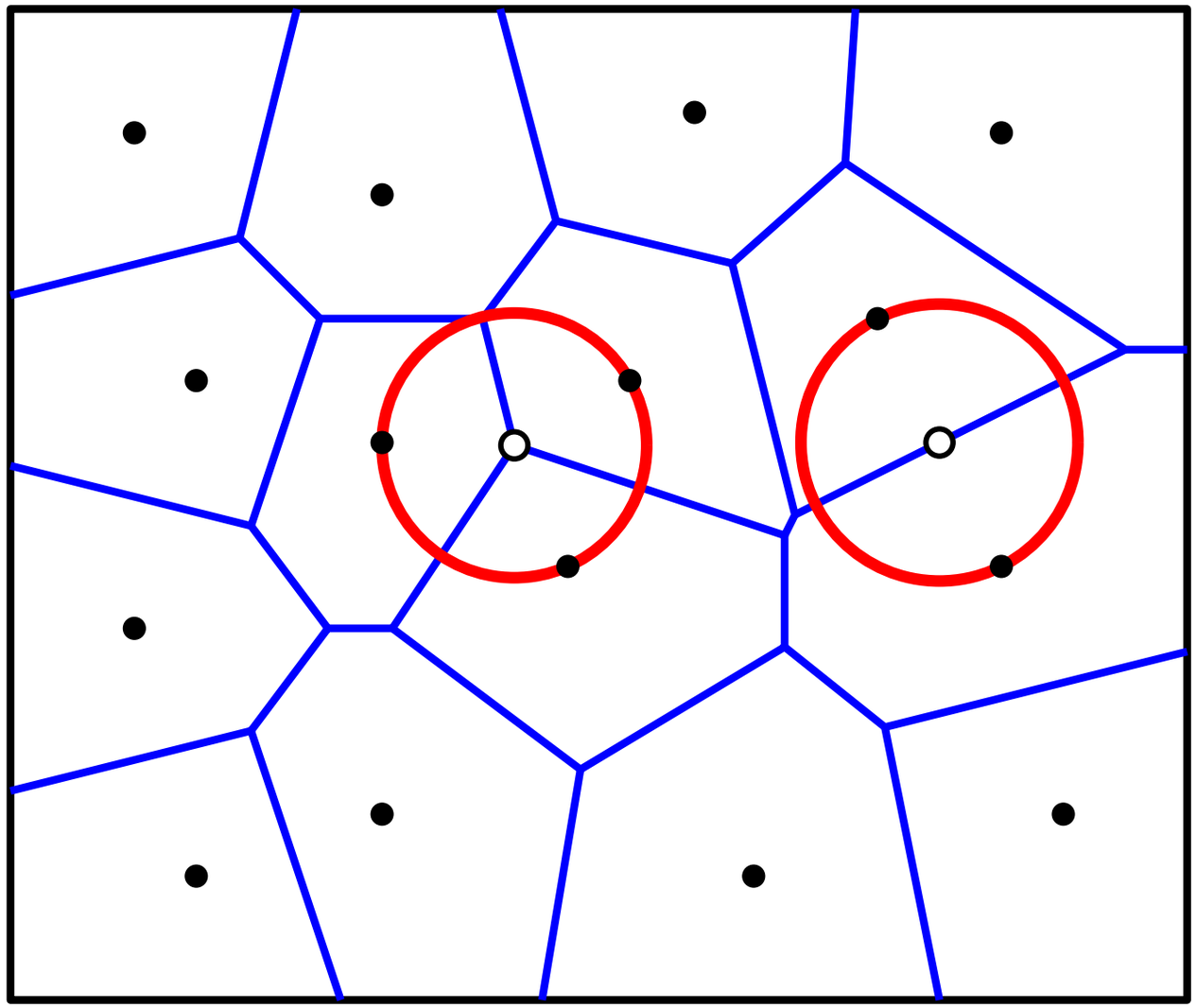

Proof:

Straightforward, because nearest neighbor to center lies on circle.

Fortune’s Algorithm - I

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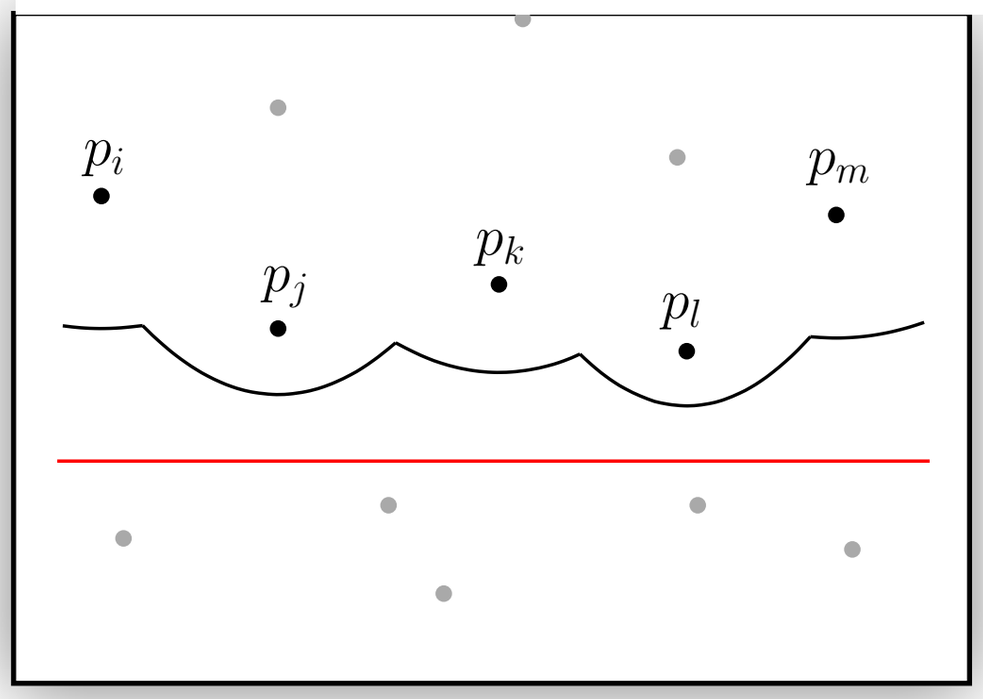

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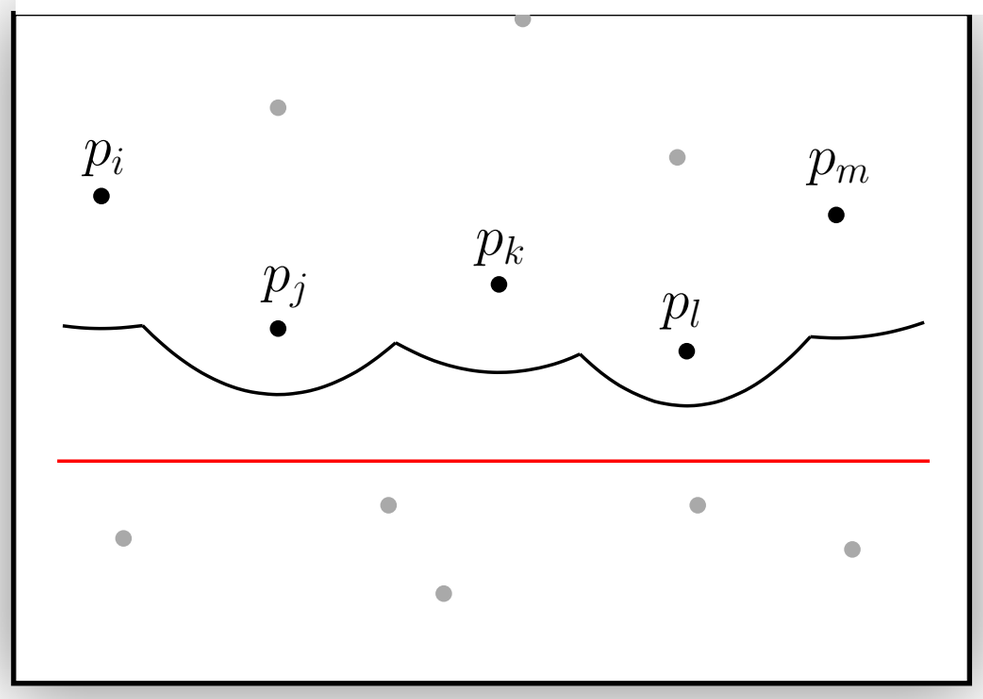

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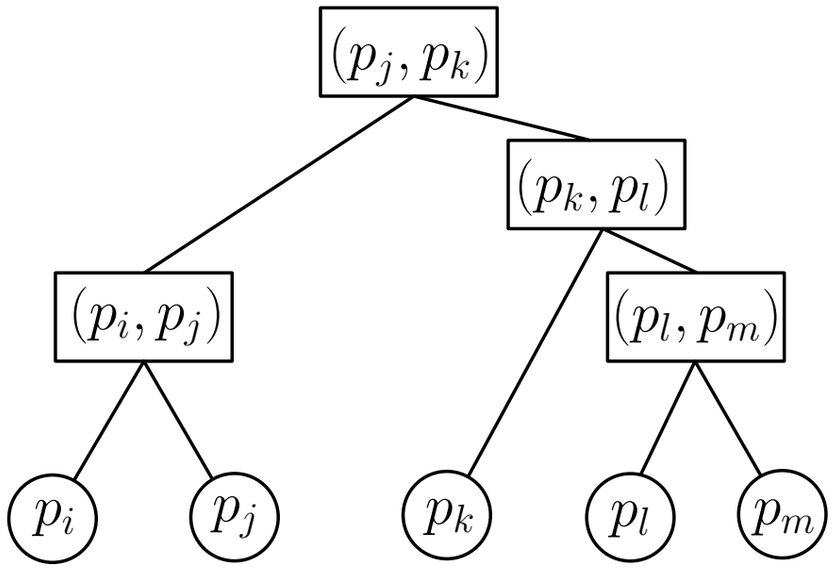

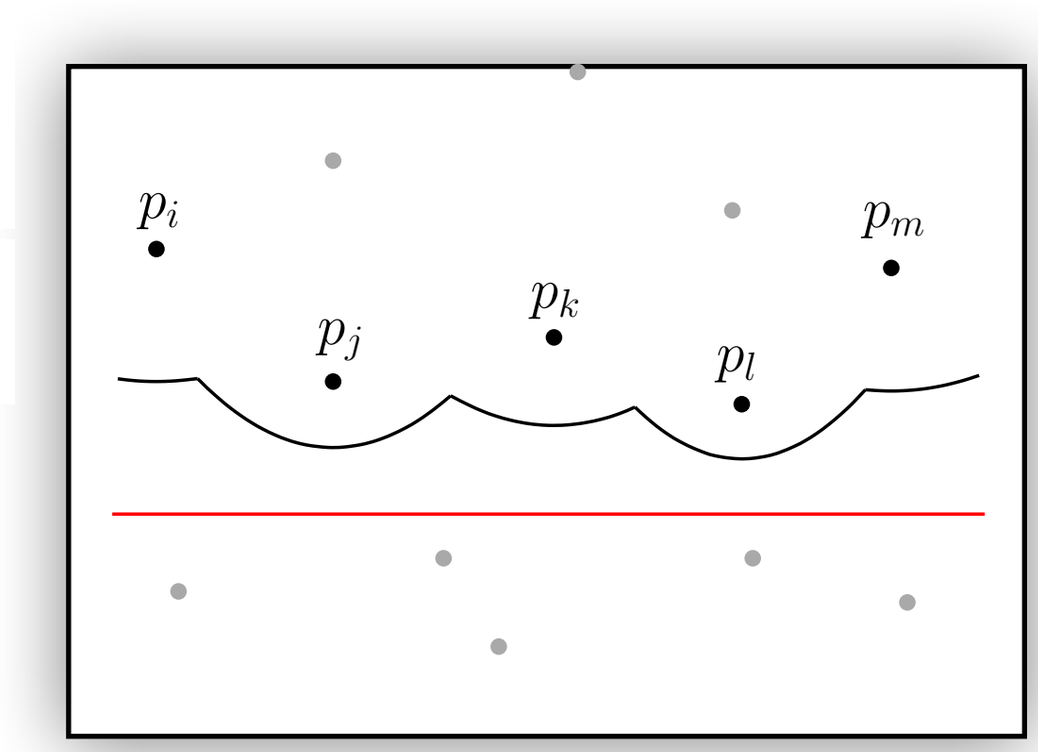

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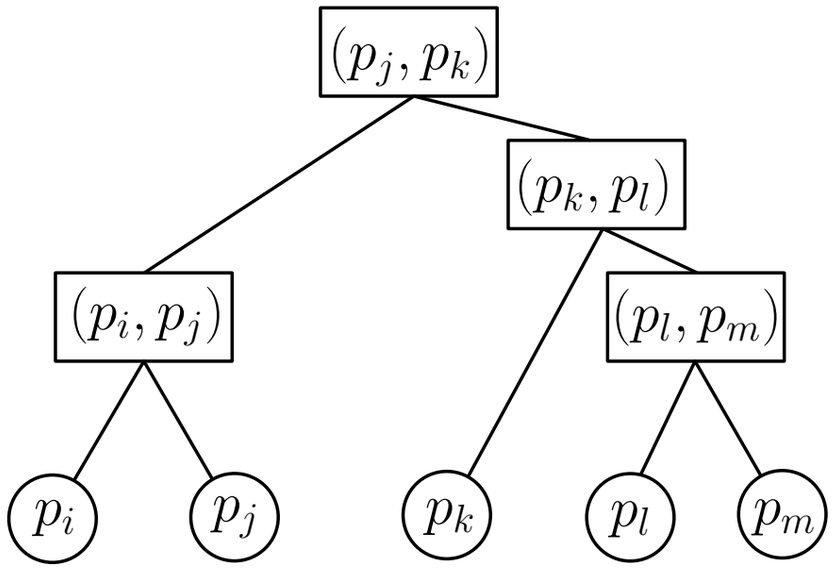

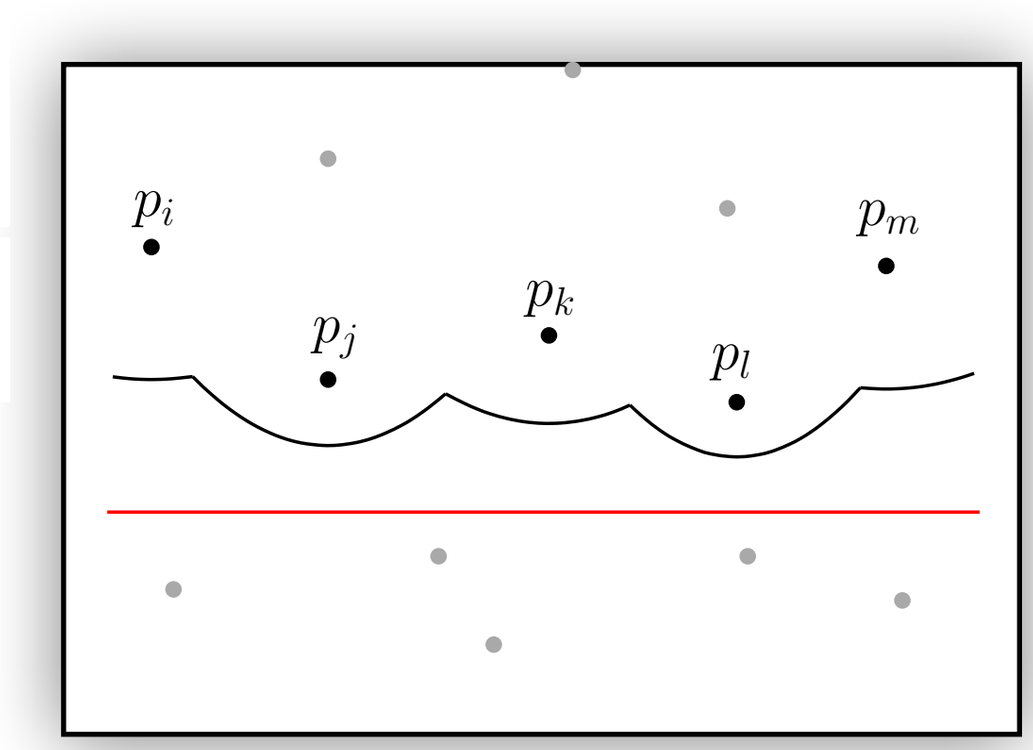

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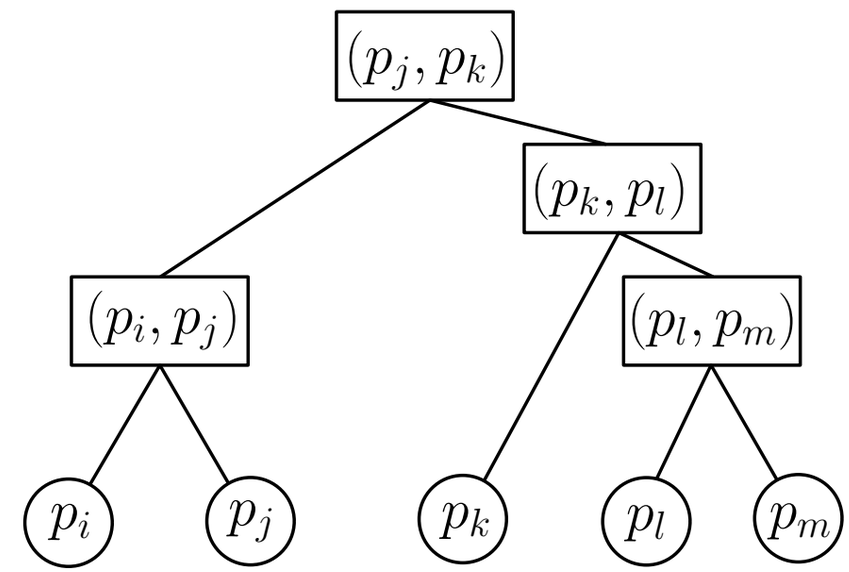

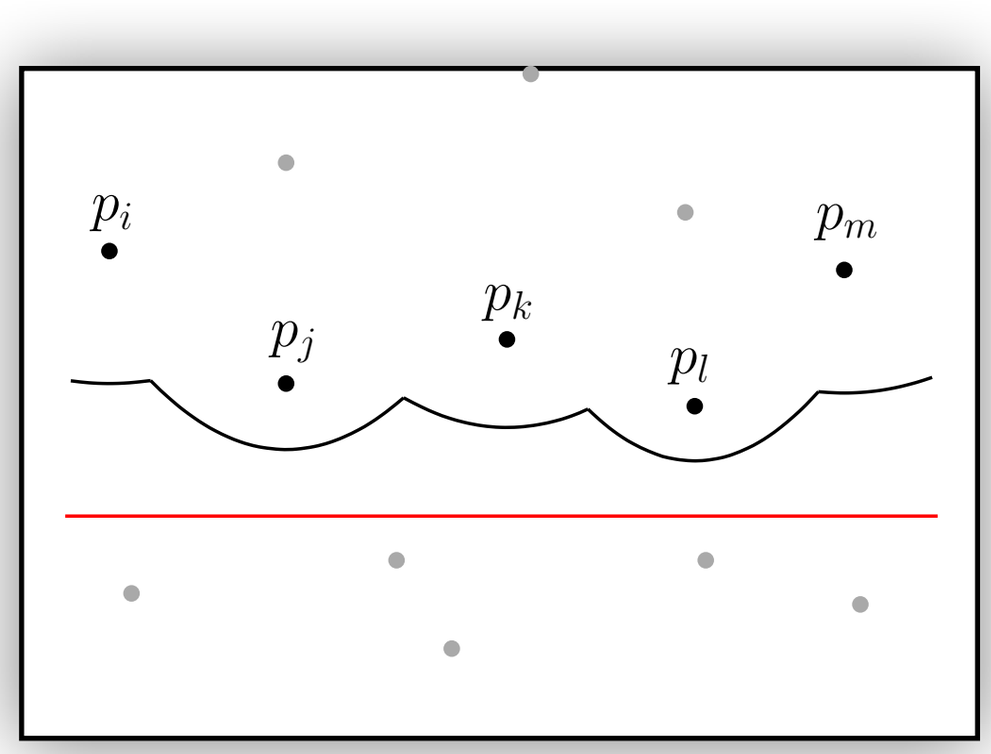

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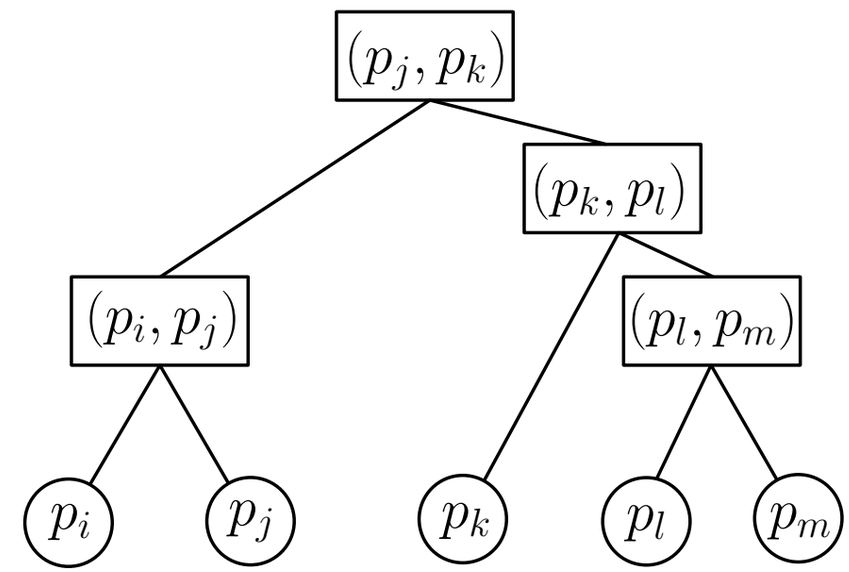

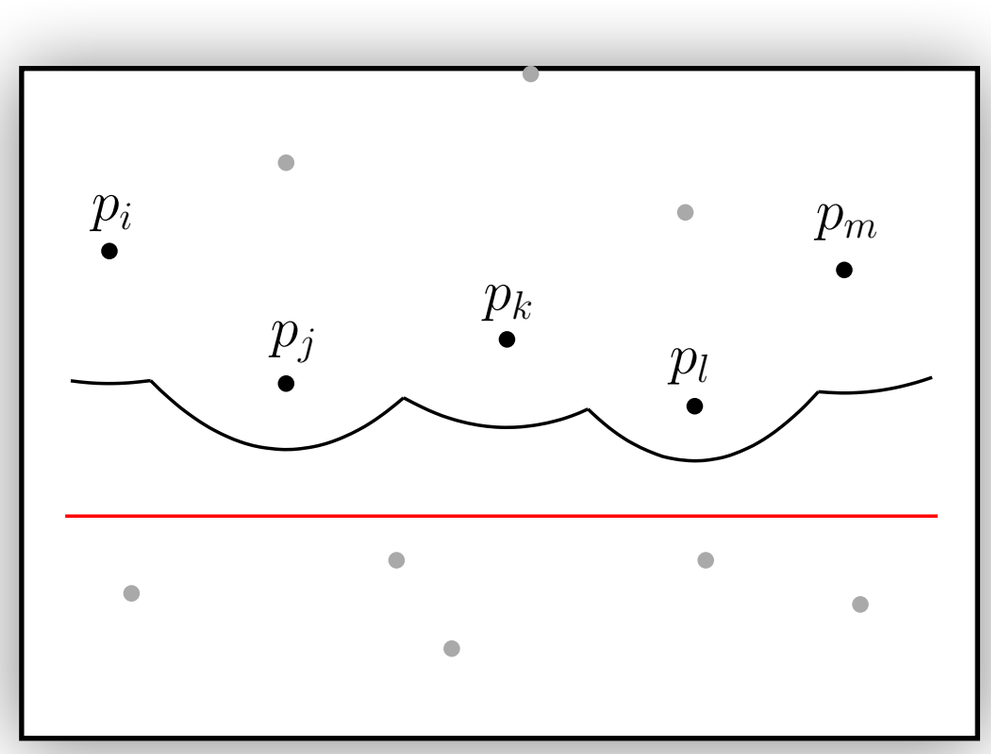

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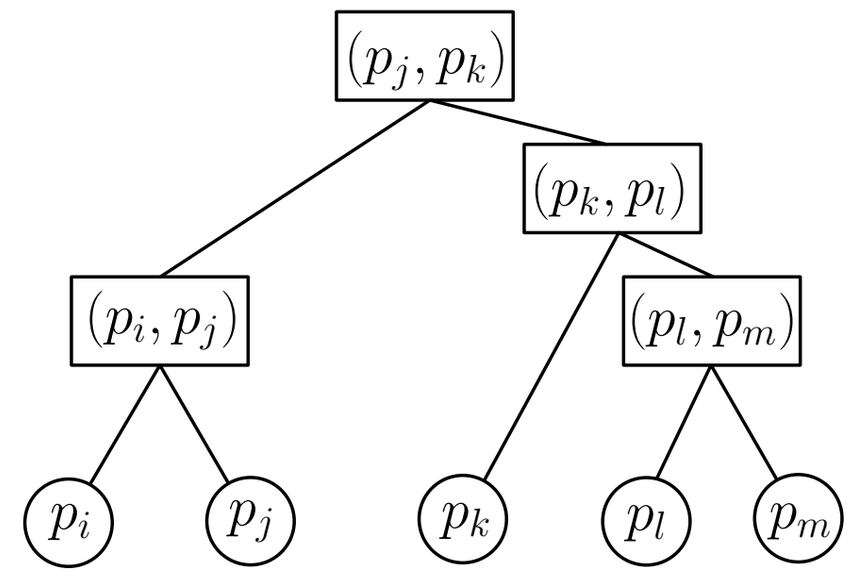

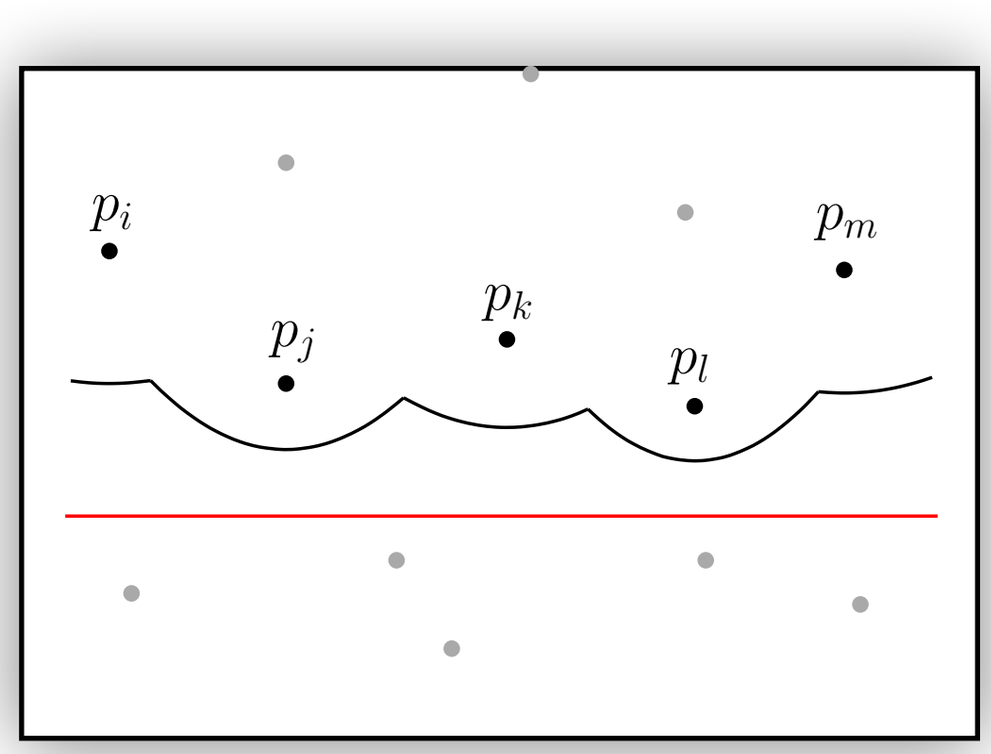

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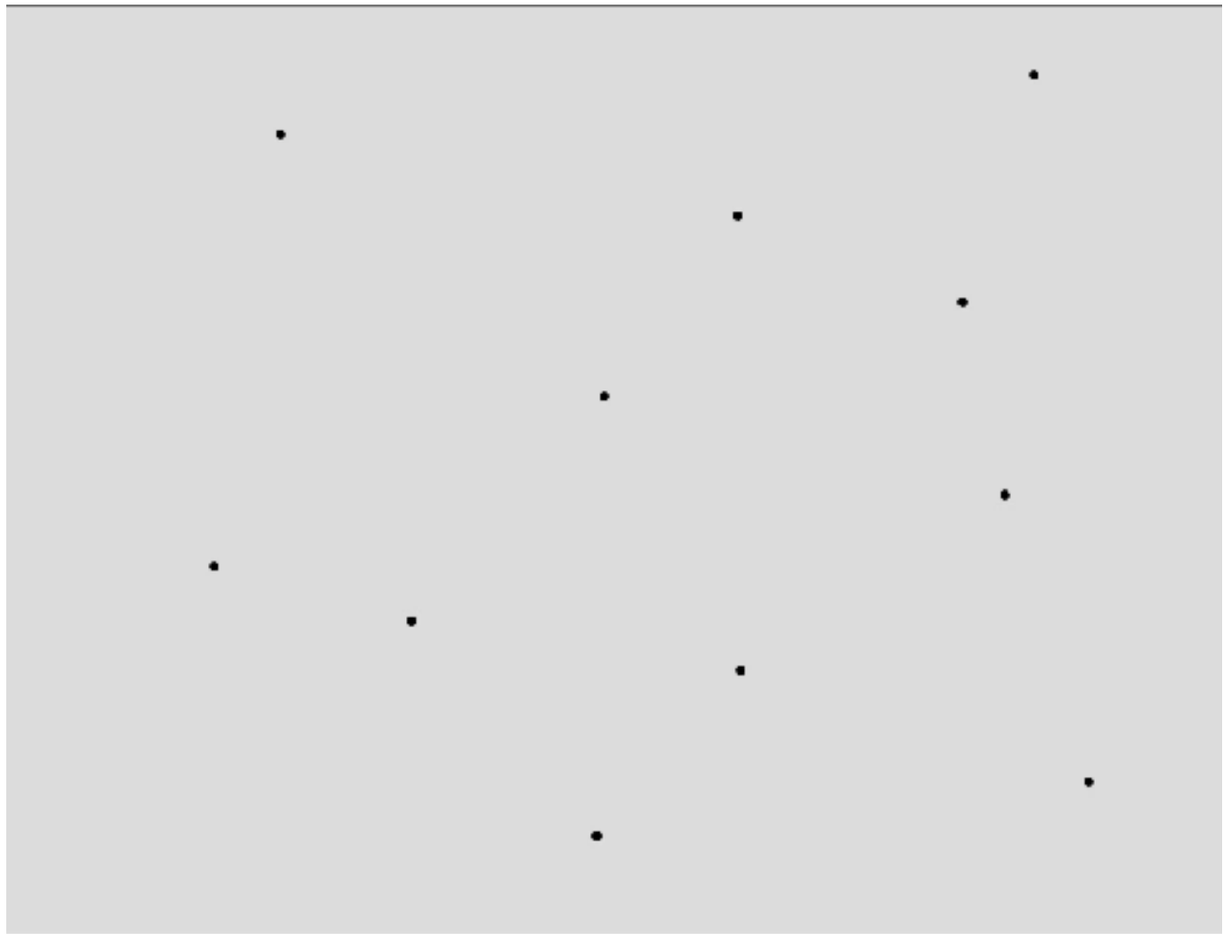

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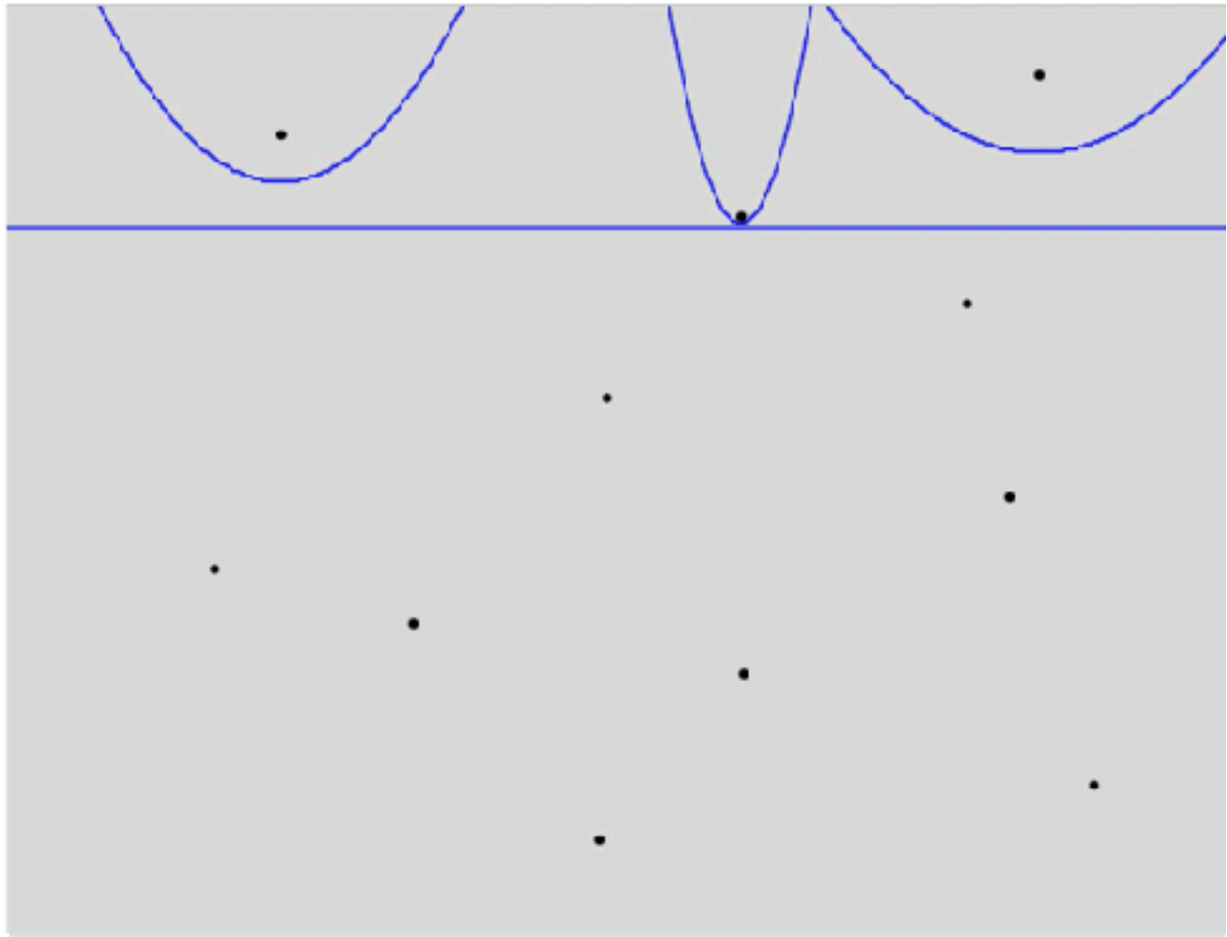

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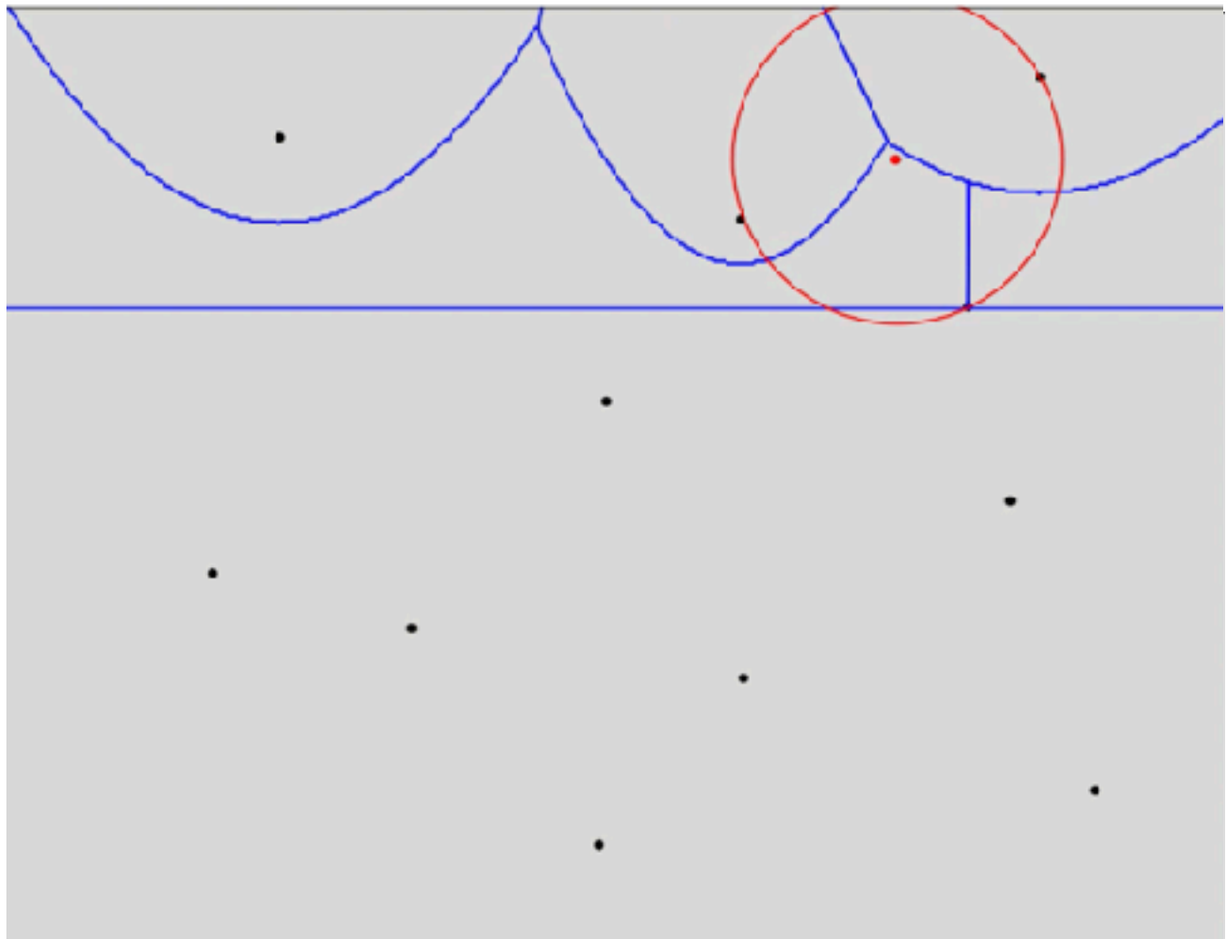

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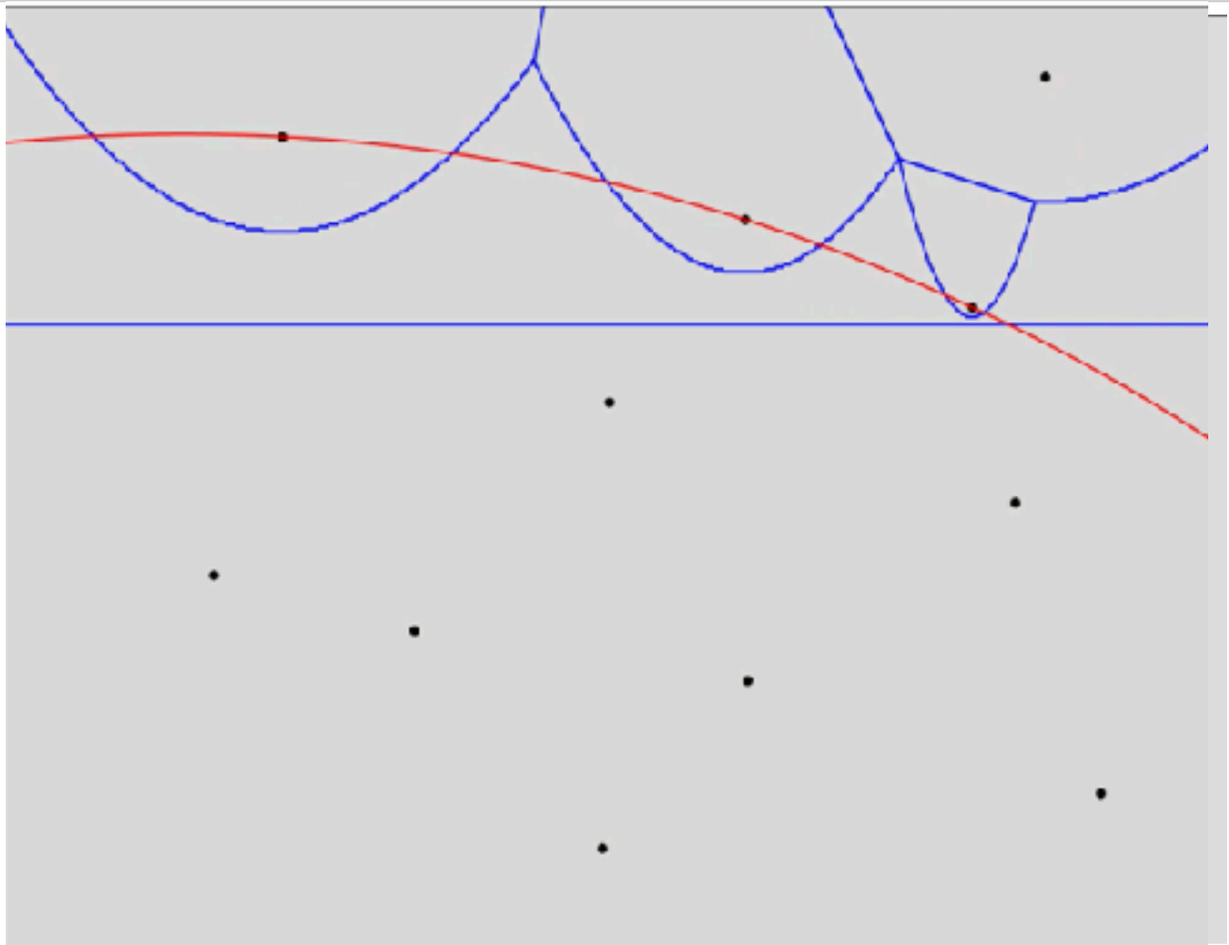

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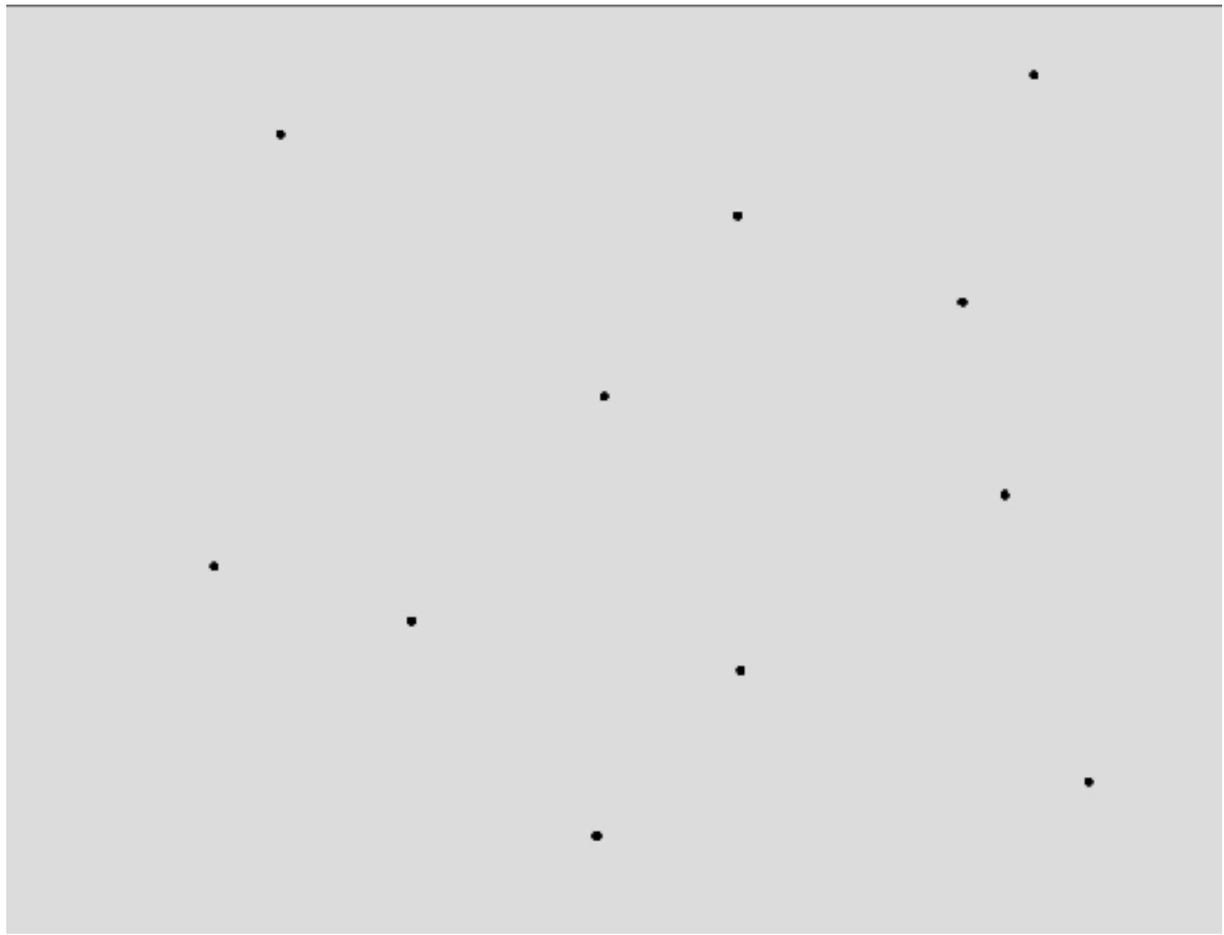

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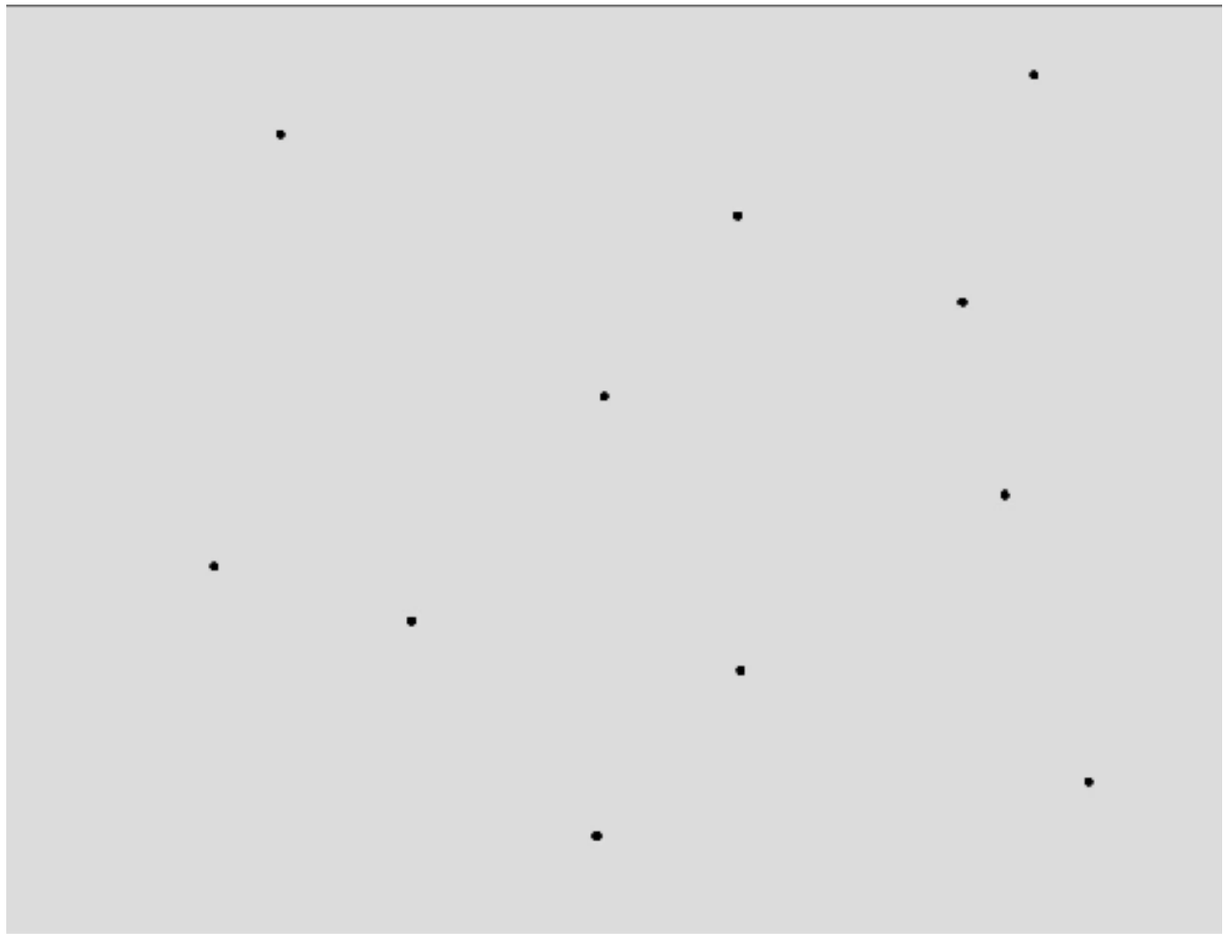

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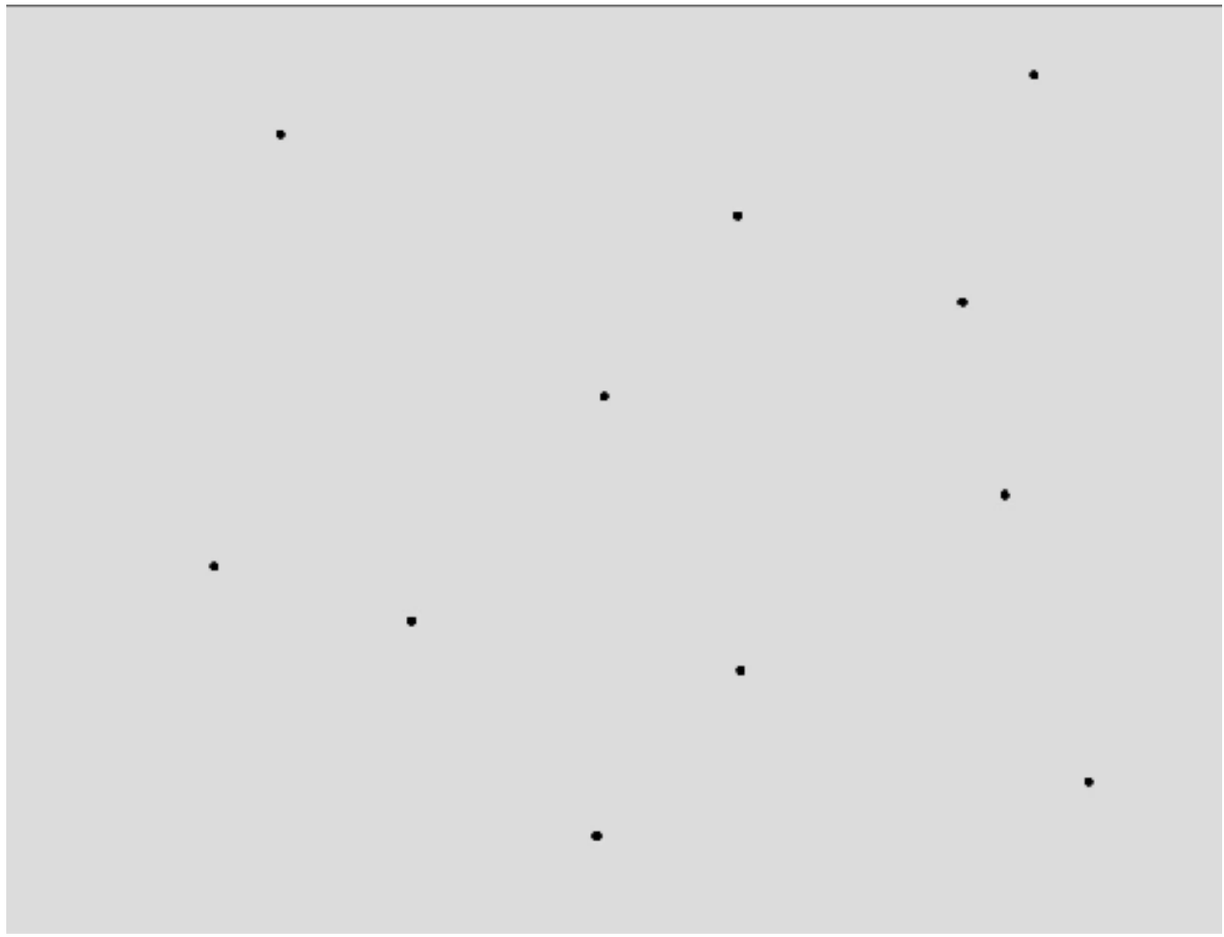

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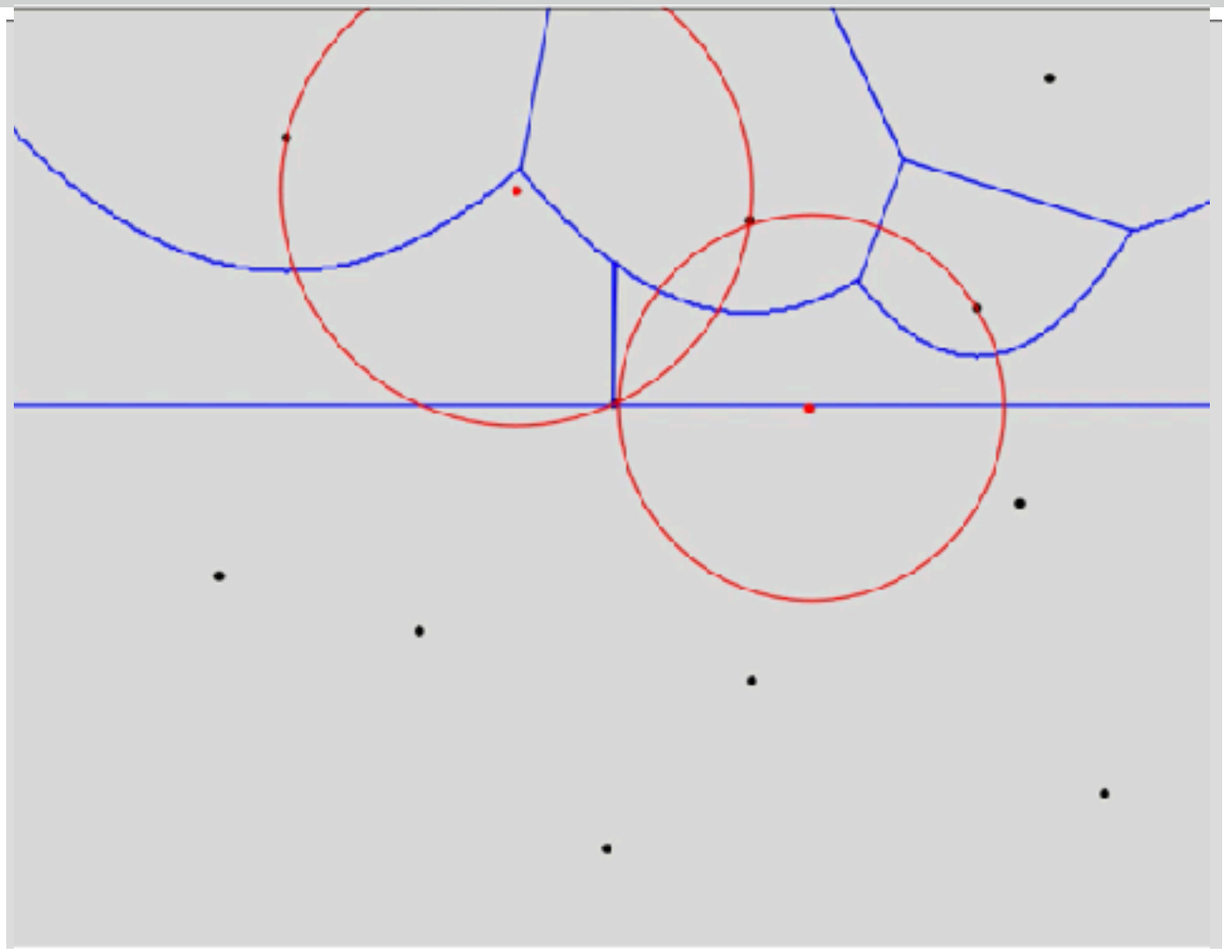

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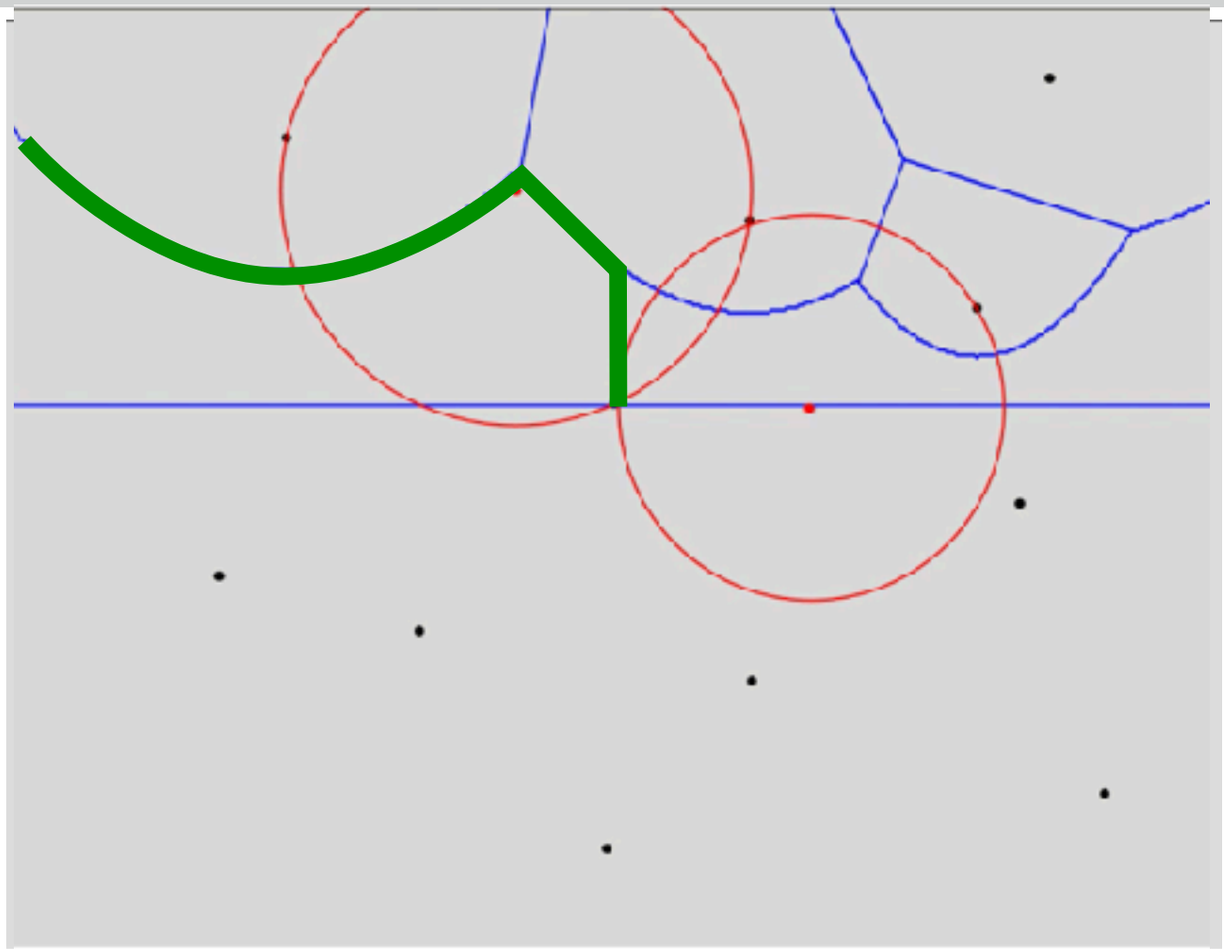

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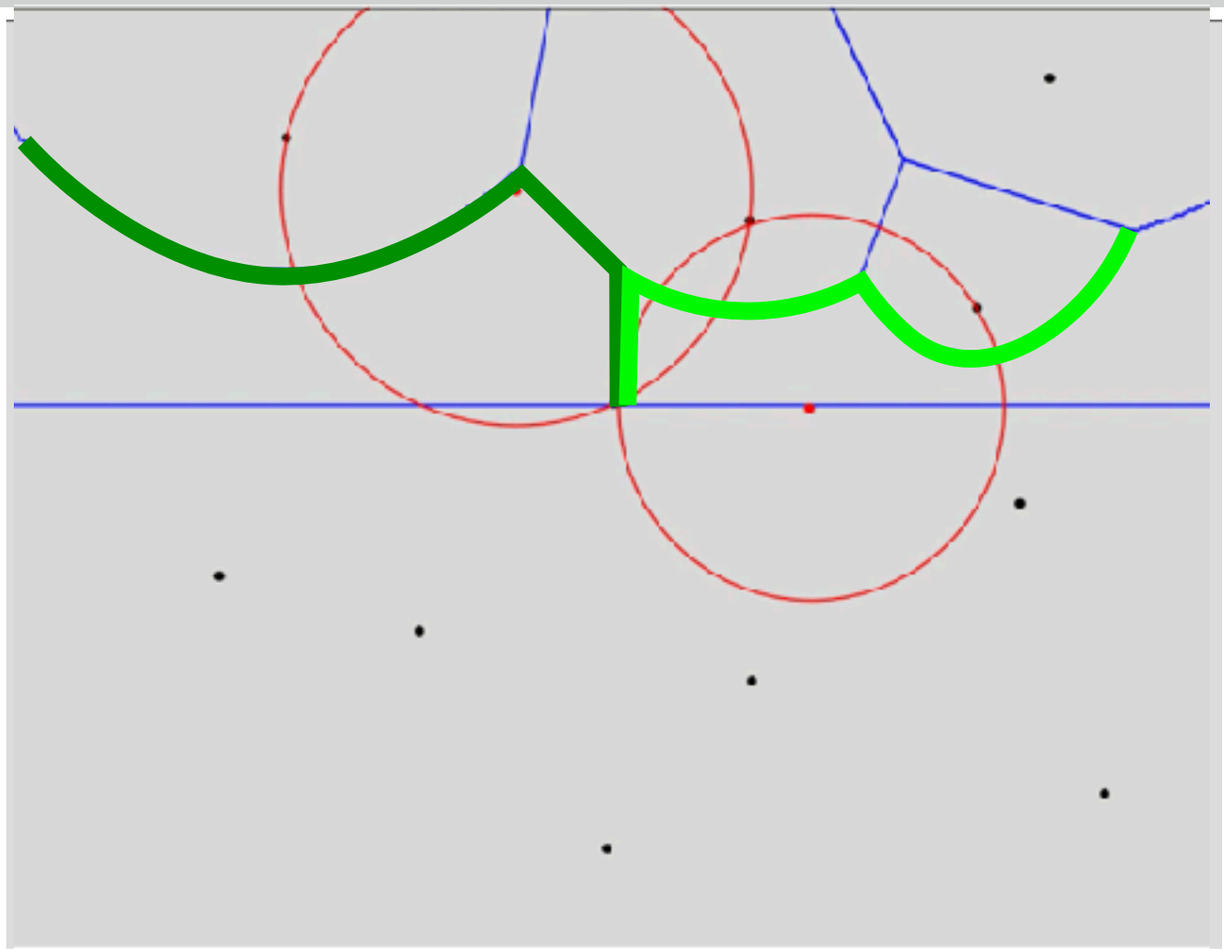

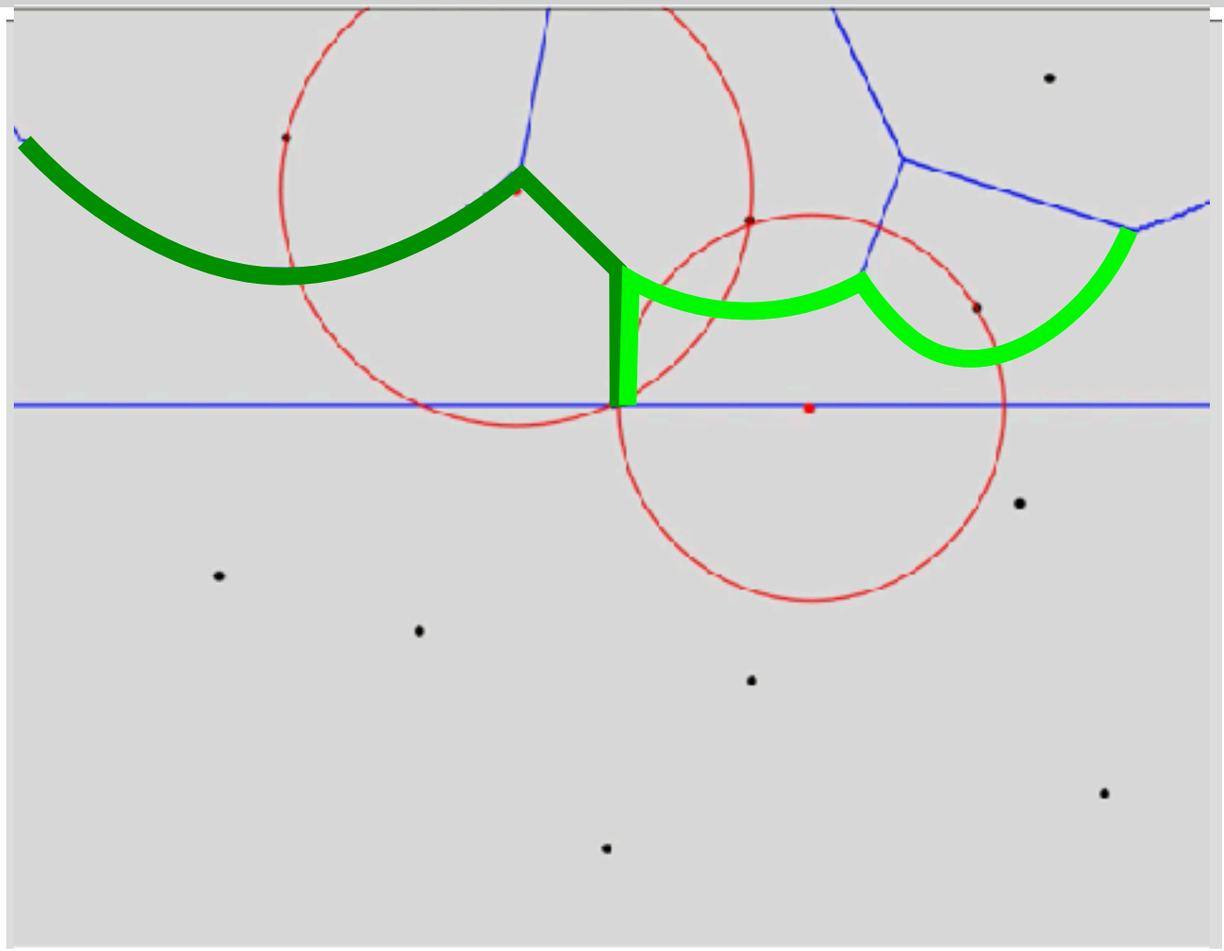

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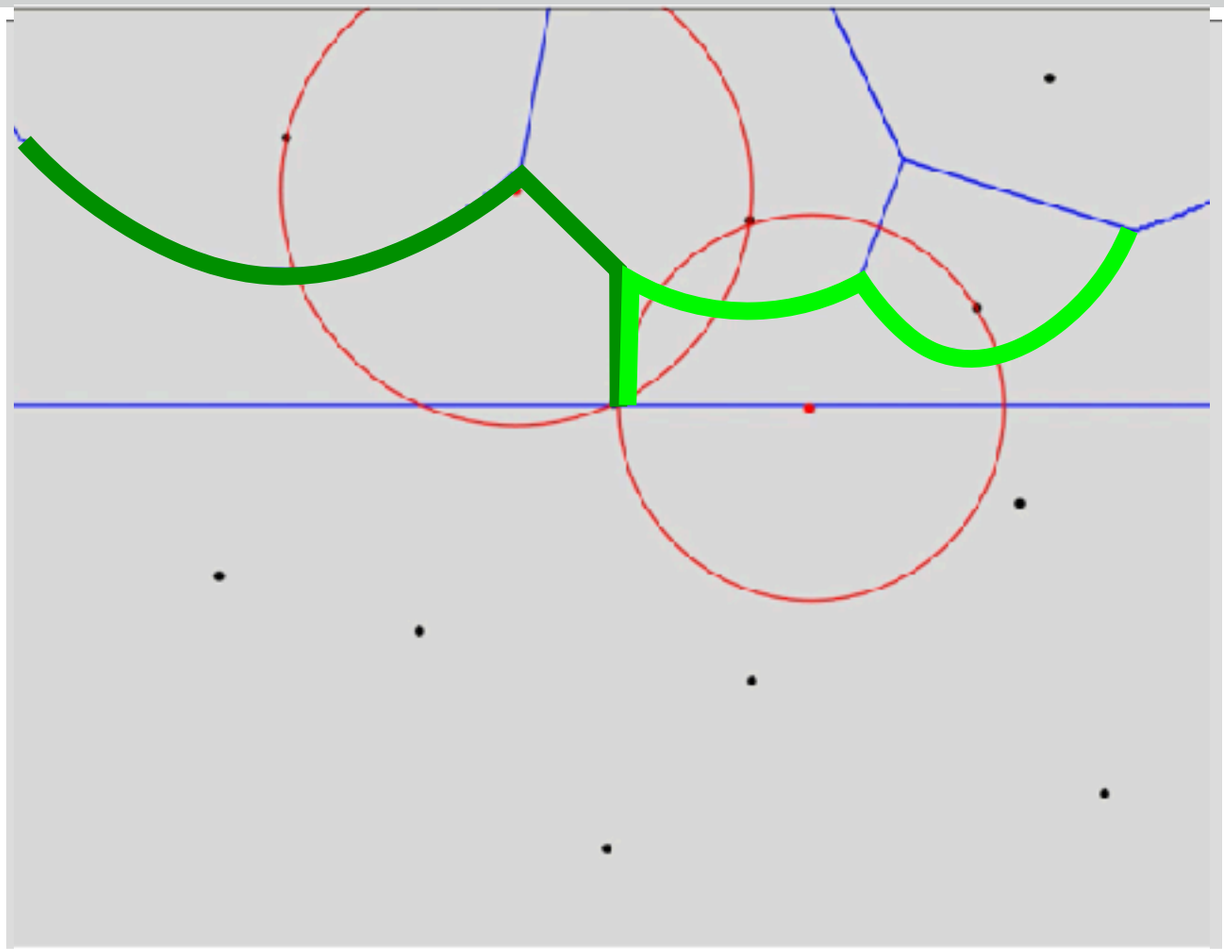

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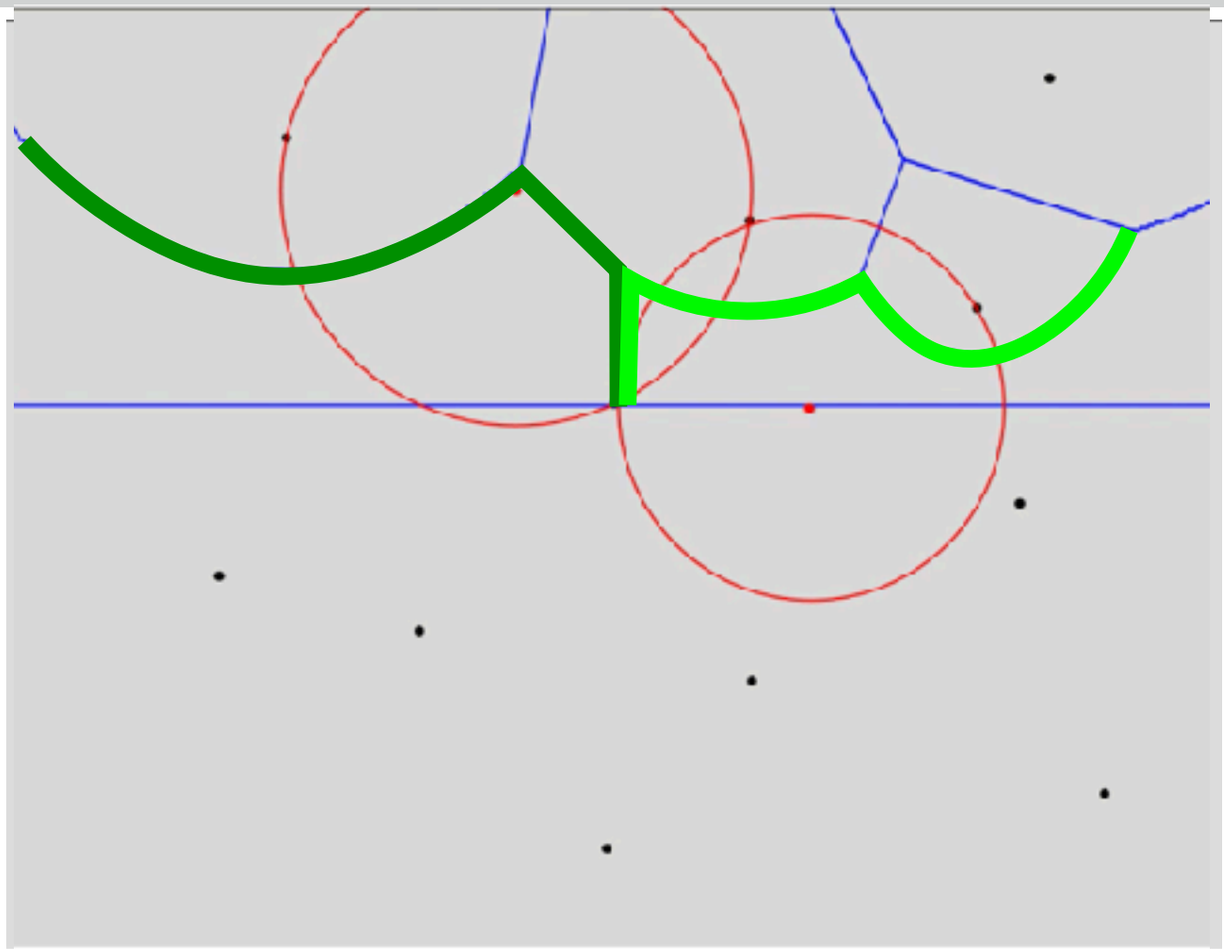
  Point events $p_i \rightarrow$ Priority by $p_i.y$.



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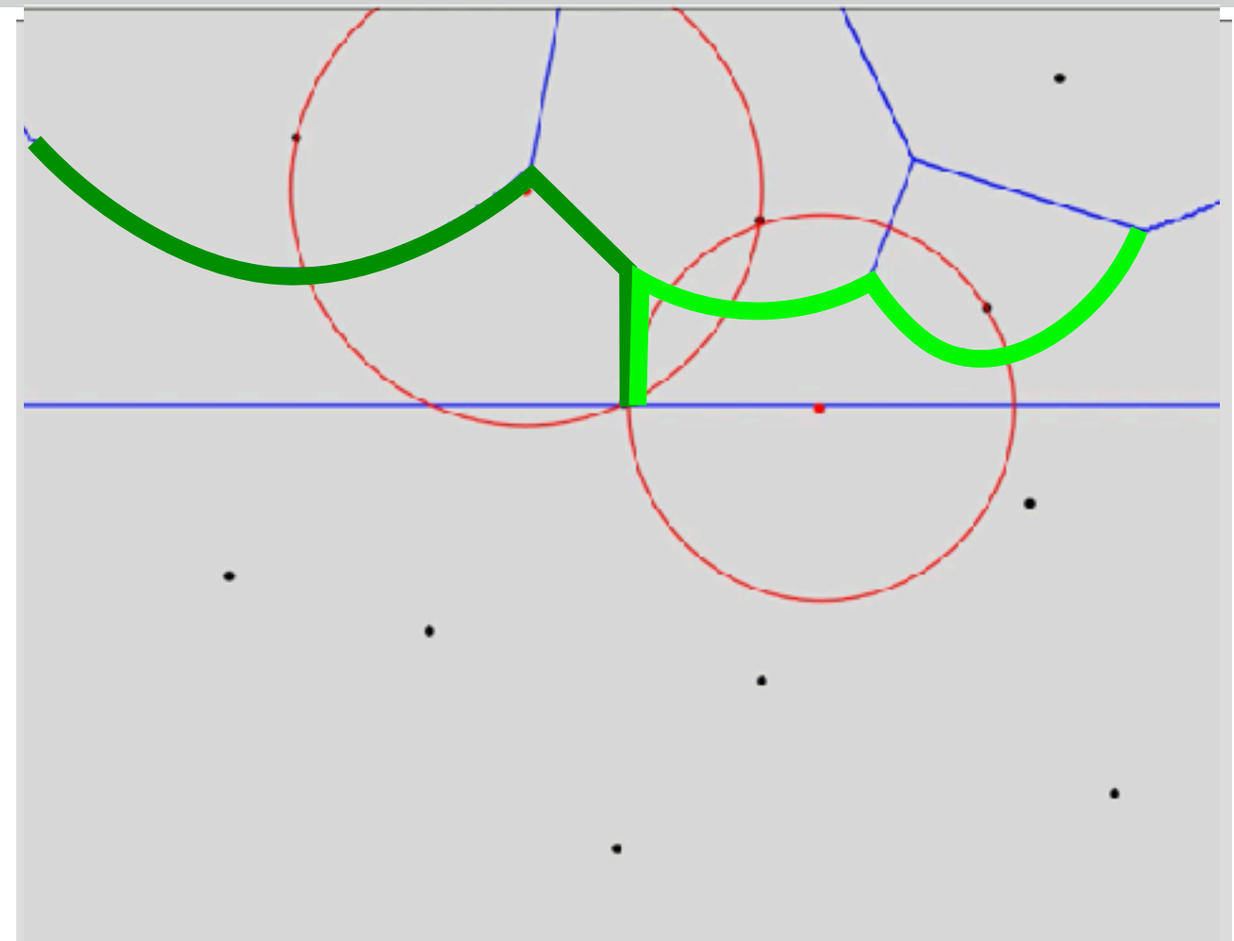
  Circle events C .



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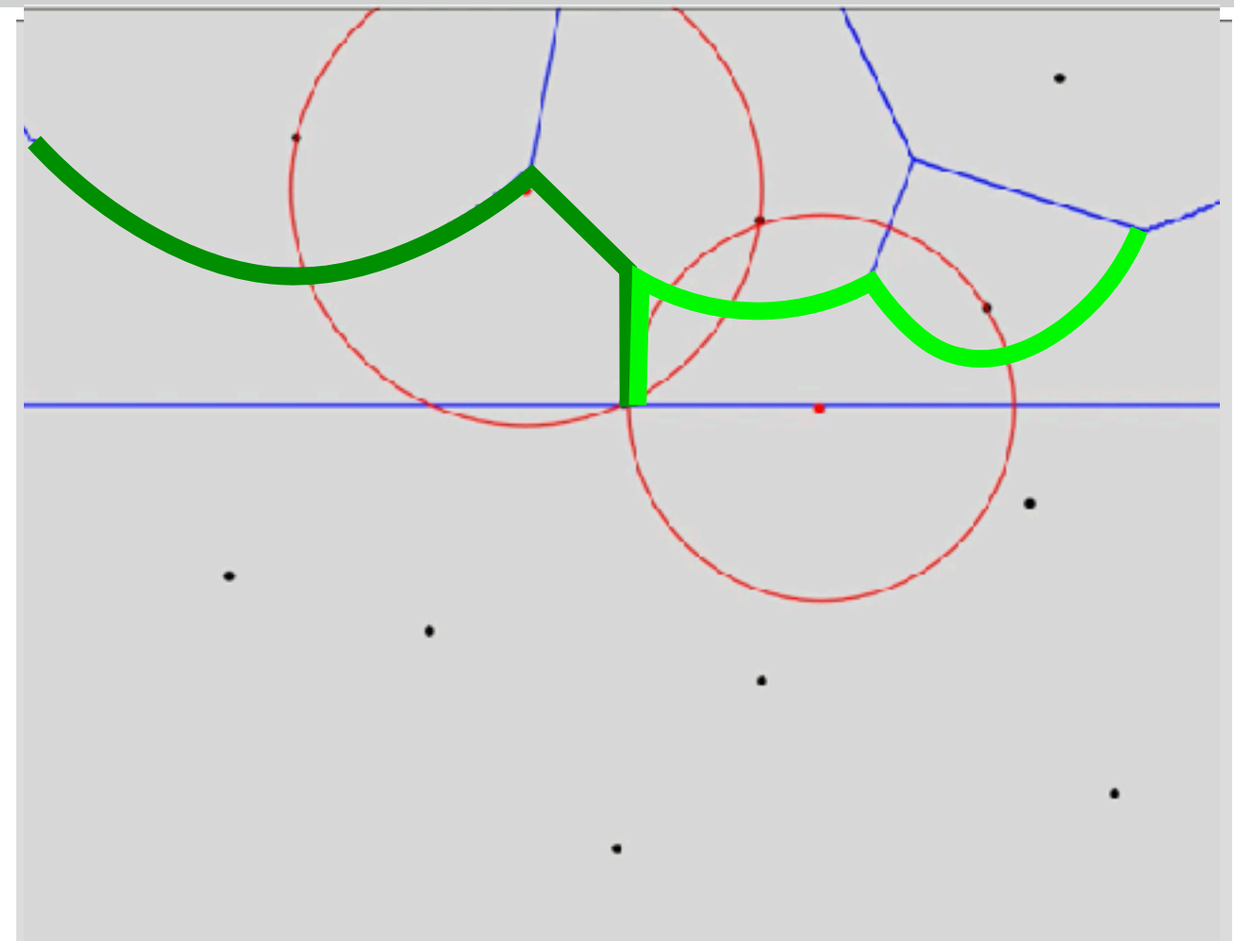

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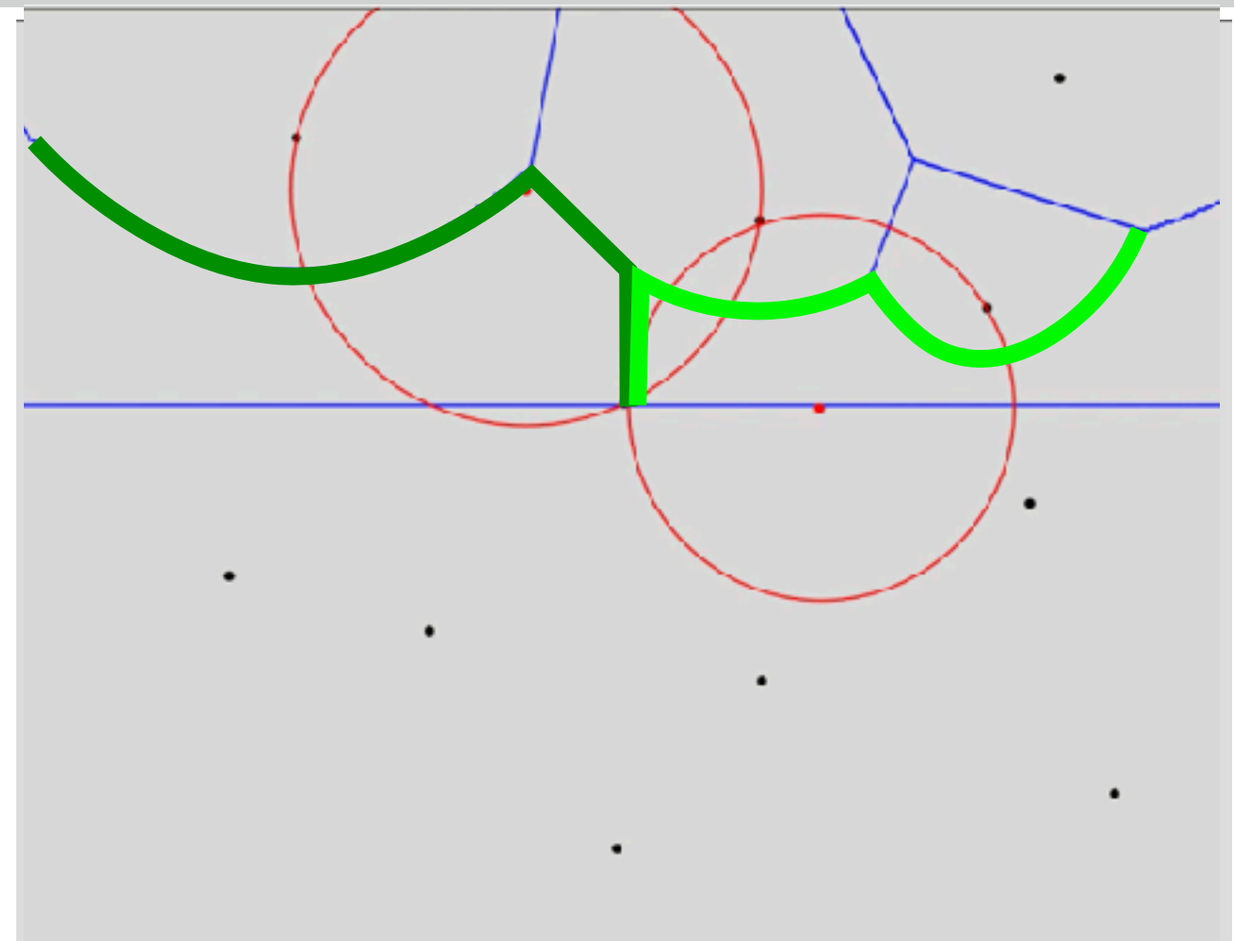
  $\rightarrow$ Pointer to arc (representing leaf $b \in B$) which may disappear.



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- Leaf $b \in B$ points to circle event $C \in Q$ for which arc β of b may disappear.

Validity of Circle Events

Observation 1:

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- Discovering $p \in C^\circ$

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- Discovering $p \in C^\circ \Rightarrow$ Circle event C becomes obsolete

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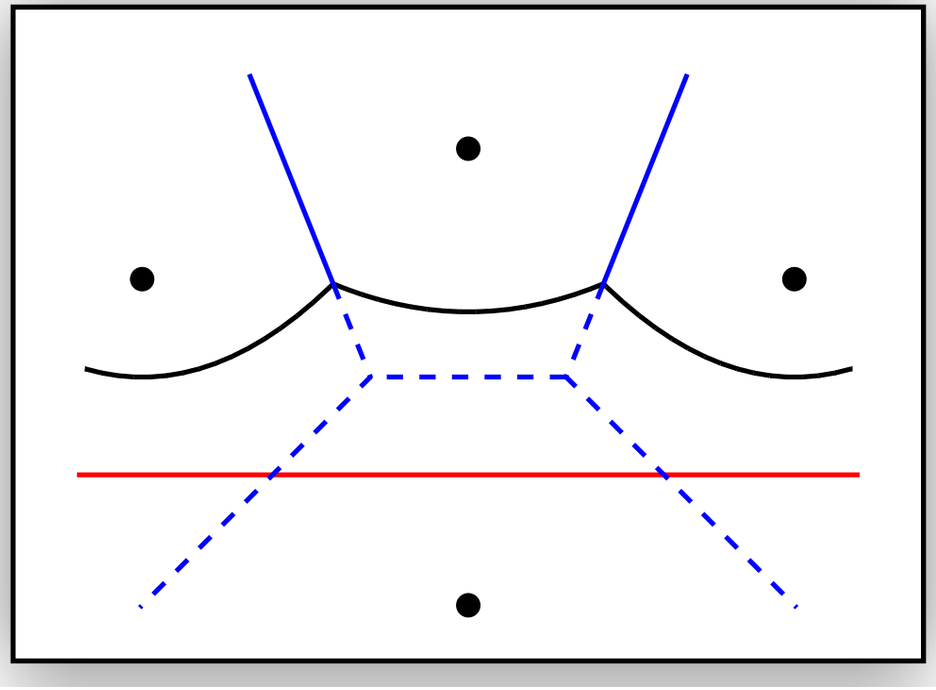
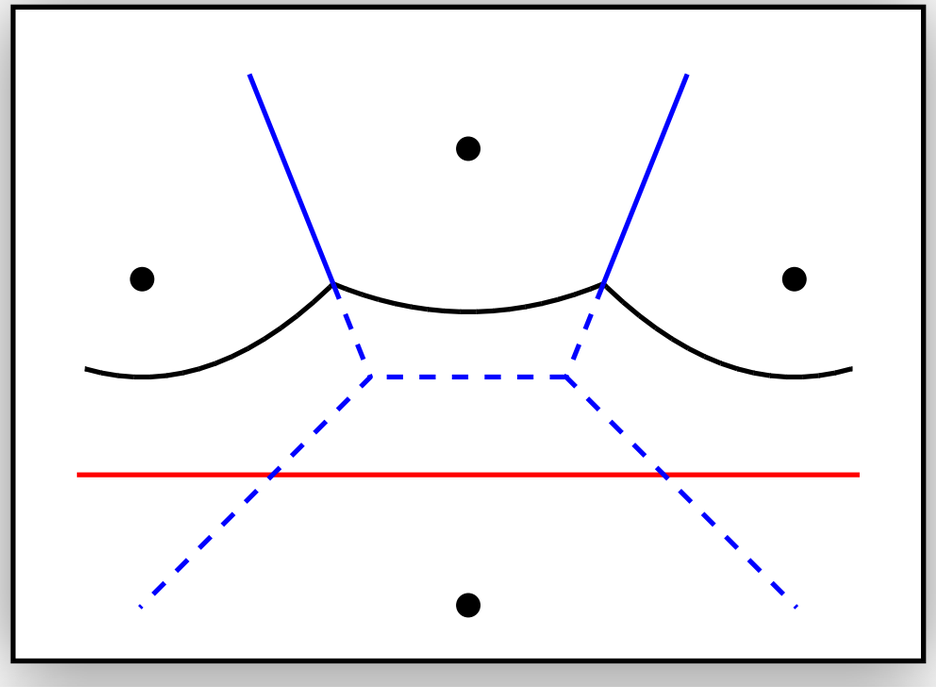


Validity of Circle Events

Observation 1:

- Discovering $p \in C^\circ \Rightarrow$ Circle event C becomes obsolete

  $\rightarrow$ Parabolic arc must know associated circle event.



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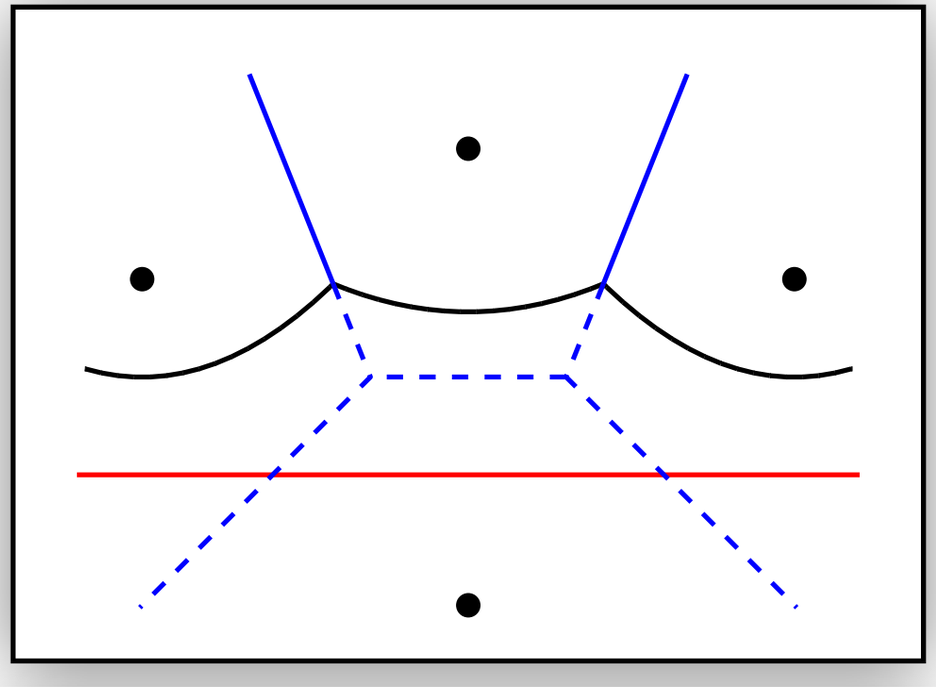

Observation 2:

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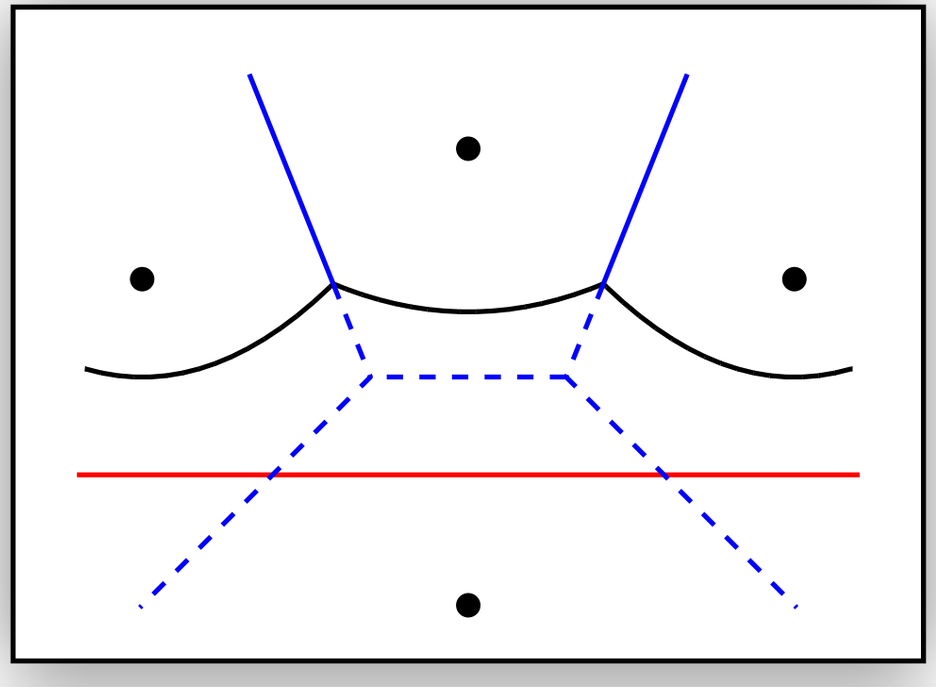

Observation 2:

- Triple of adjacent arcs do not always define a circle event.

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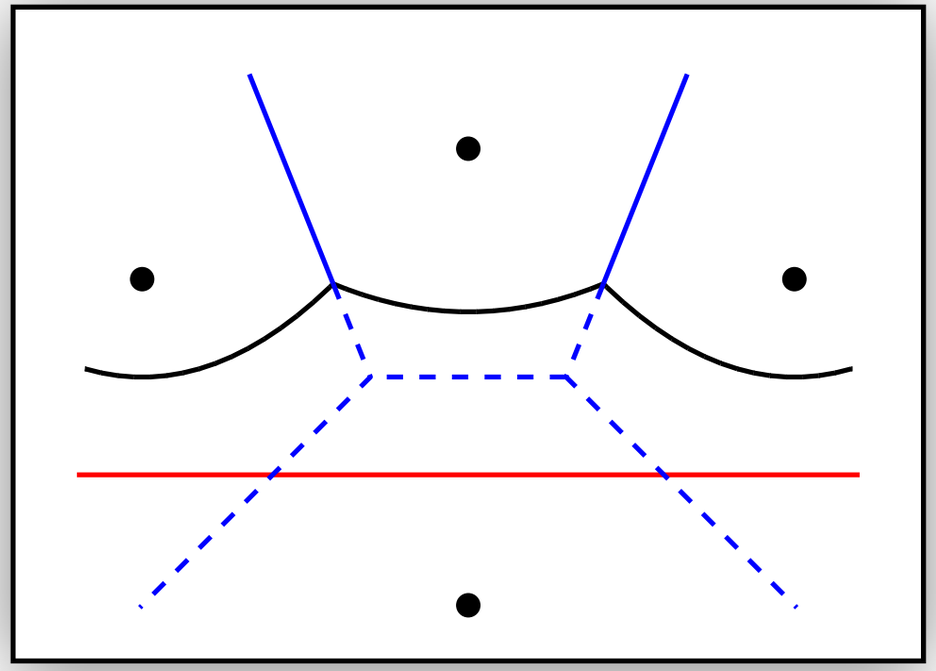

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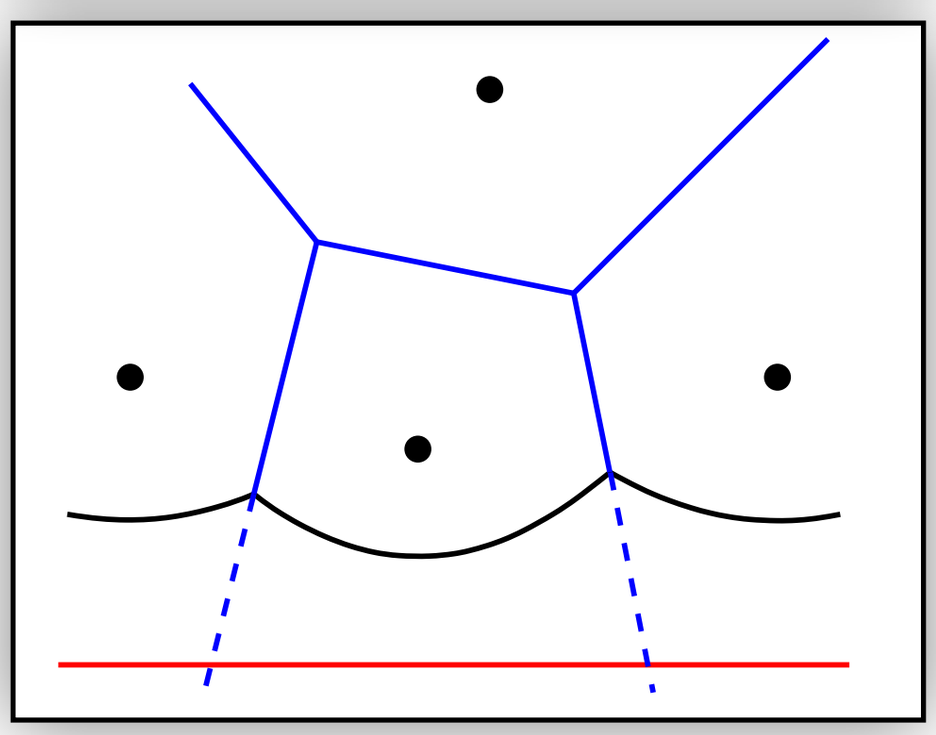

Processing Point Events - I

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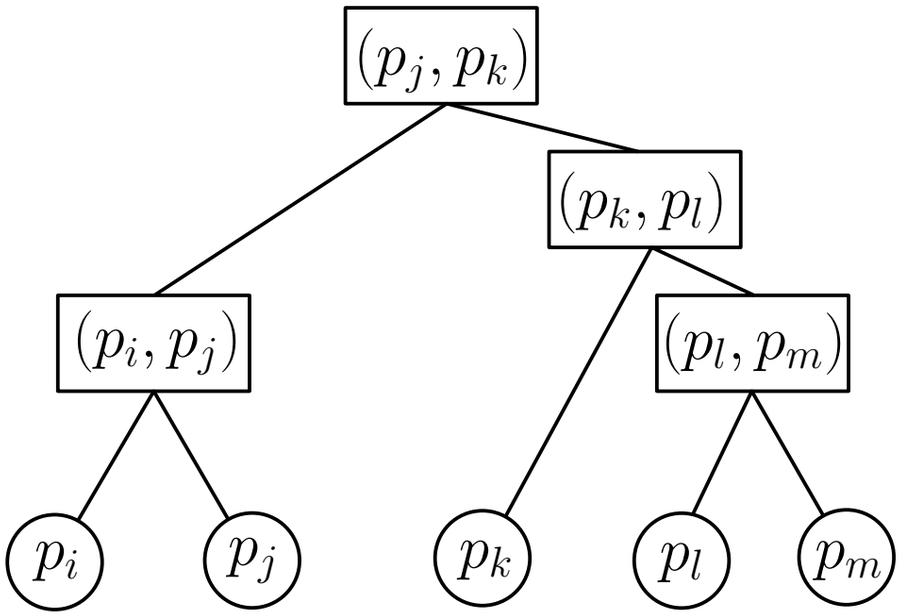

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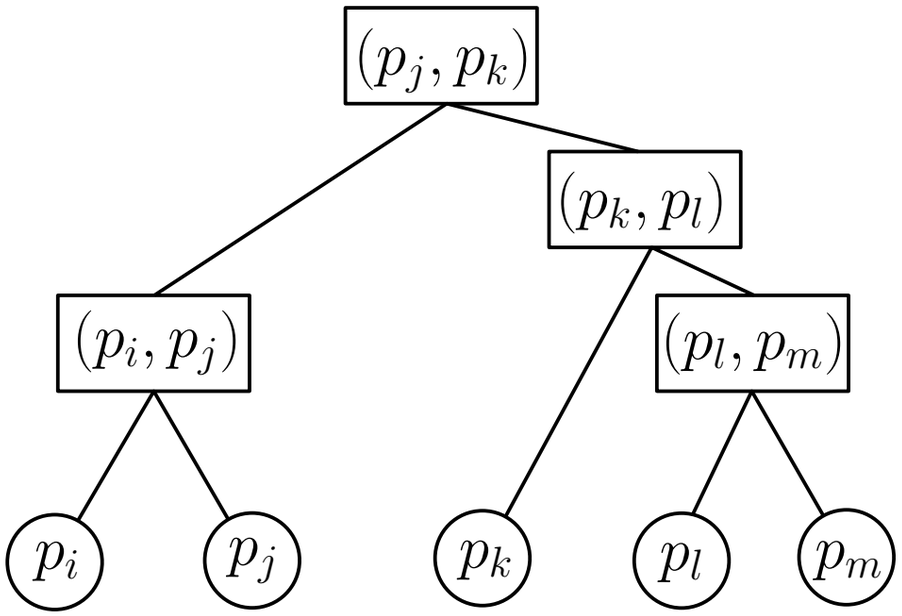

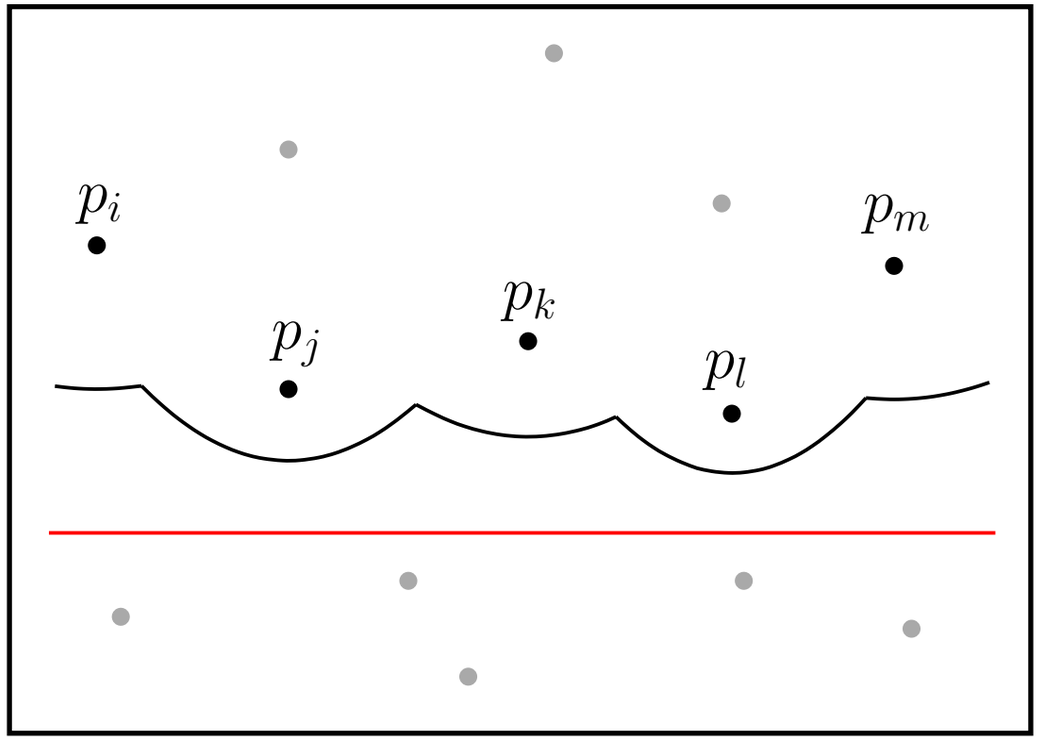

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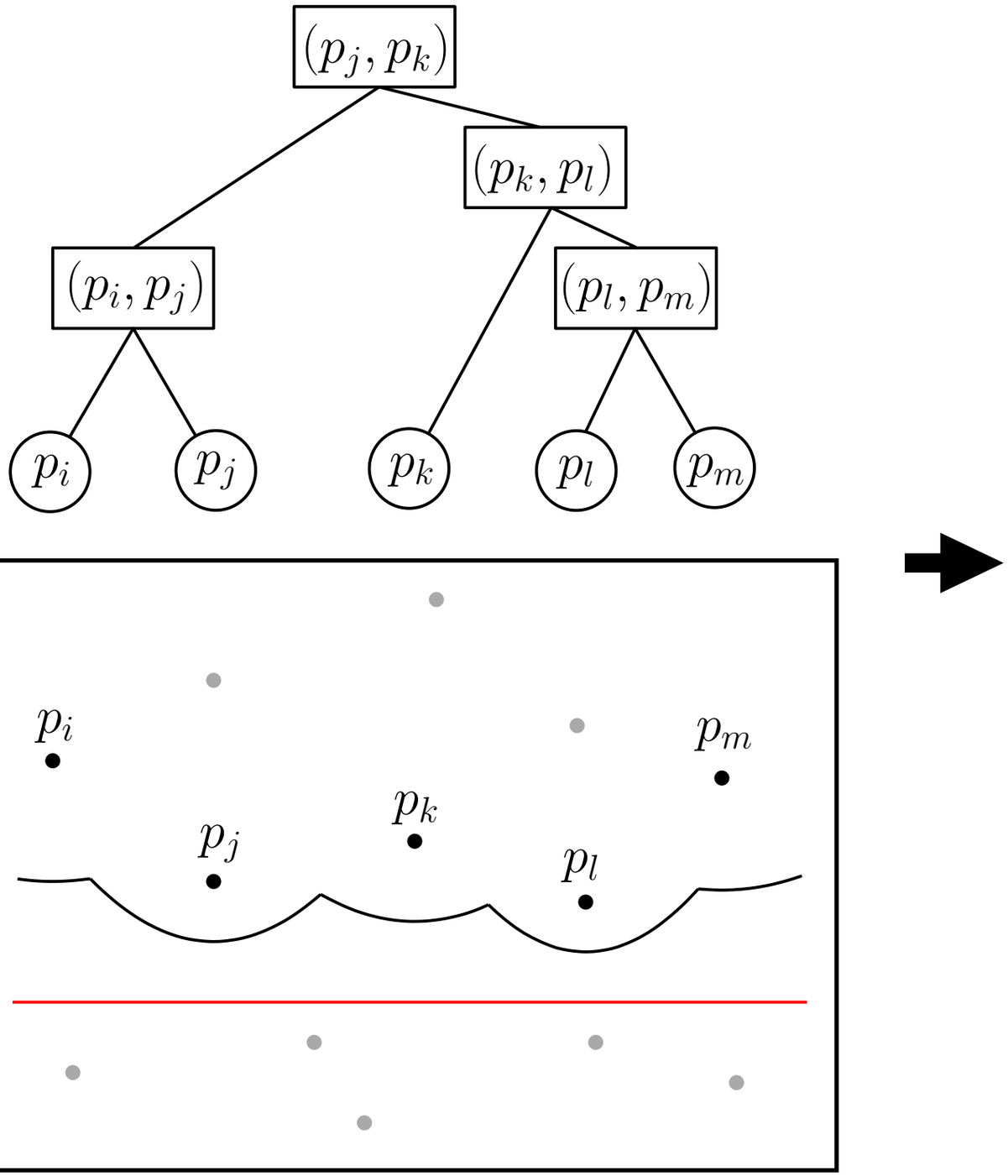
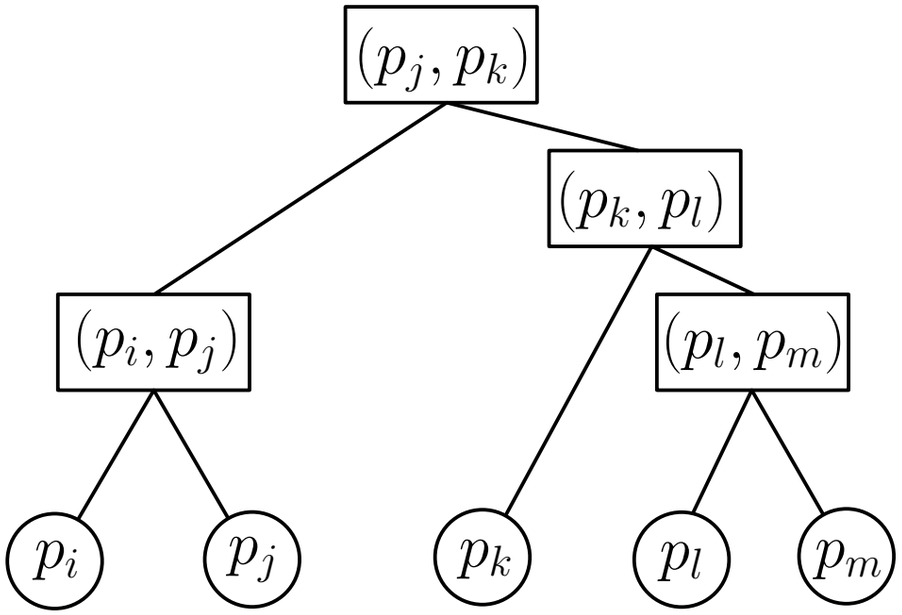
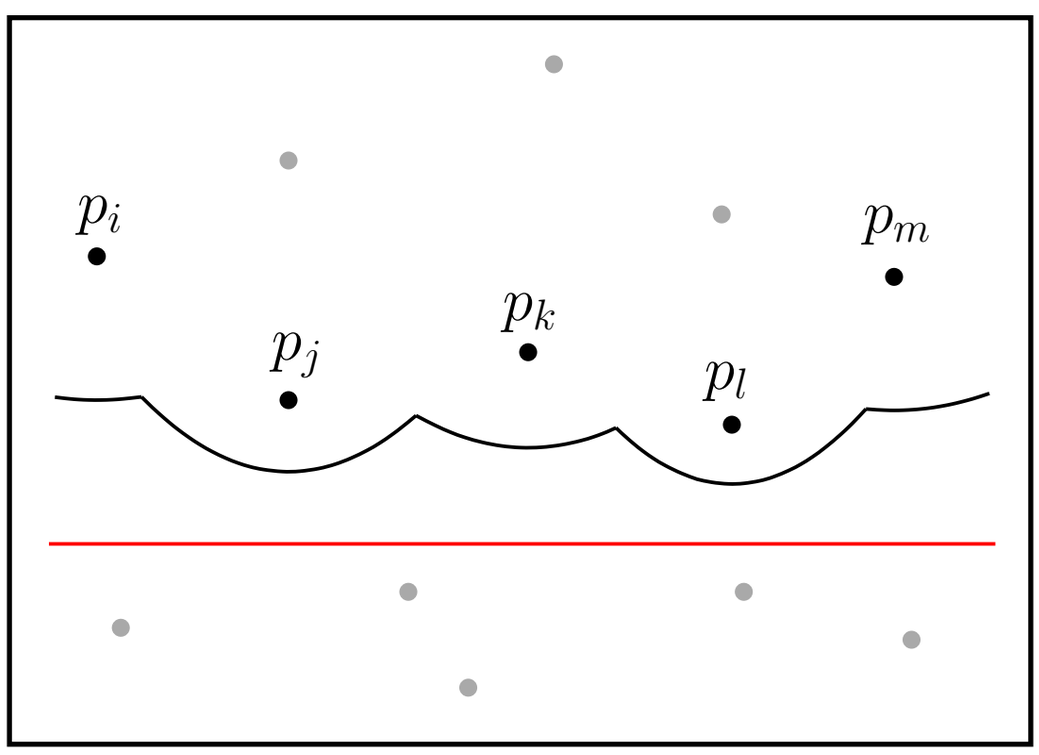
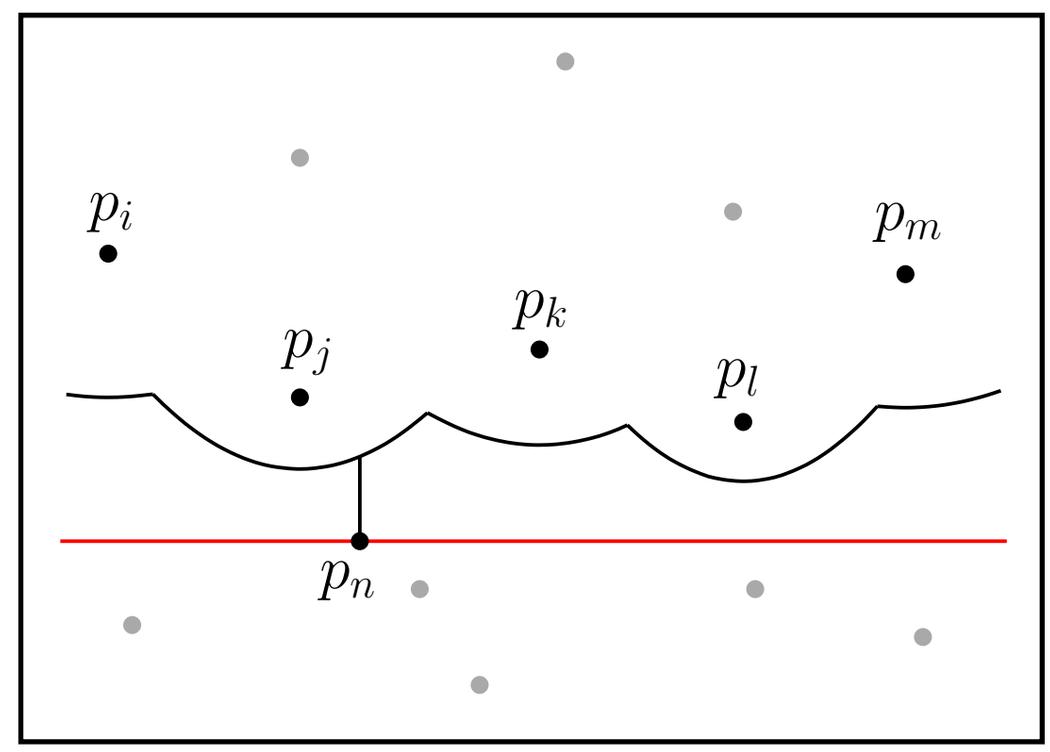


Processing Point Events - I







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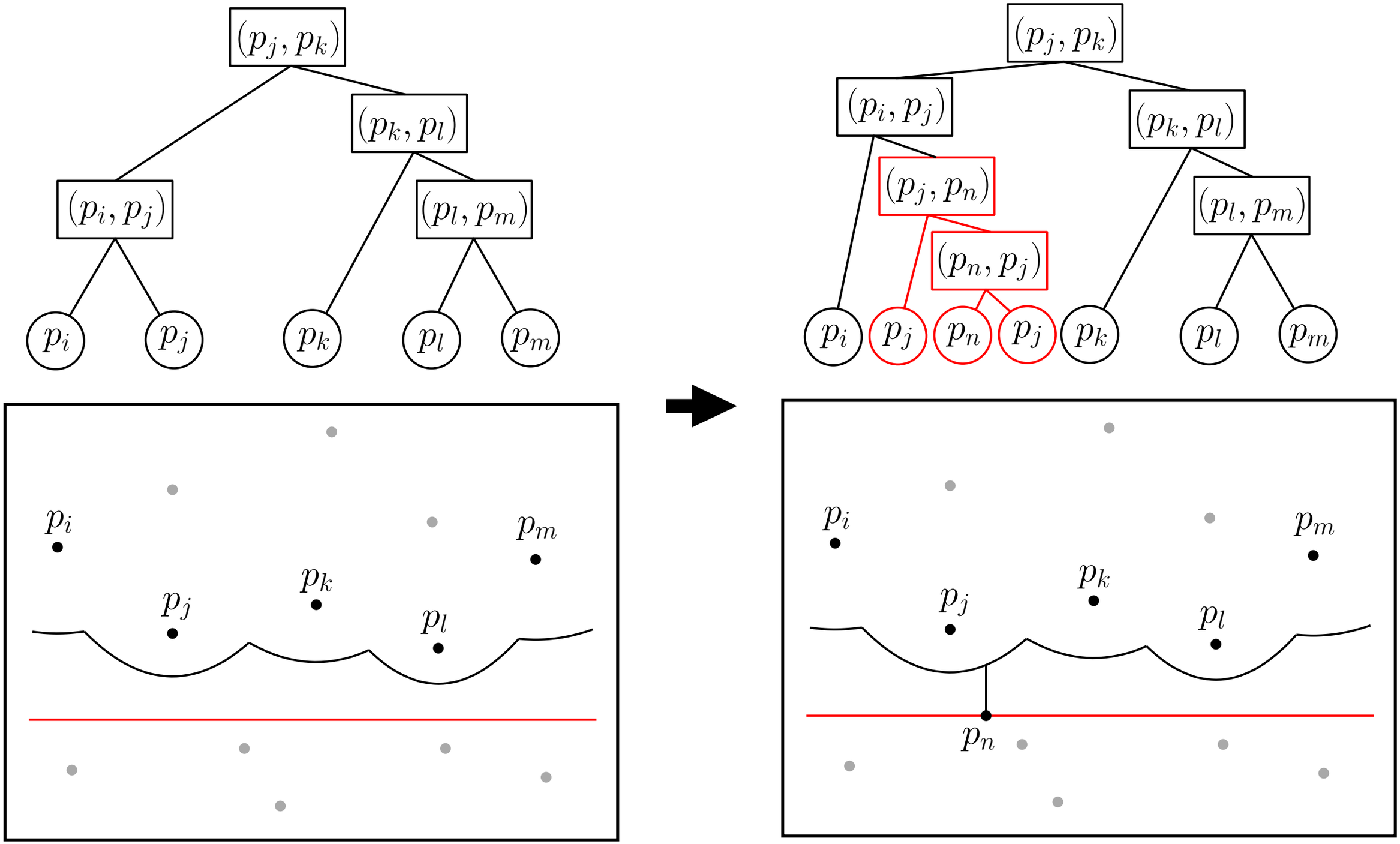


Processing Point Events - II

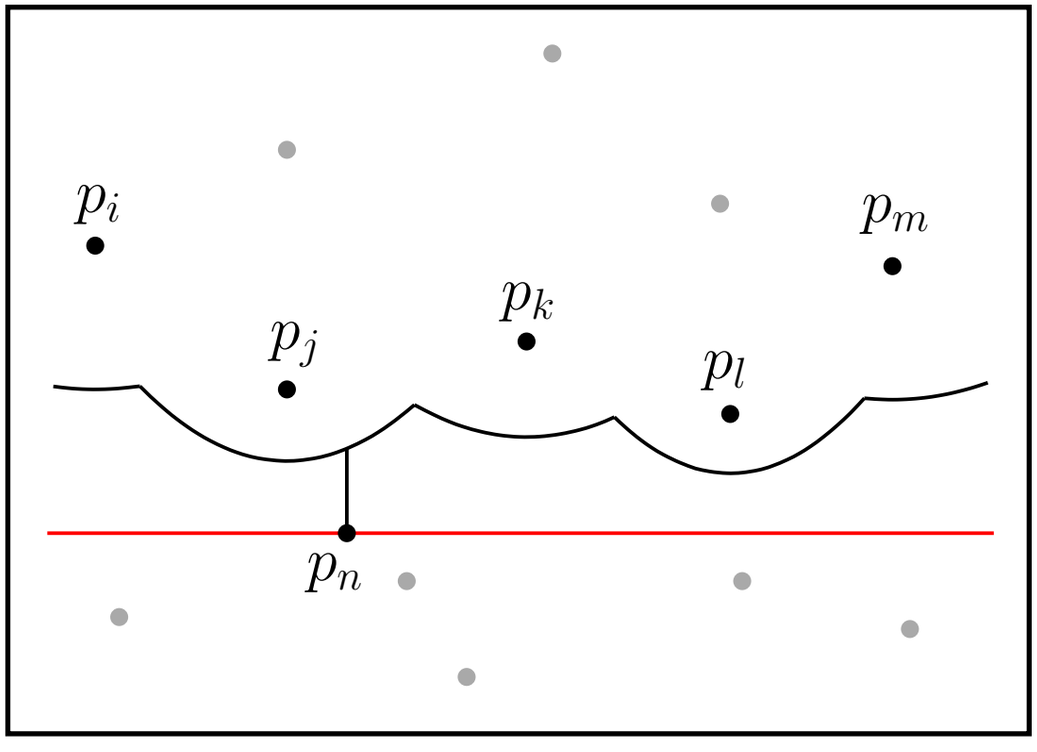
Introducing an arc

Introducing an arc

- Defined by point p_n

Introducing an arc

- Defined by point p_n



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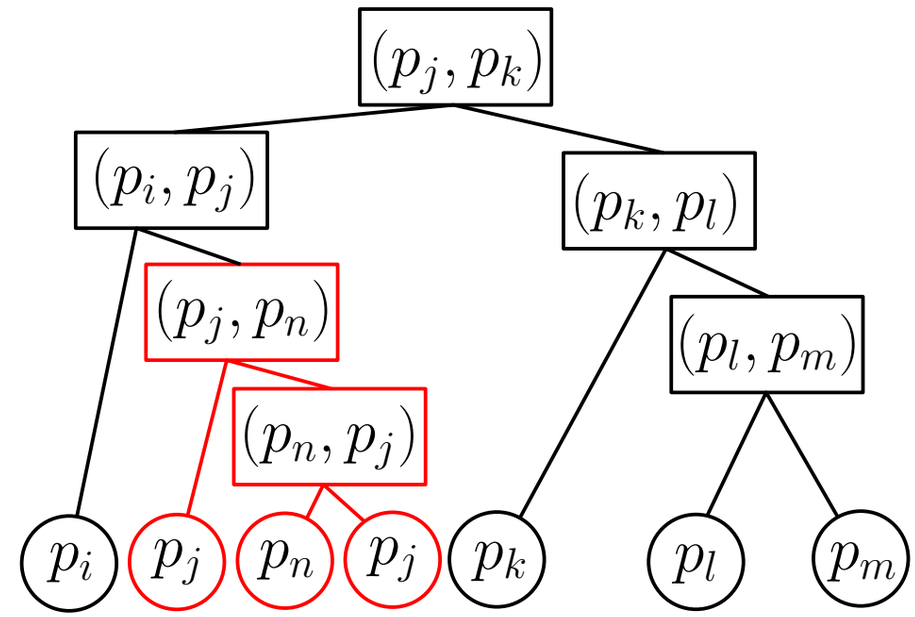
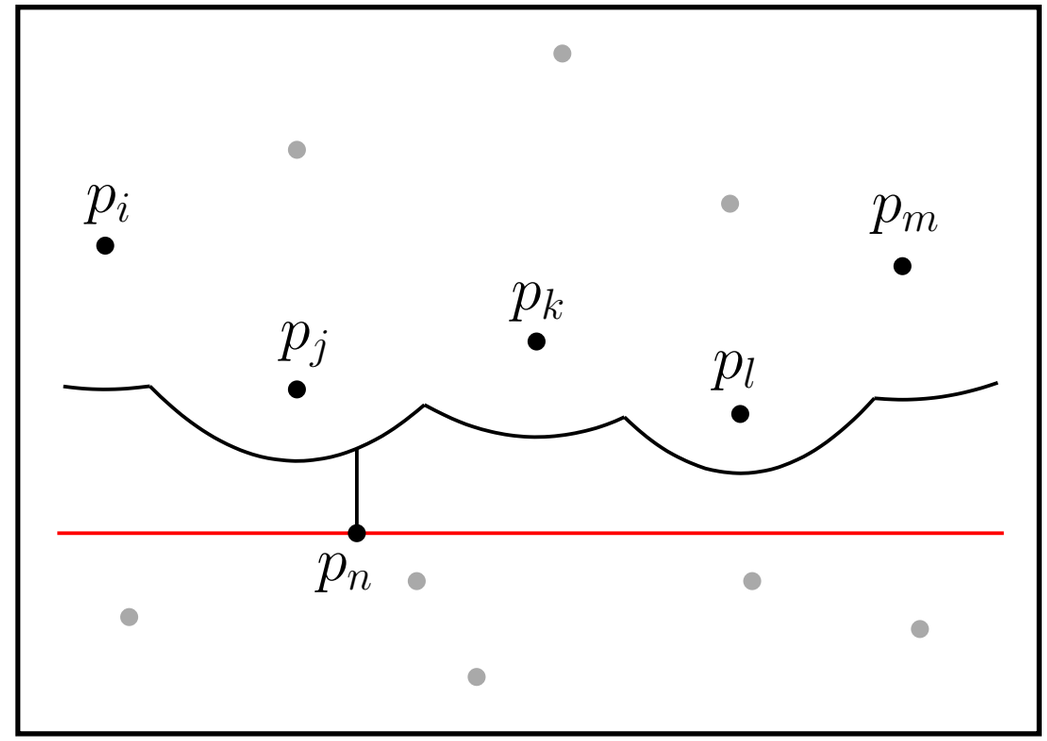
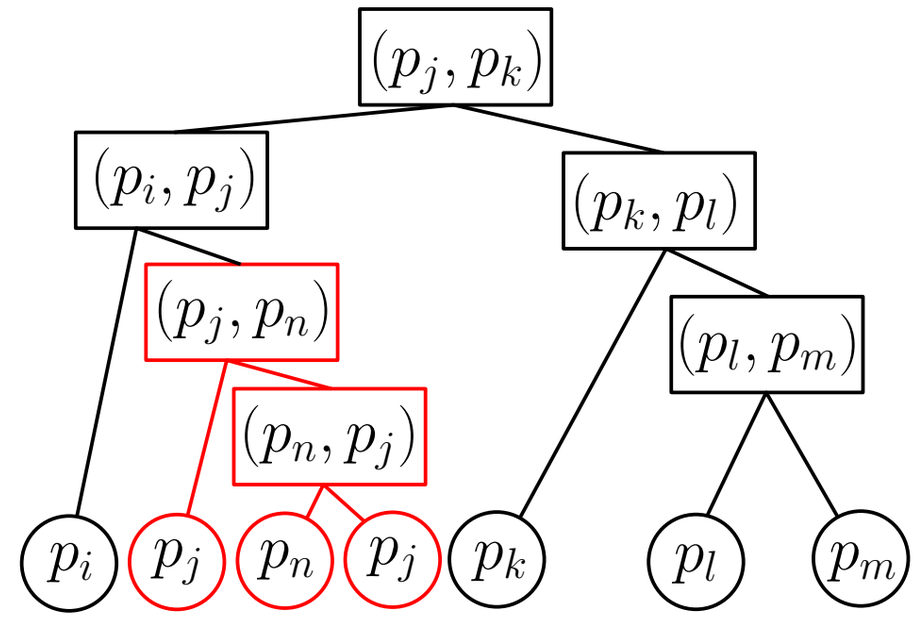
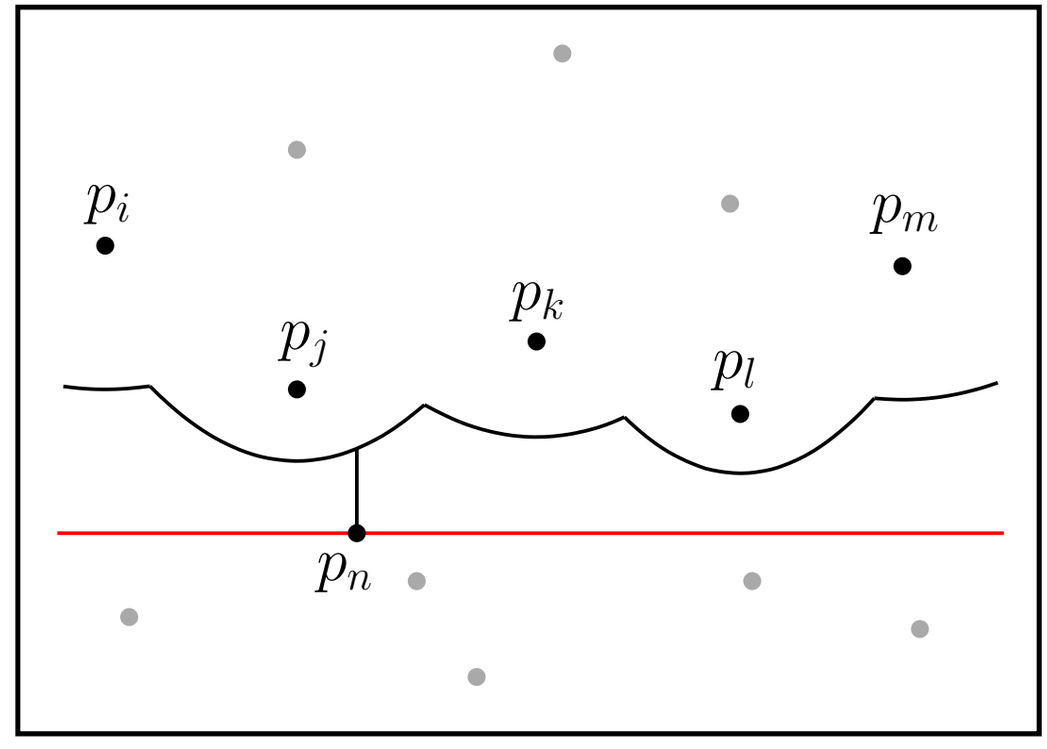




Introducing an arc

- Defined by point p_n
- Search for B at insert position.

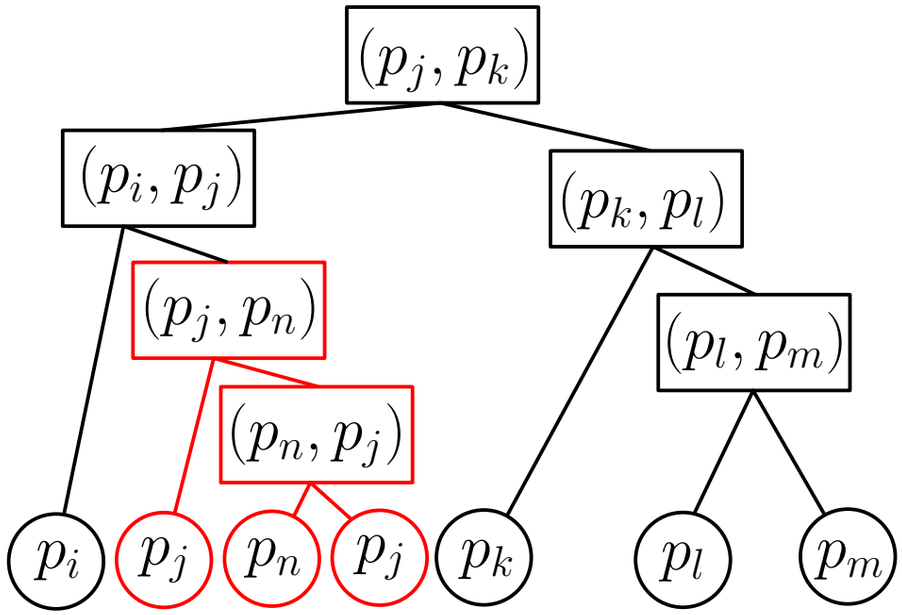


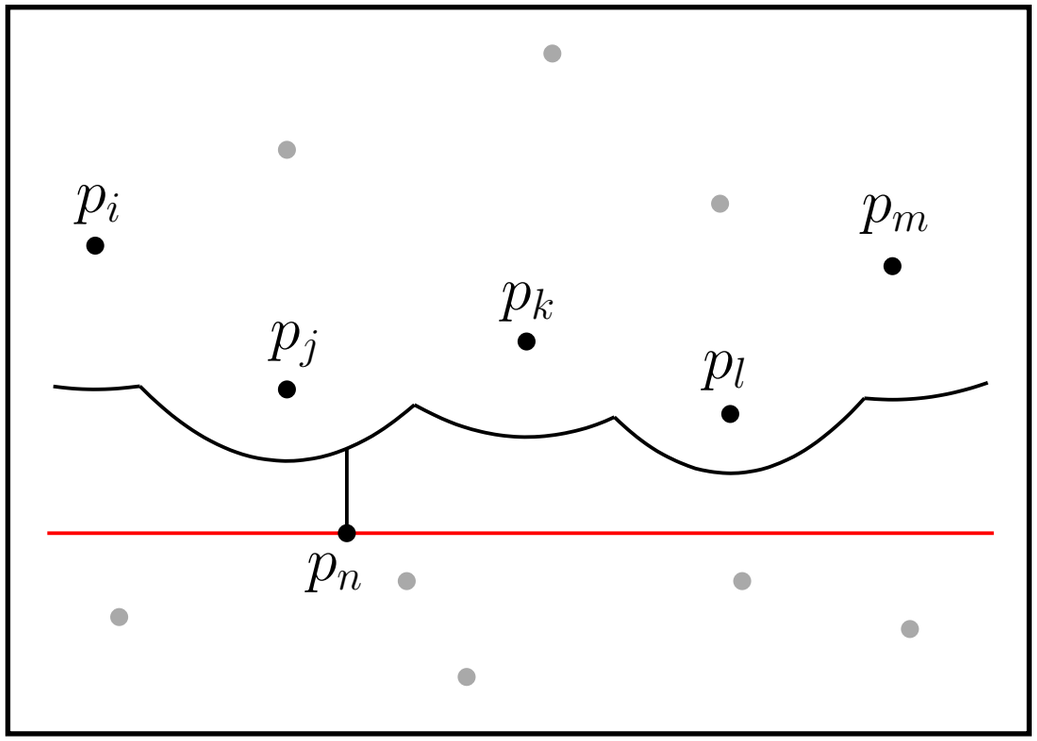


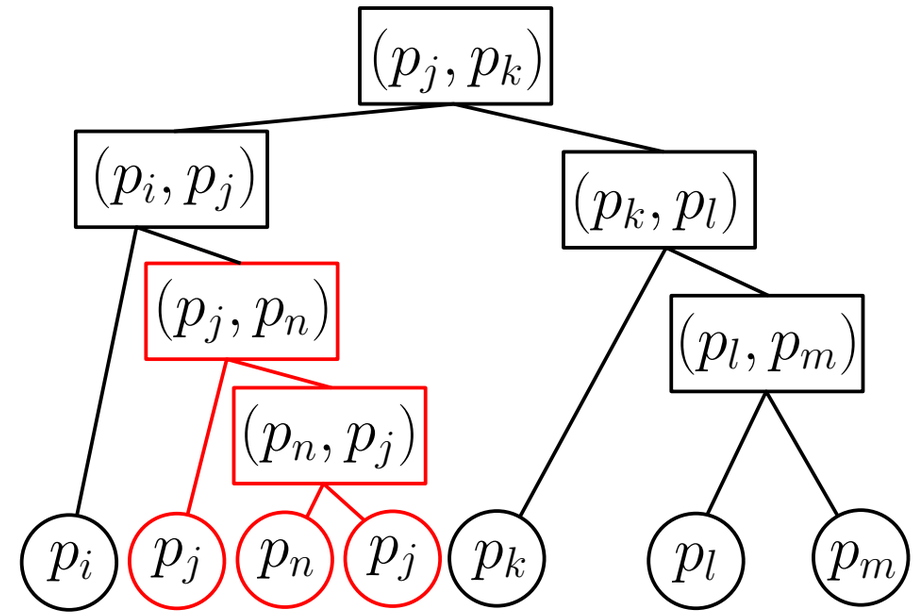
Introducing an arc

- Defined by point p_n
- Search for B at insert position.

  $\rightarrow$ Intersection points stored implicitly



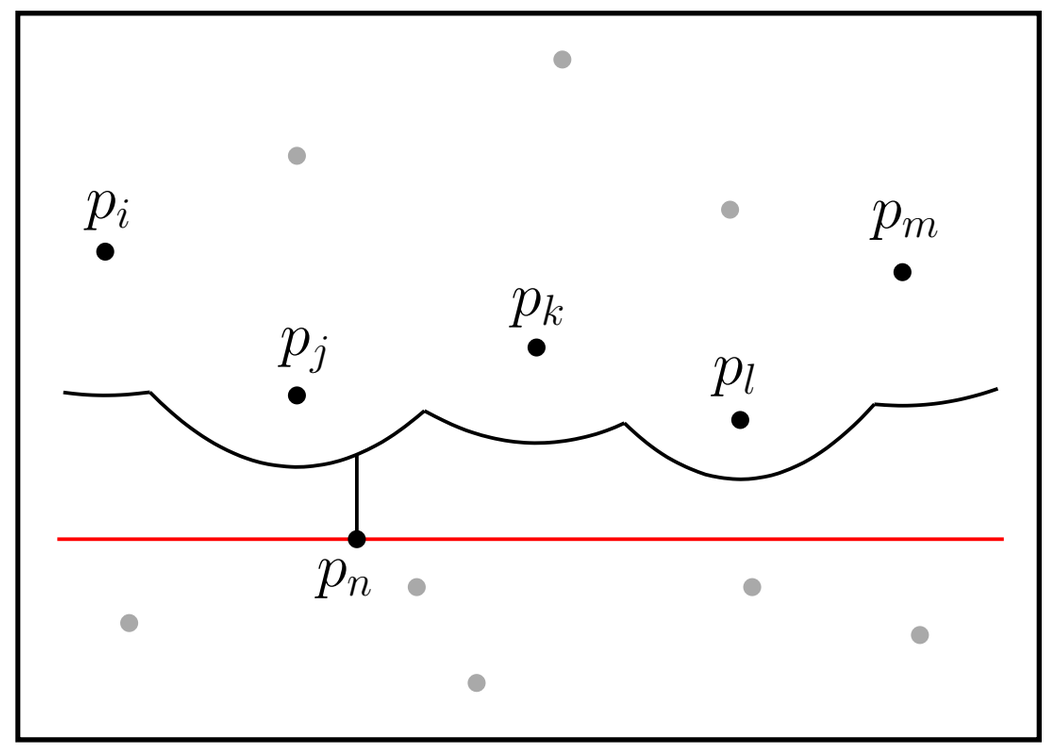


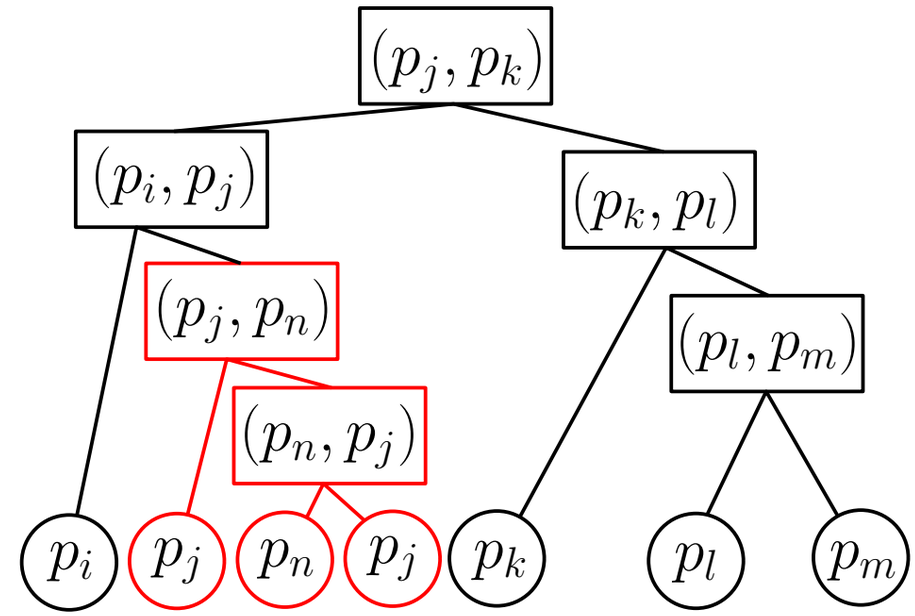


Introducing an arc

- Defined by point p_n
- Search for B at insert position.

  $\rightarrow$ Intersection points stored implicitly
- Splitting an arc β
  (Special case: p_n below arc intersection)

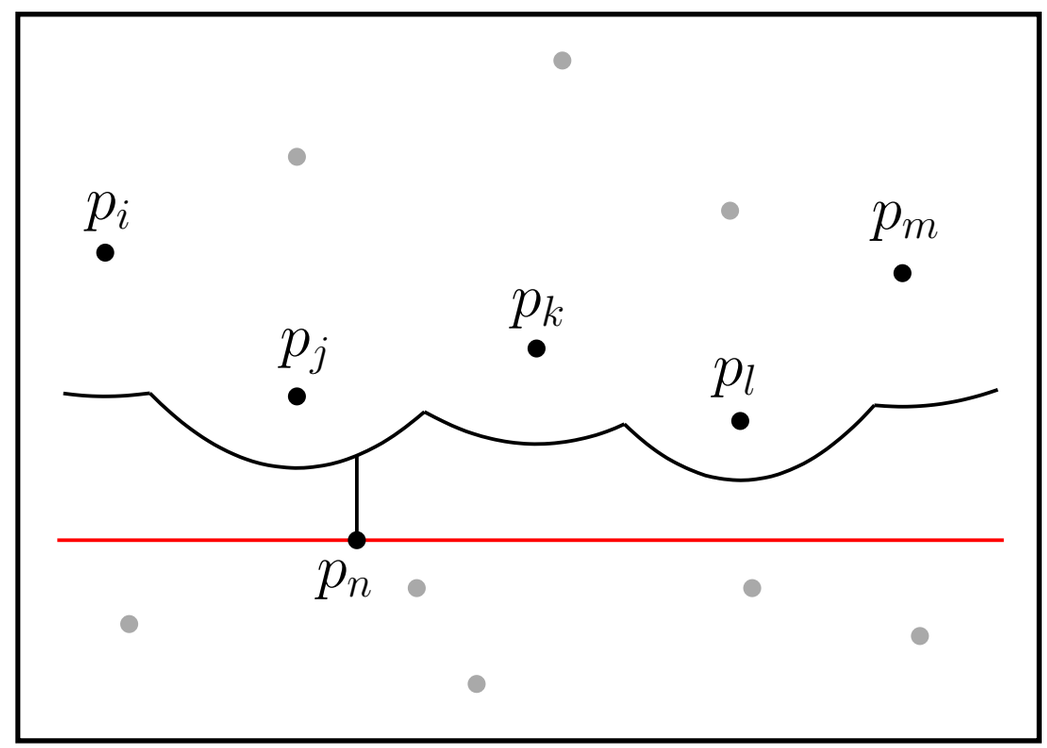




Introducing an arc

- Defined by point p_n
- Search for B at insert position.

  $\rightarrow$ Intersection points stored implicitly
- Splitting an arc β
  (Special case: p_n below arc intersection)
- Splitting: $\beta \rightarrow \beta_1, \beta_n, \beta_2$
  (rebalancing if necessary)



Processing Point Events - III

Generating circle events:

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- Before insertion of β_n(defined byp_n):

$$\ldots \beta_i \beta_j \beta_k \beta_l \ldots$$

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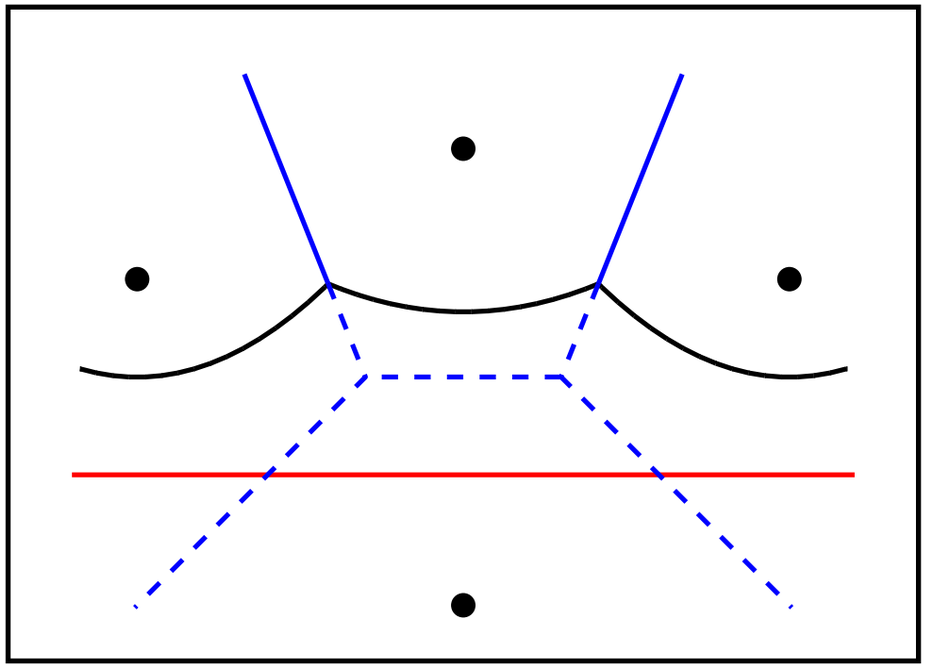


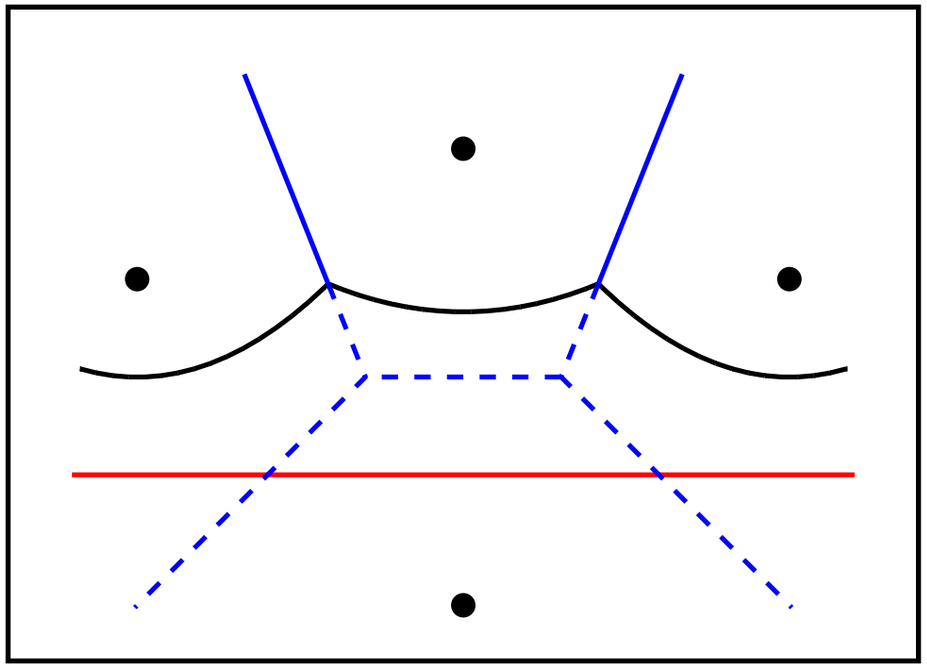
Generating circle events:

- Before insertion of β_n(defined byp_n):

$$\ldots \beta_i \beta_j \beta_k \beta_l \ldots$$

- After insertion:

$$\ldots \beta_i \beta_{j,1} \beta_n \beta_{j,2} \beta_k \beta_l \ldots$$



Generating circle events:

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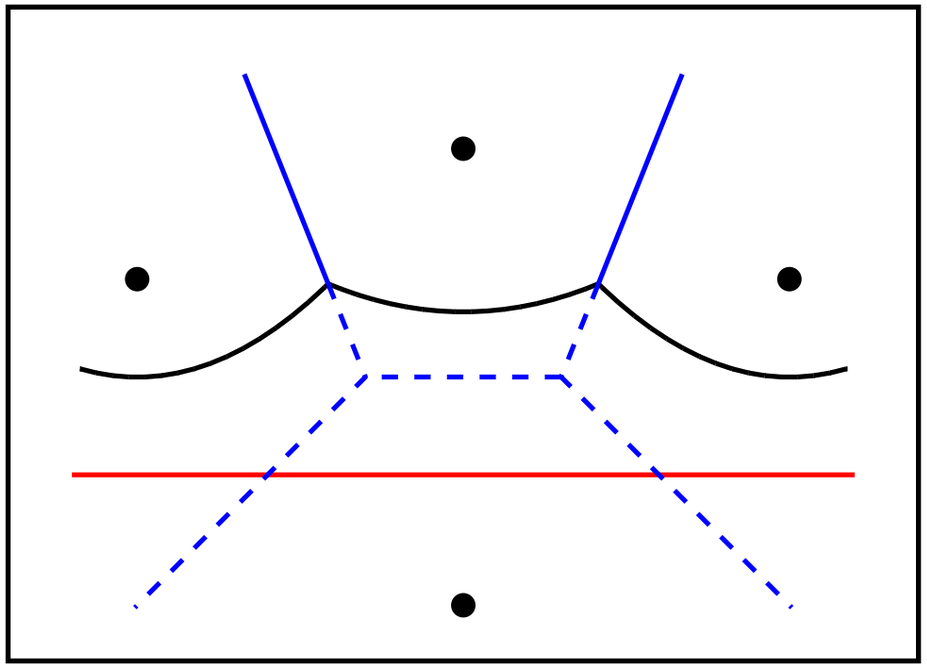
- After insertion:

$$\ldots \beta_i \beta_{j,1} \beta_n \beta_{j,2} \beta_k \beta_l \ldots$$

- Possibly deletion of circle events (e.g., defined by $(\beta_i, \beta_j, \beta_k)$).



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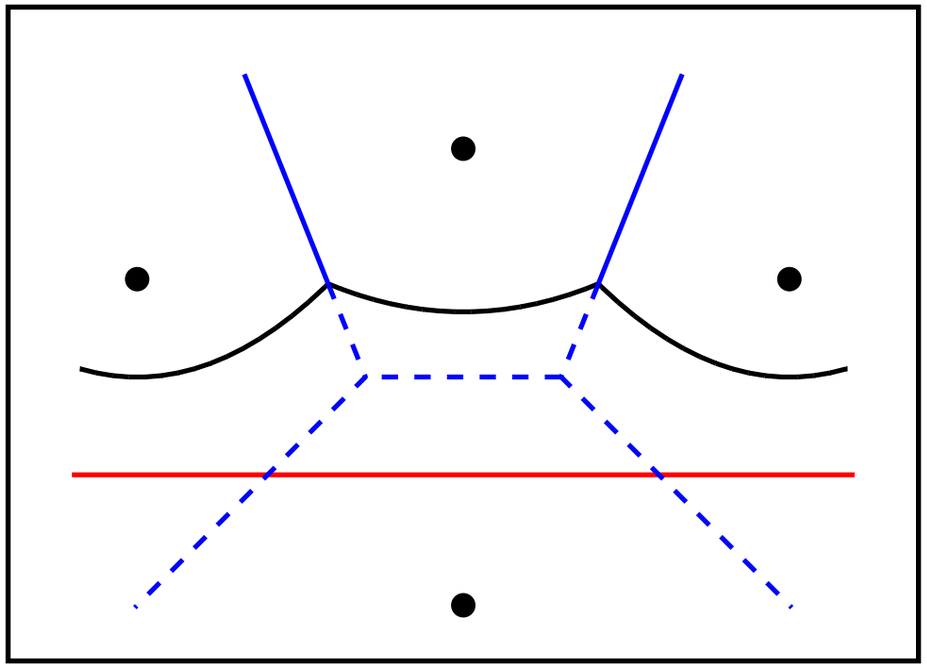

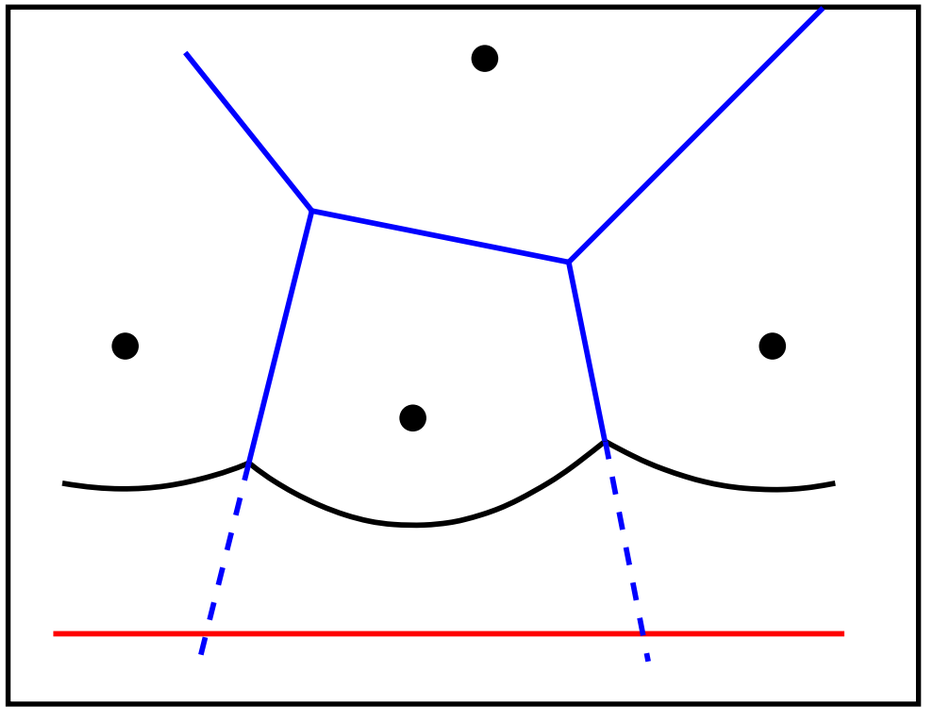

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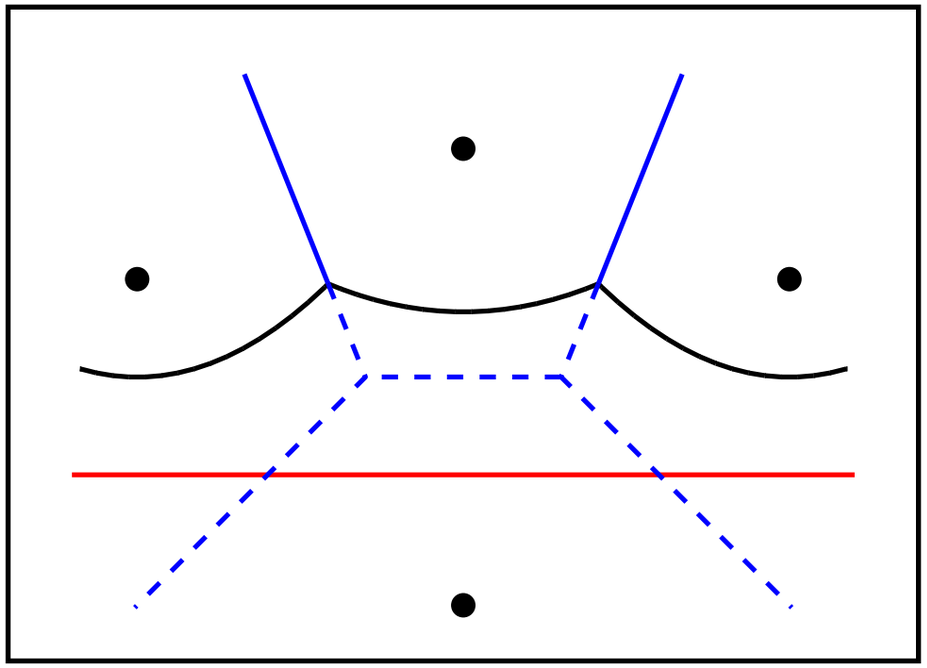
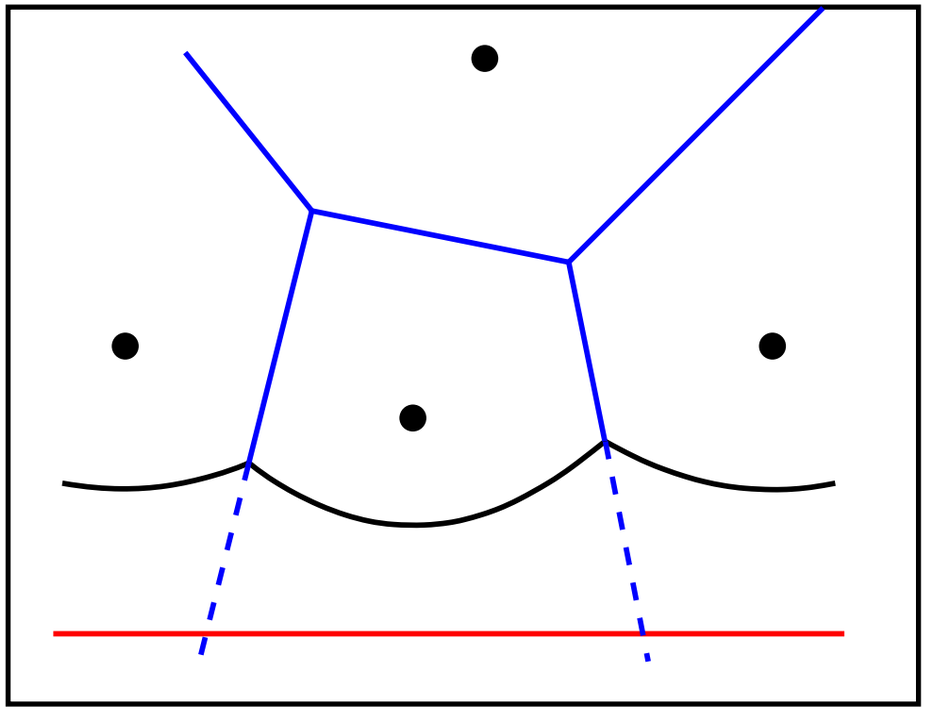
- After insertion:

$$\ldots \beta_i \beta_{j,1} \beta_n \beta_{j,2} \beta_k \beta_l \ldots$$

- Possibly deletion of circle events (e.g., defined by $(\beta_i, \beta_j, \beta_k)$).

- Test all newly adjacent triples.





Generating circle events:

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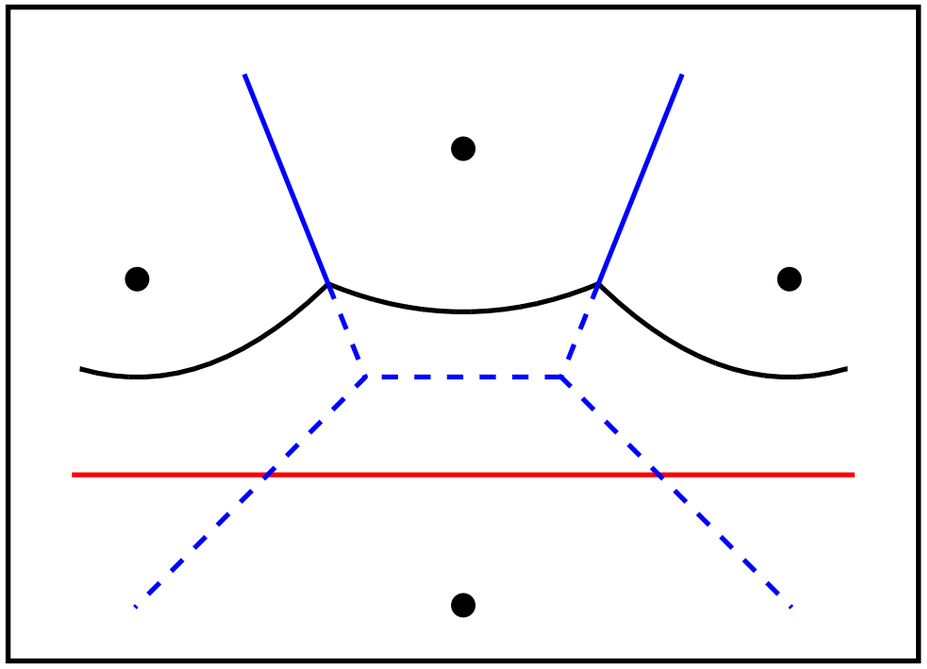
- After insertion:

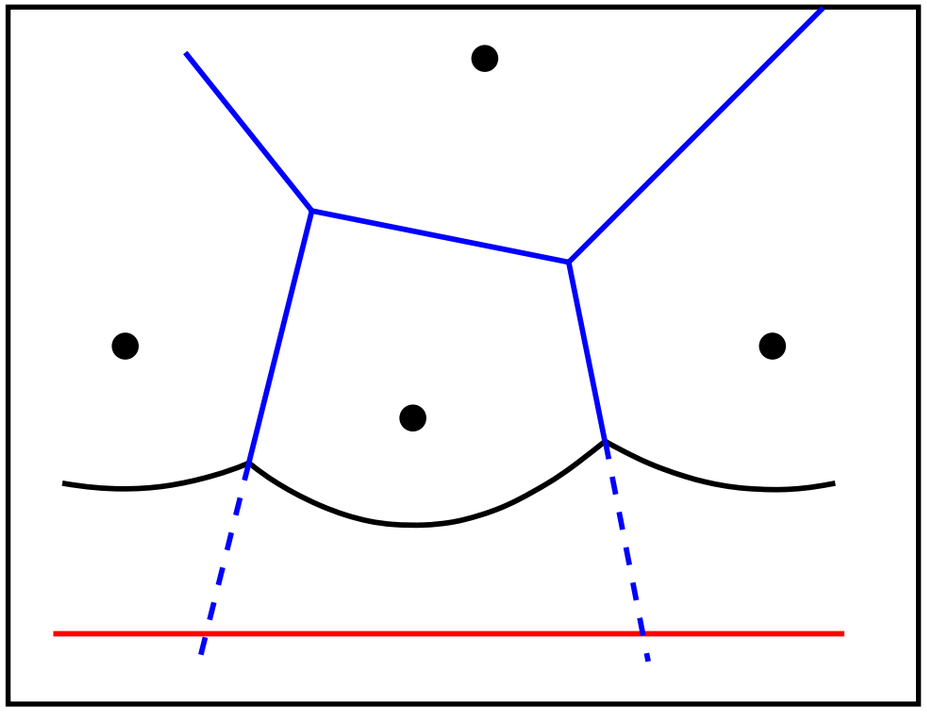
$$\ldots \beta_i \beta_{j,1} \beta_n \beta_{j,2} \beta_k \beta_l \ldots$$

- Possibly deletion of circle events (e.g., defined by $(\beta_i, \beta_j, \beta_k)$).

- Test all newly adjacent triples.

- Insert new events into Q.

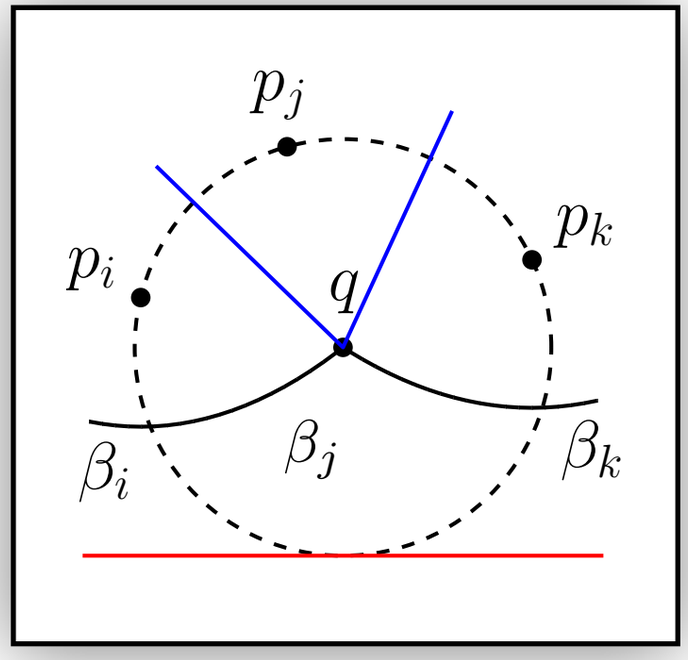




Processing Circle Events - I

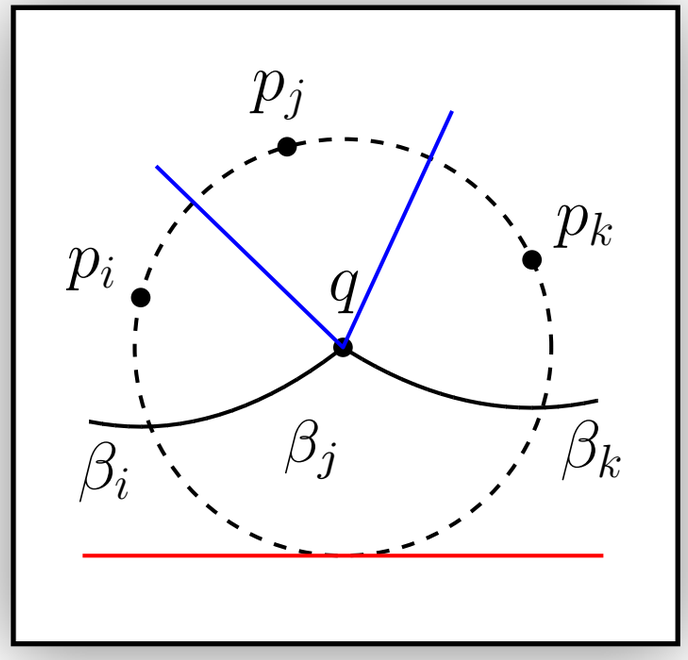
Updates

Updates



Updates

- Delete β_j from B.



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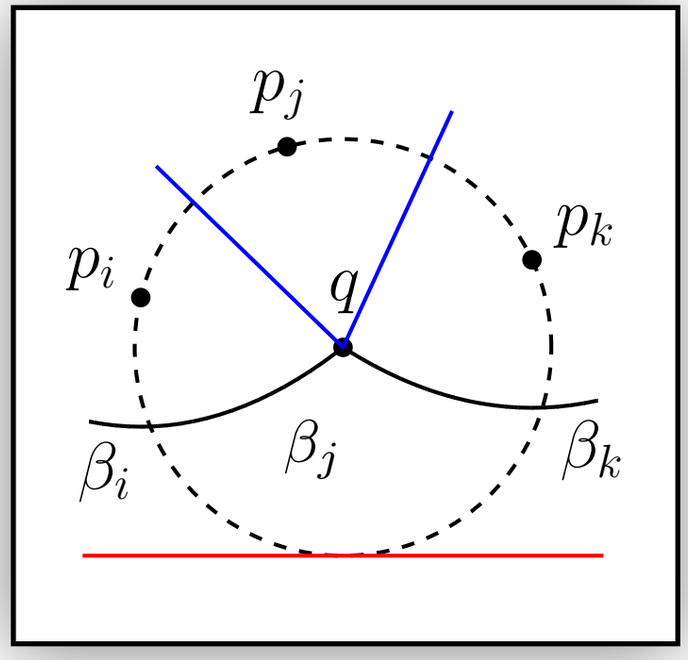

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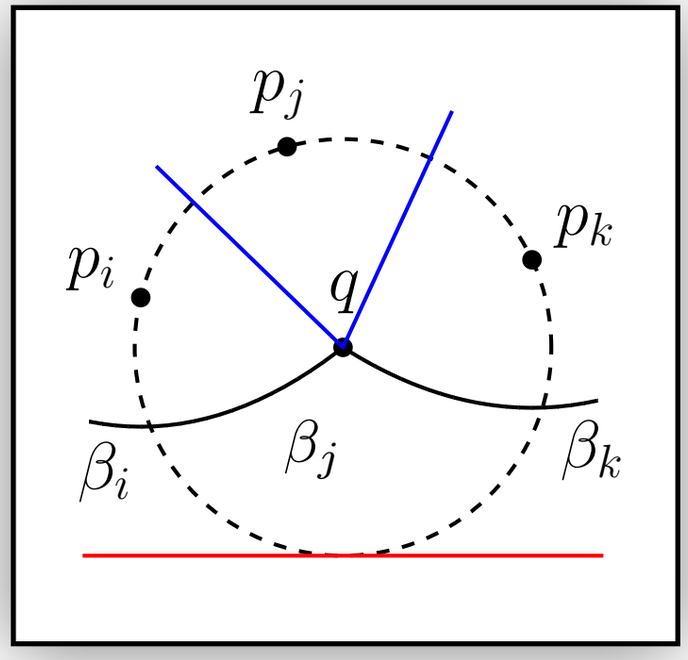

Updates

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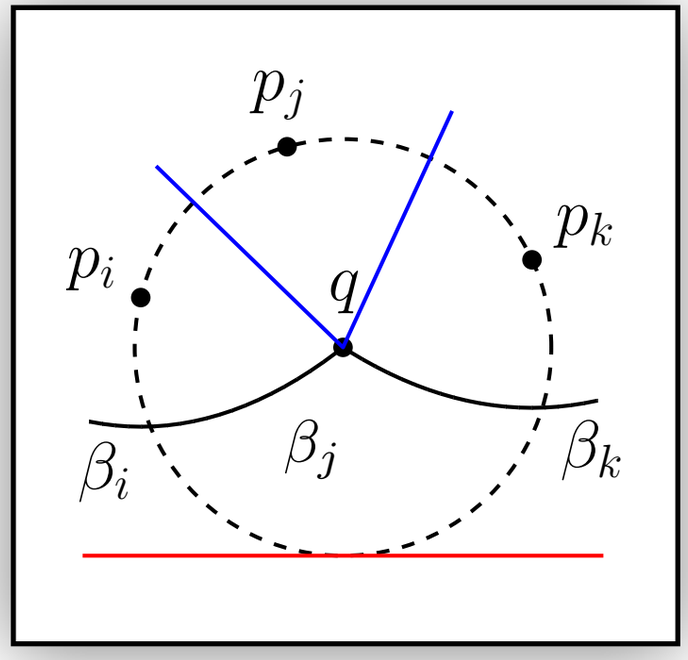
- Delete circle events defined by β_j
- New adjacent triples
  $\rightarrow$ Possibly insert new circle events



Correctness - I

Lemma 4.21

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- Theorem 4.20 $\Rightarrow \exists p_i, p_j, p_k \in \mathcal{P}$:

$$p_i, p_j, p_k \in \partial C$$

$$C \cap \mathcal{P} = \{p_i, p_j, p_k\}$$

for circle C with center q and radius $d(p_i, q) = d(p_j, q) = d(p_k, q)$.

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Theorem 4.20:

1. $x \in \mathbb{R}^2$ Voronoi vertex

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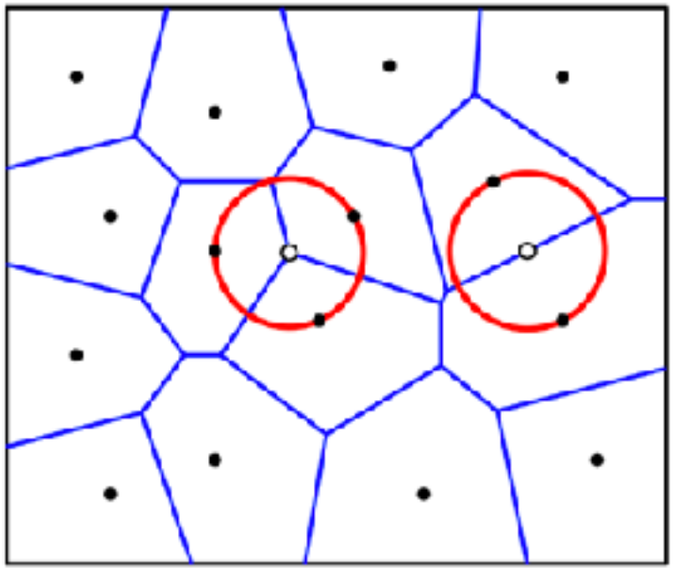

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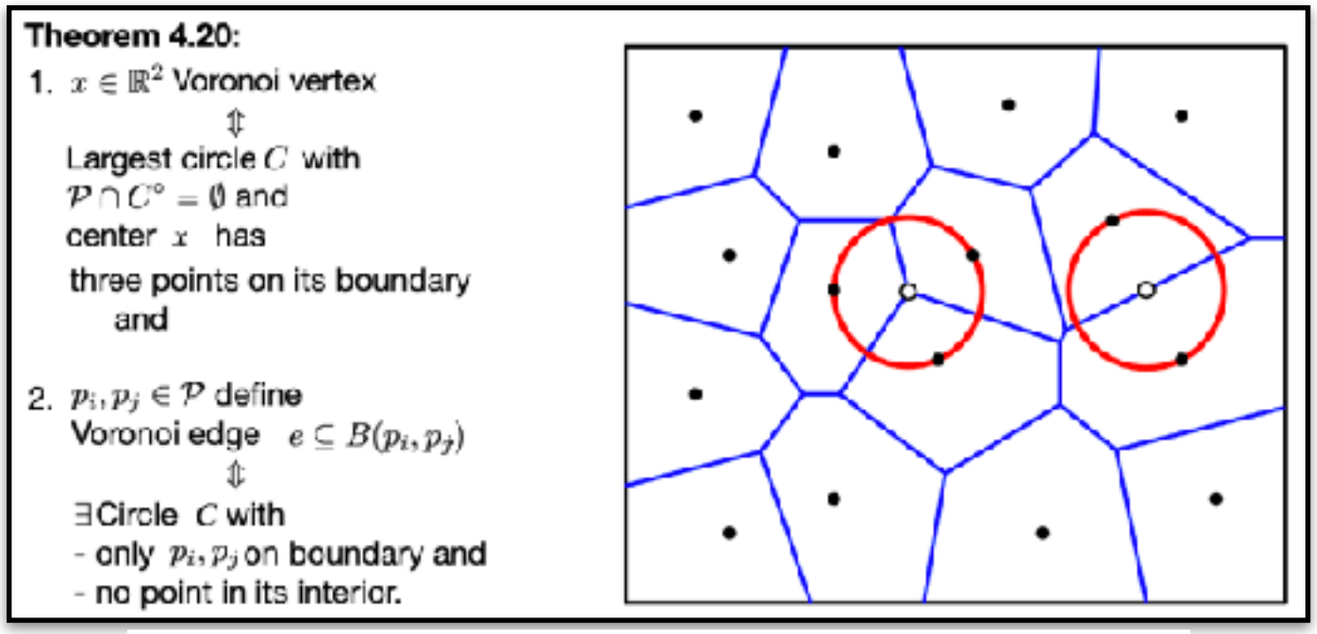

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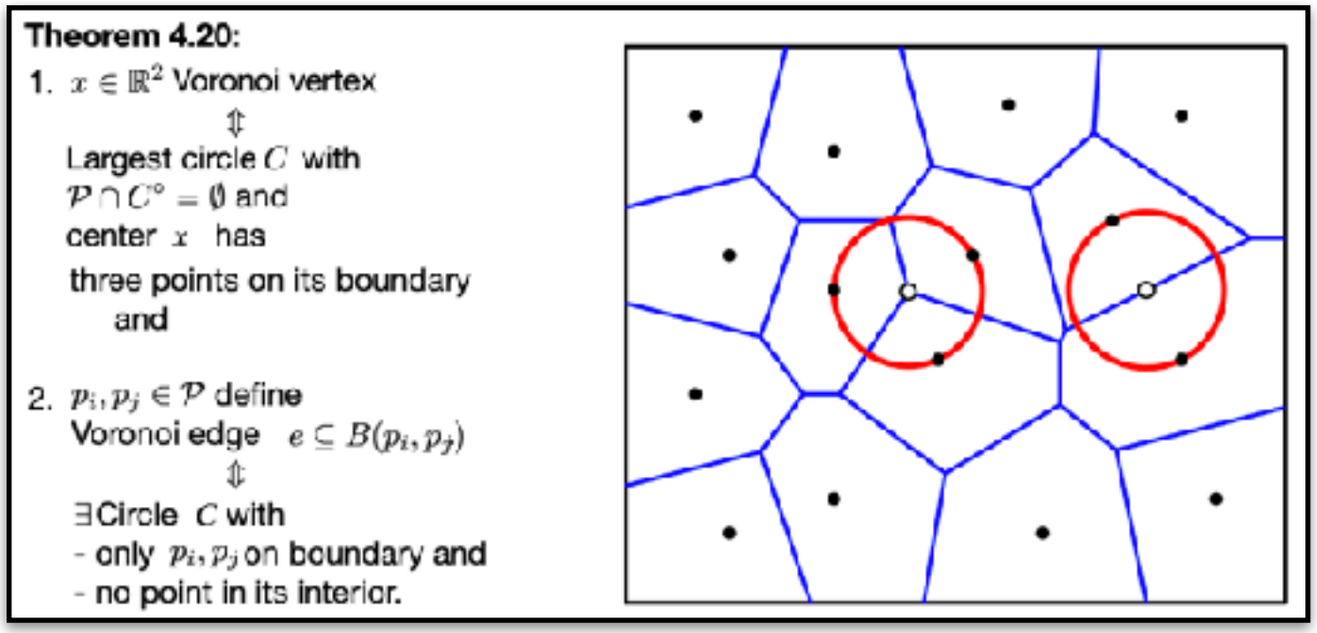

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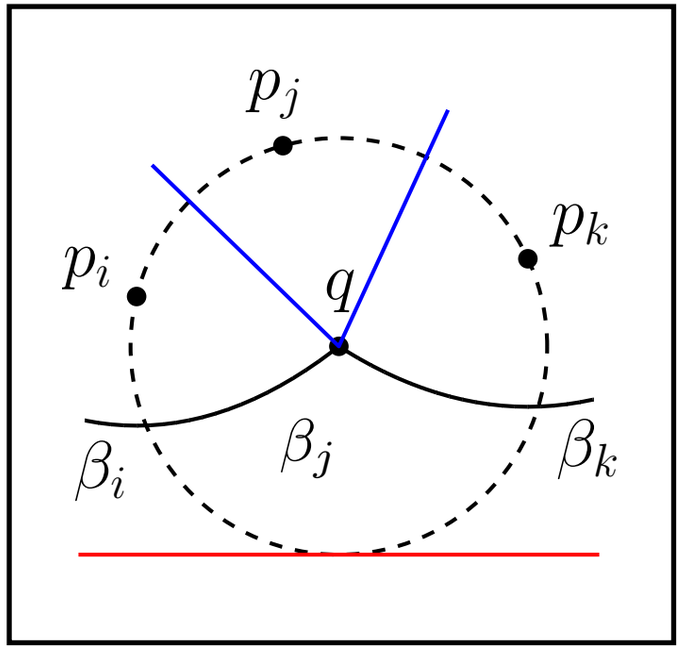

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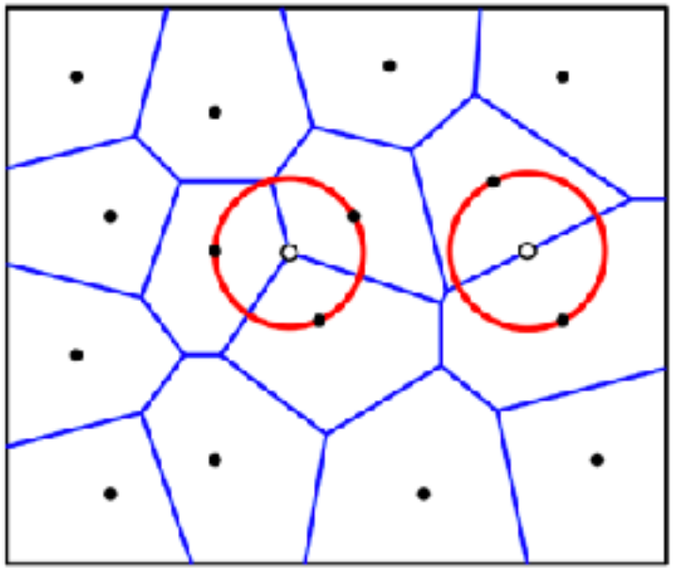

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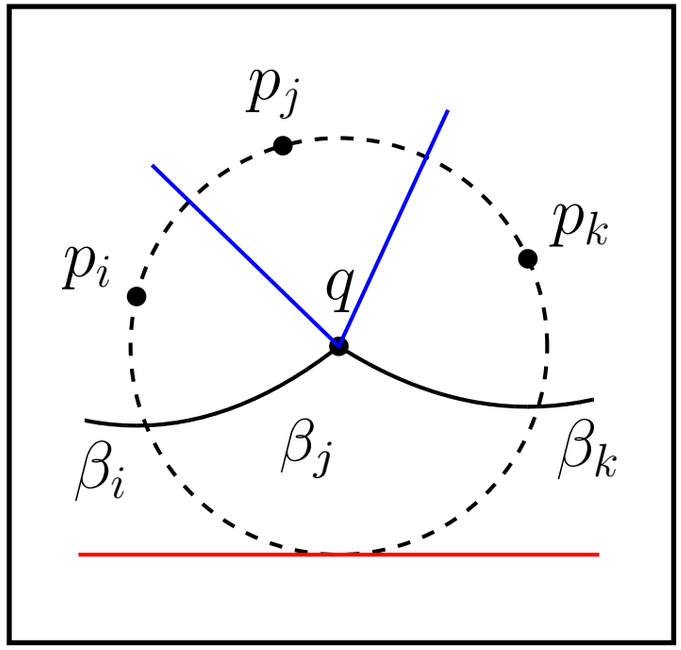

$\square$

Constructing the Voronoi Diagram - I

Goal

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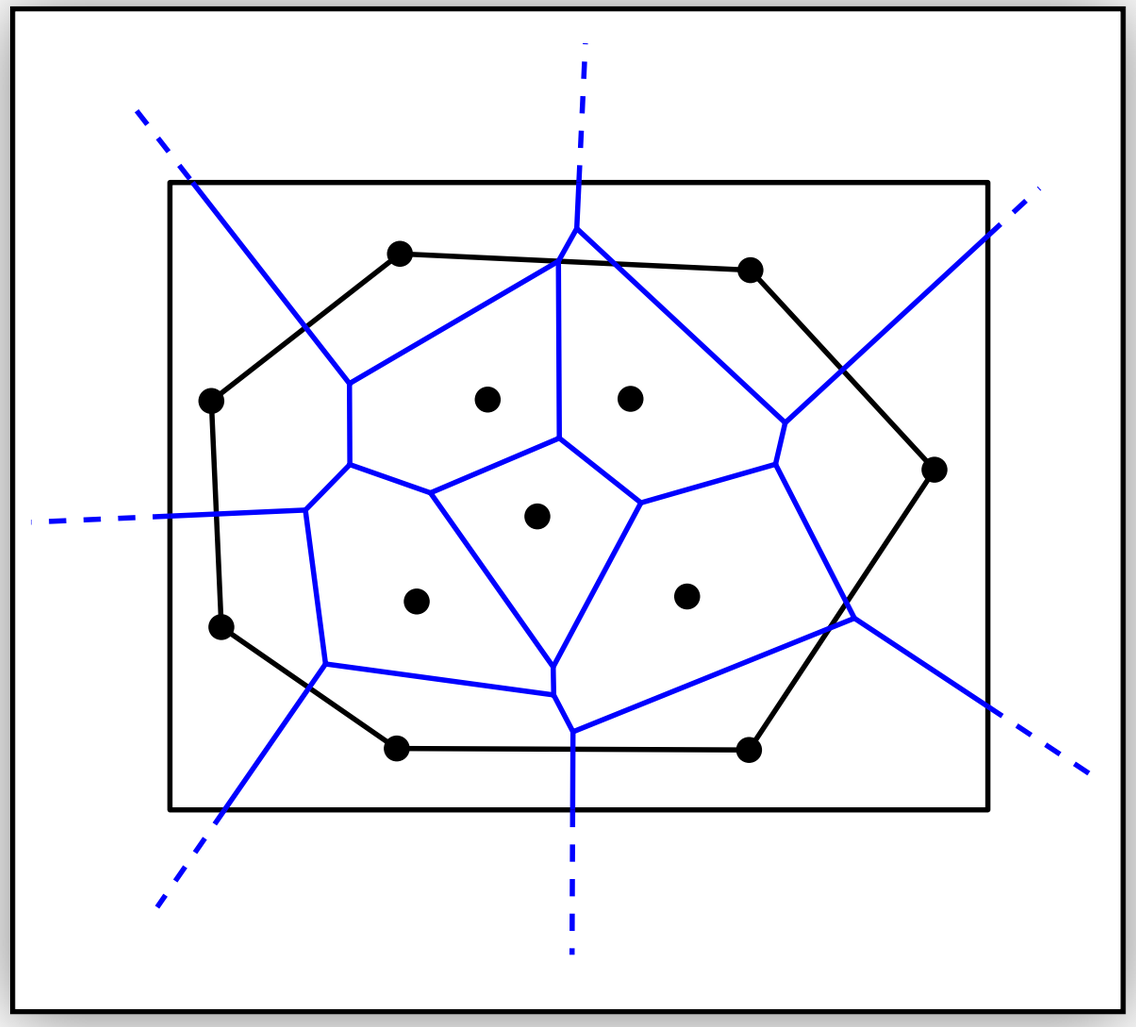

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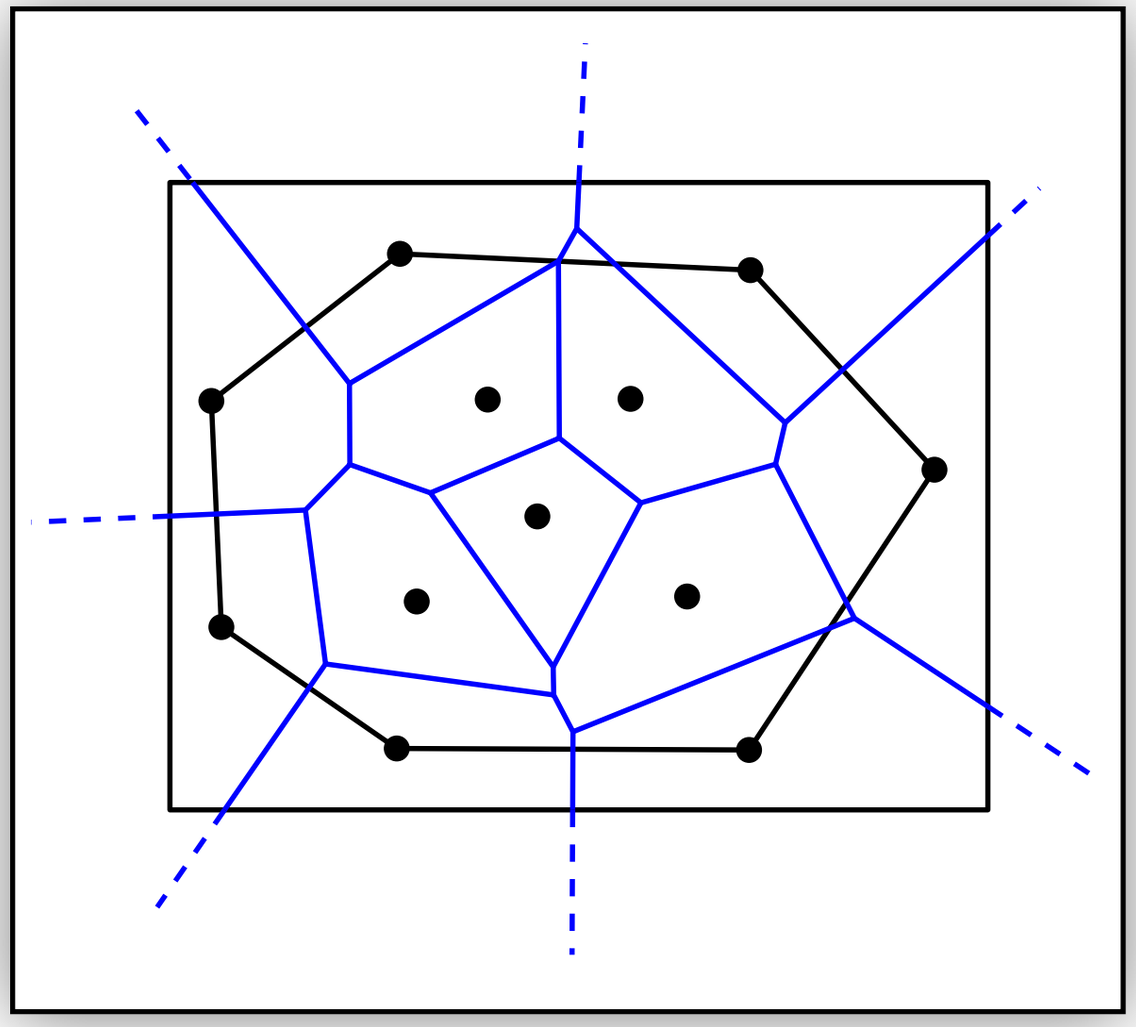

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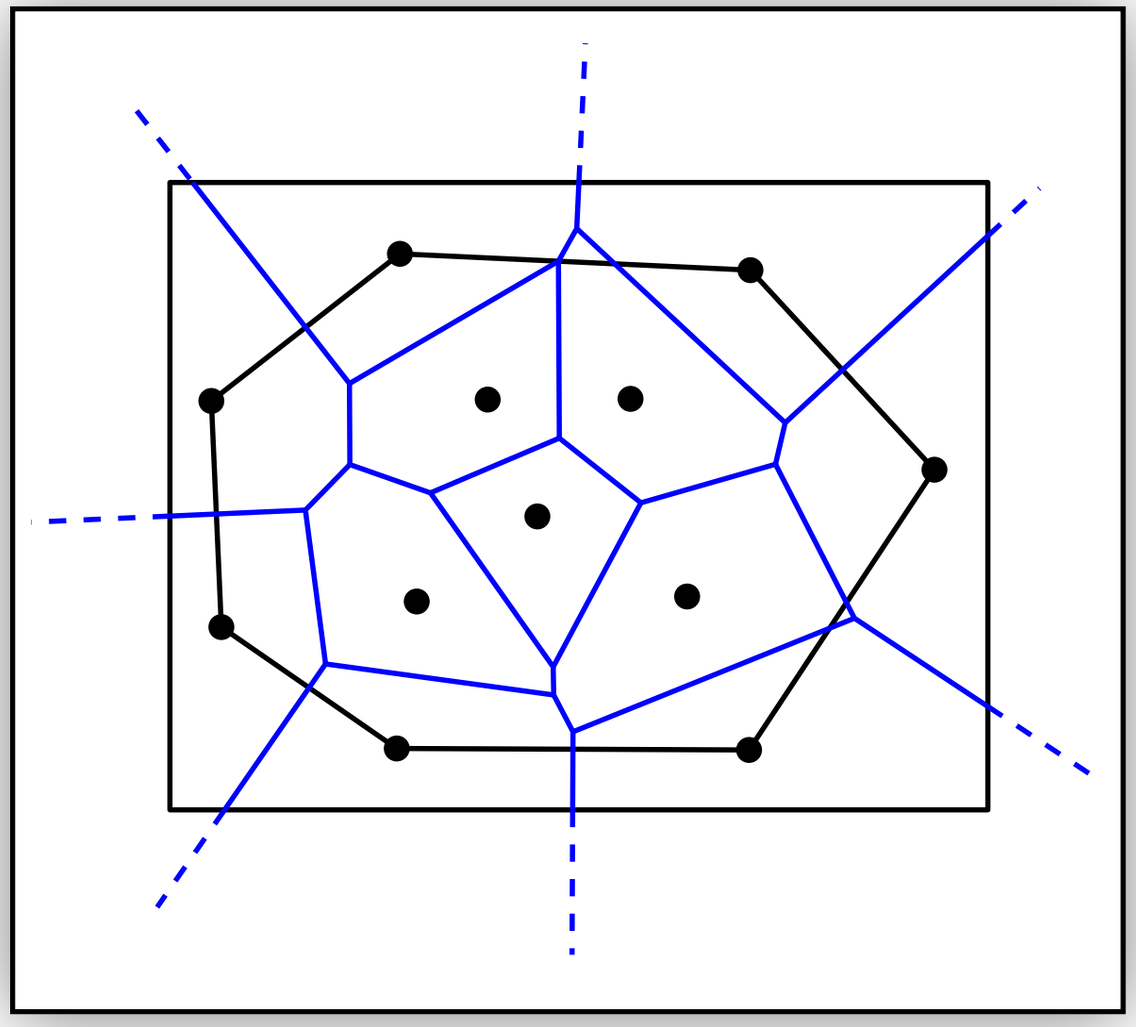

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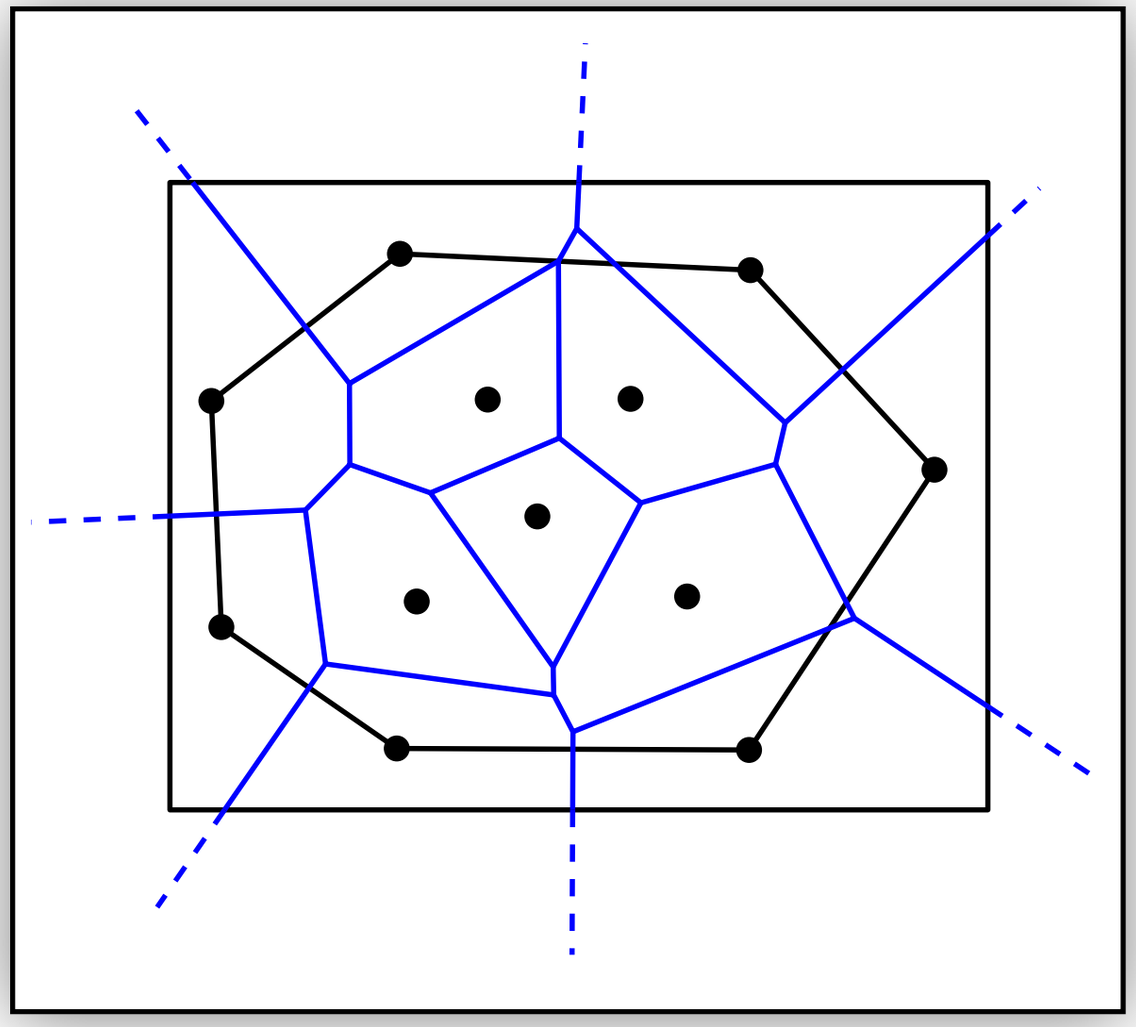
- Constructing $Vor(\mathcal{P})$: second run



Constructing the Voronoi Diagram - II

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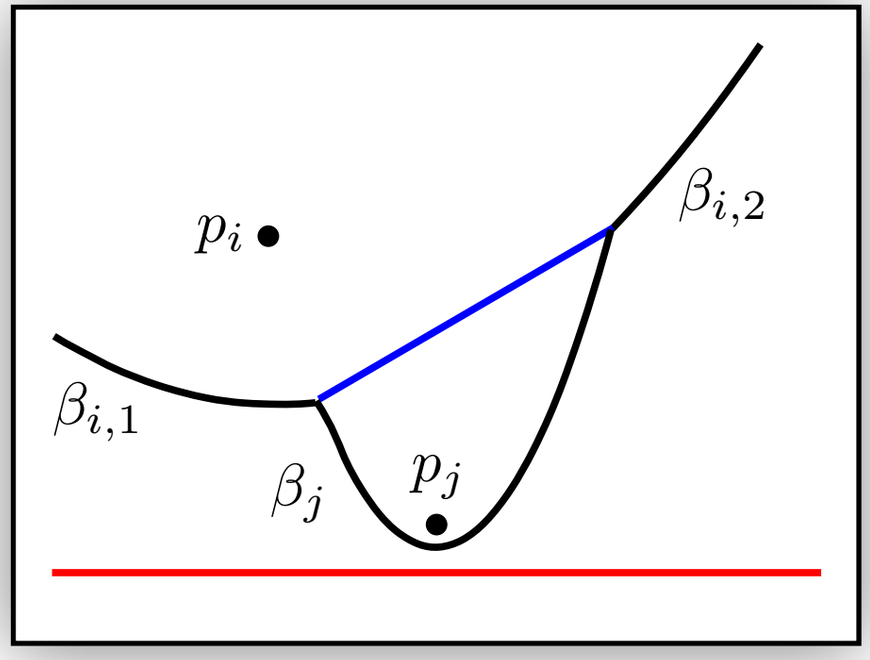

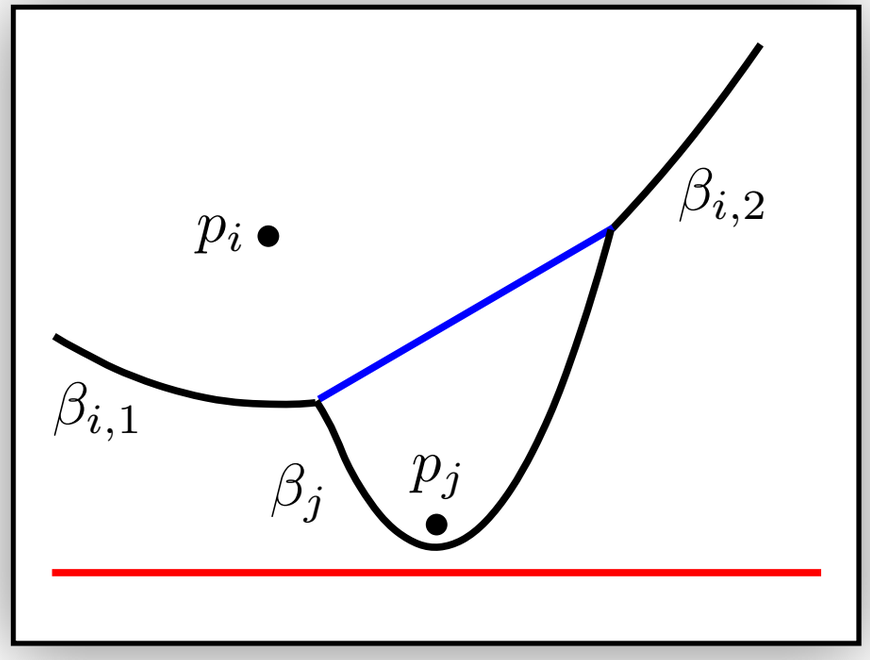

Constructing the DCEL:

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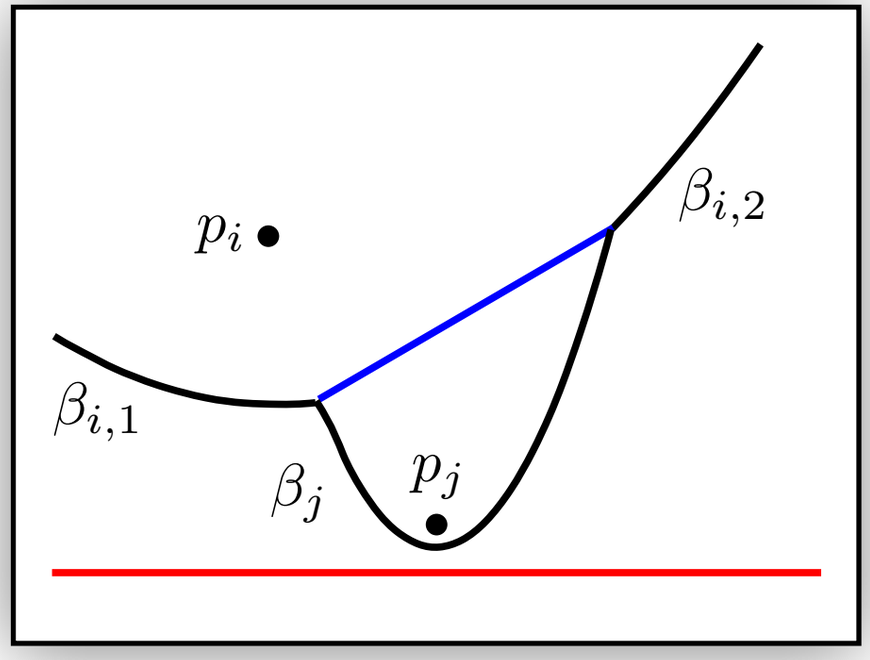

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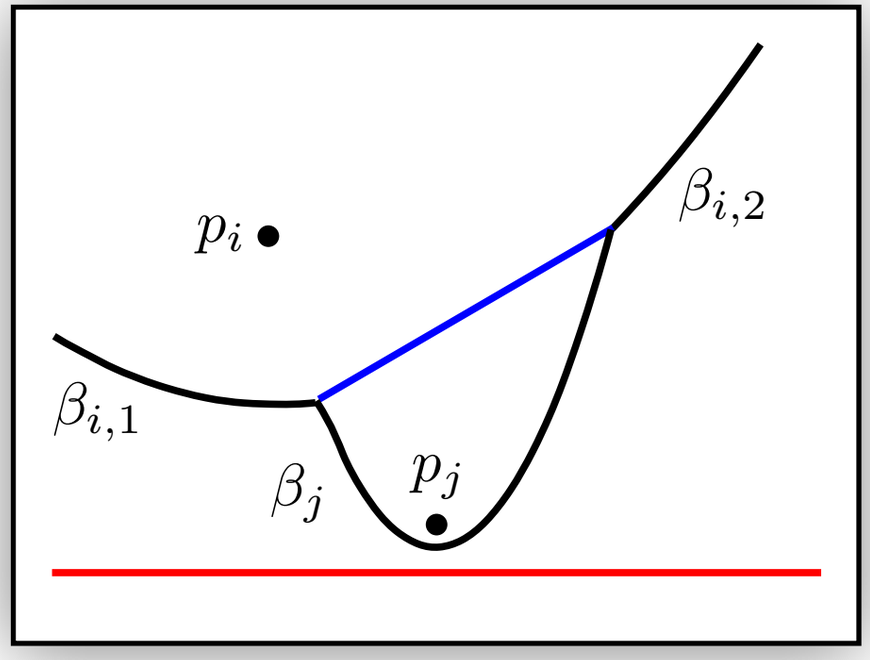

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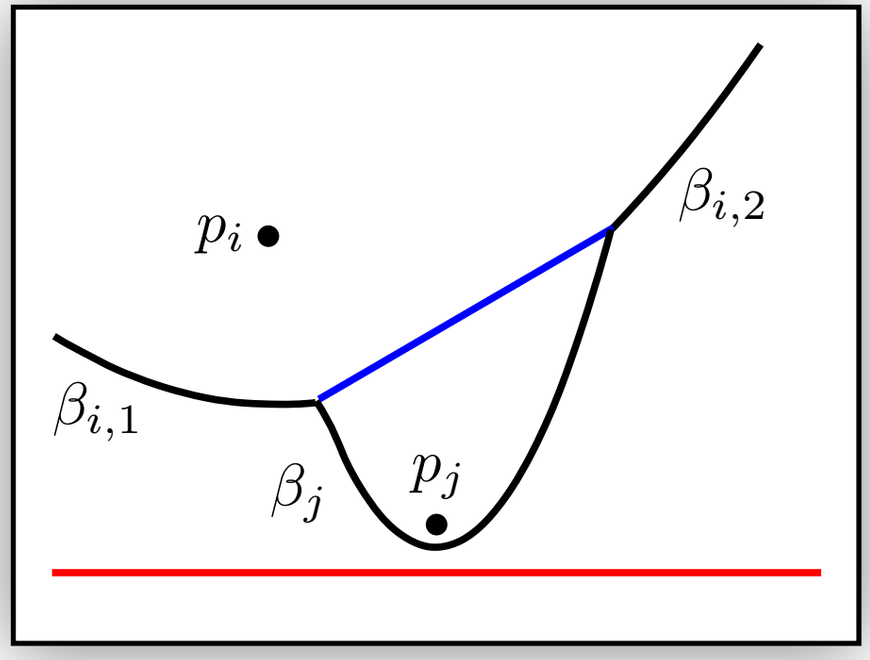

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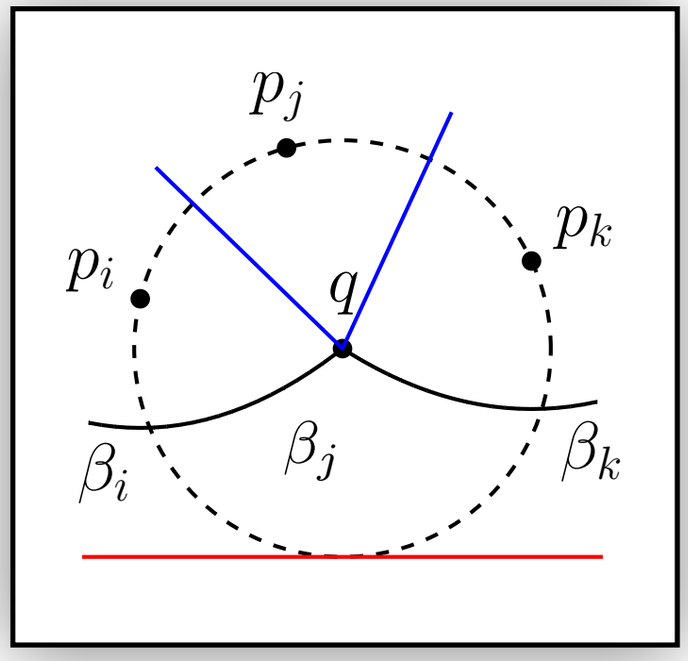

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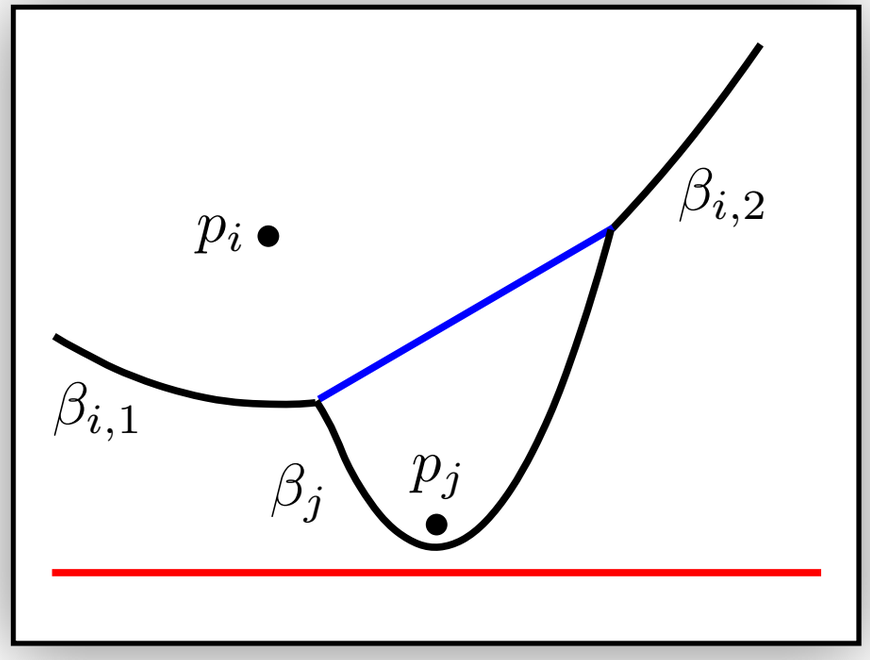

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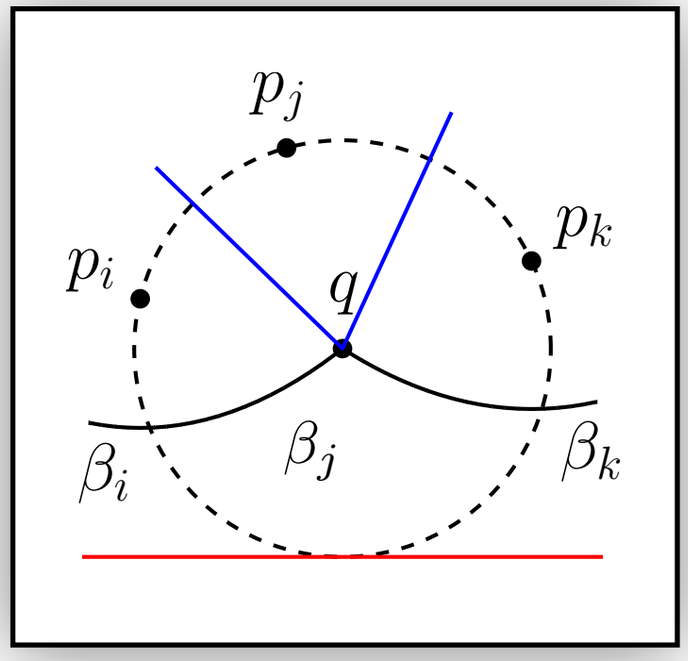
  $\rightarrow$ new edge $e_3 \subset B(p_i, p_k)$ incident to q.



Constructing the Voronoi Diagram - III

Unbounded edges:

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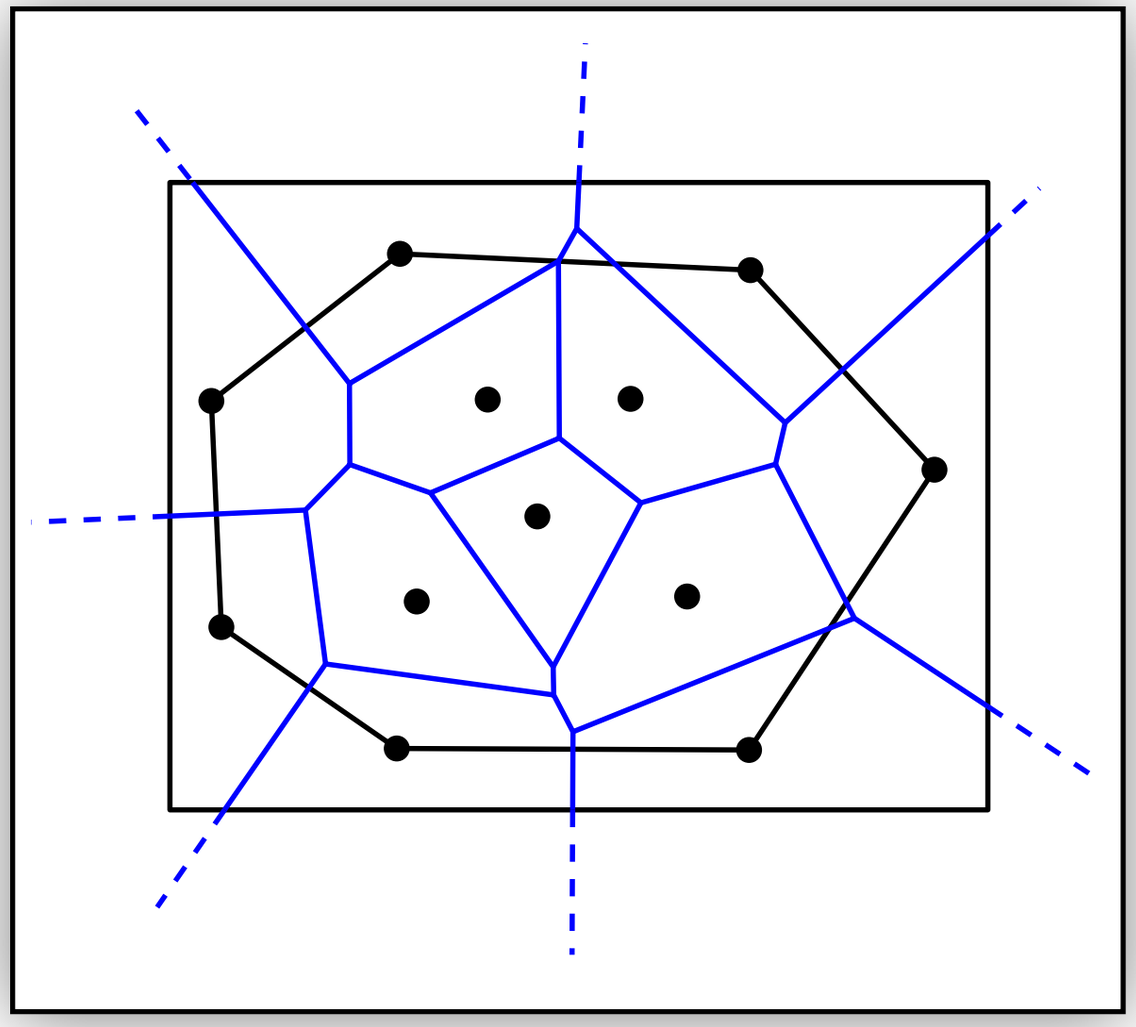
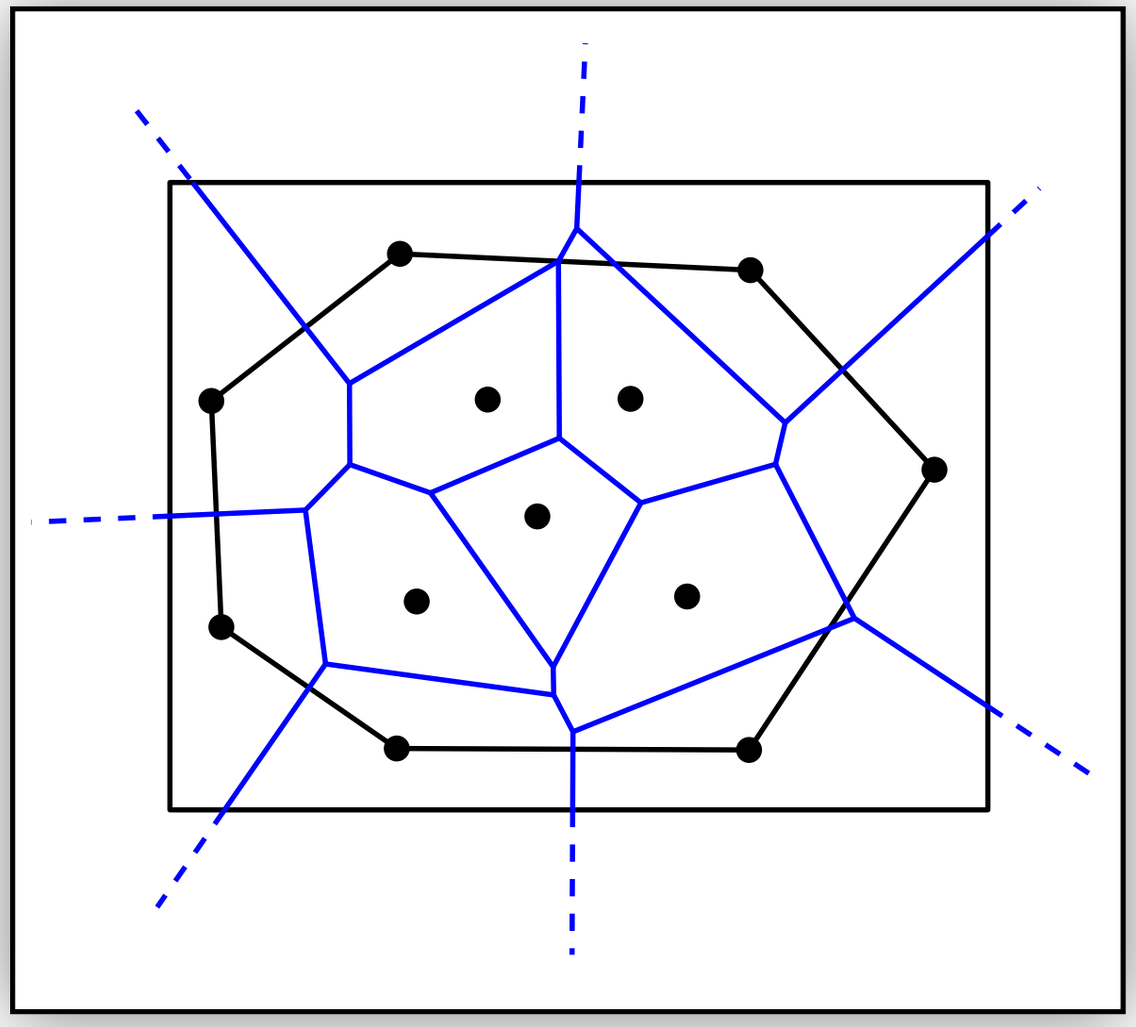


Unbounded edges:

- After last event in Q: $|B| \geq 2$



Unbounded edges:

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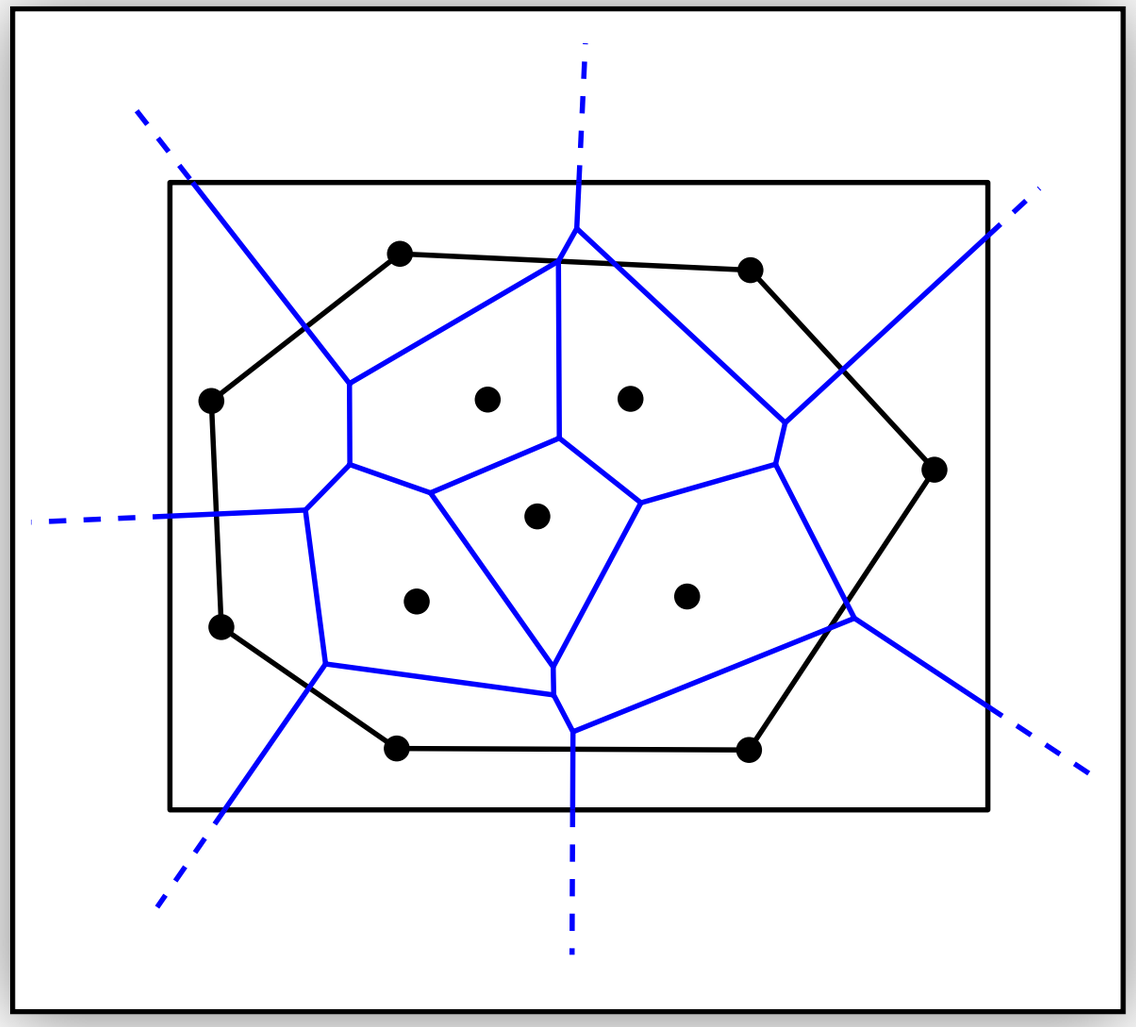
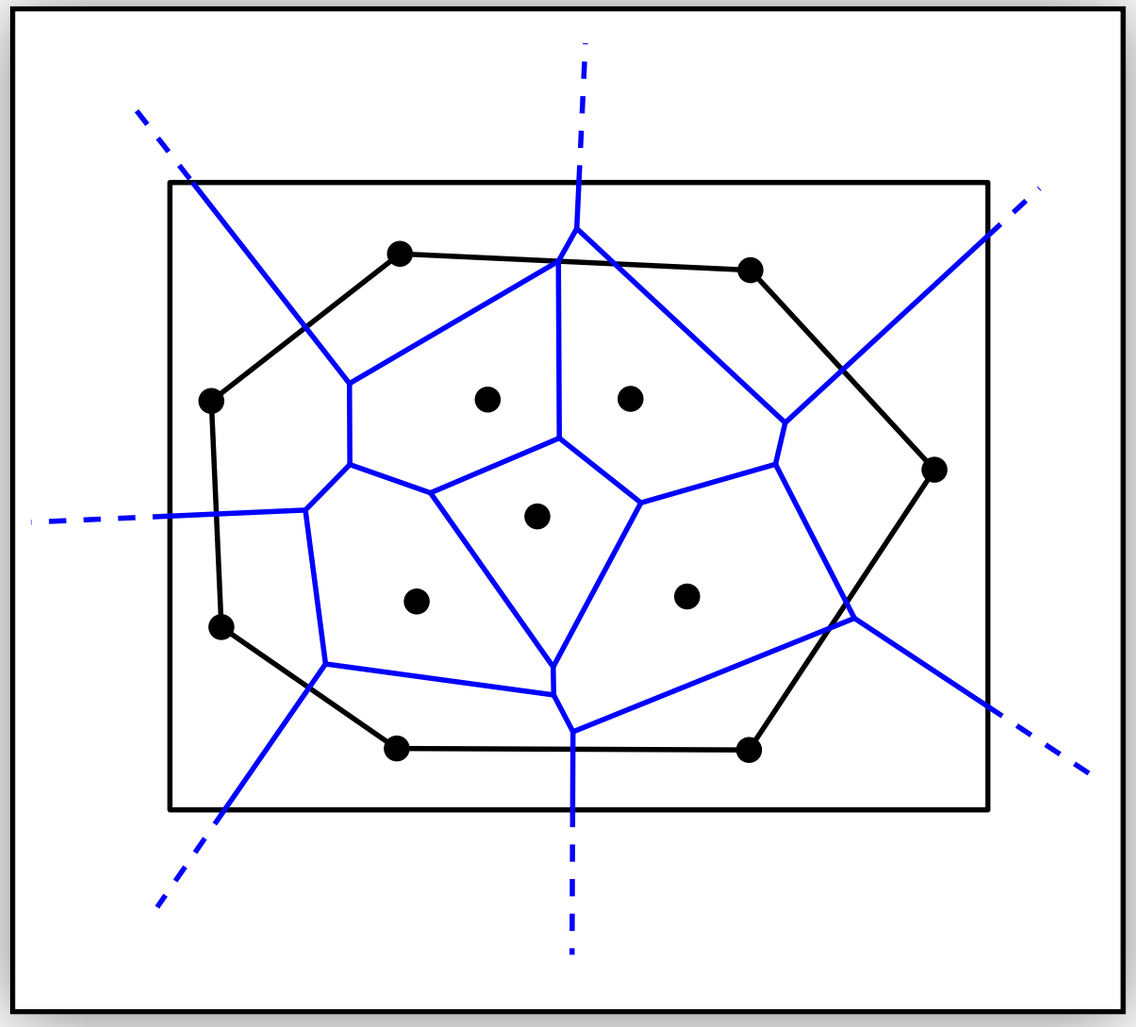


Unbounded edges:

- After last event in Q: $|B| \geq 2$
- $\rightarrow$ Two $p_1, p_2 \in Q$ form unbounded $e \subset B(p_1, p_2)$
- Connect such edges with the bounding box.



Degeneracy

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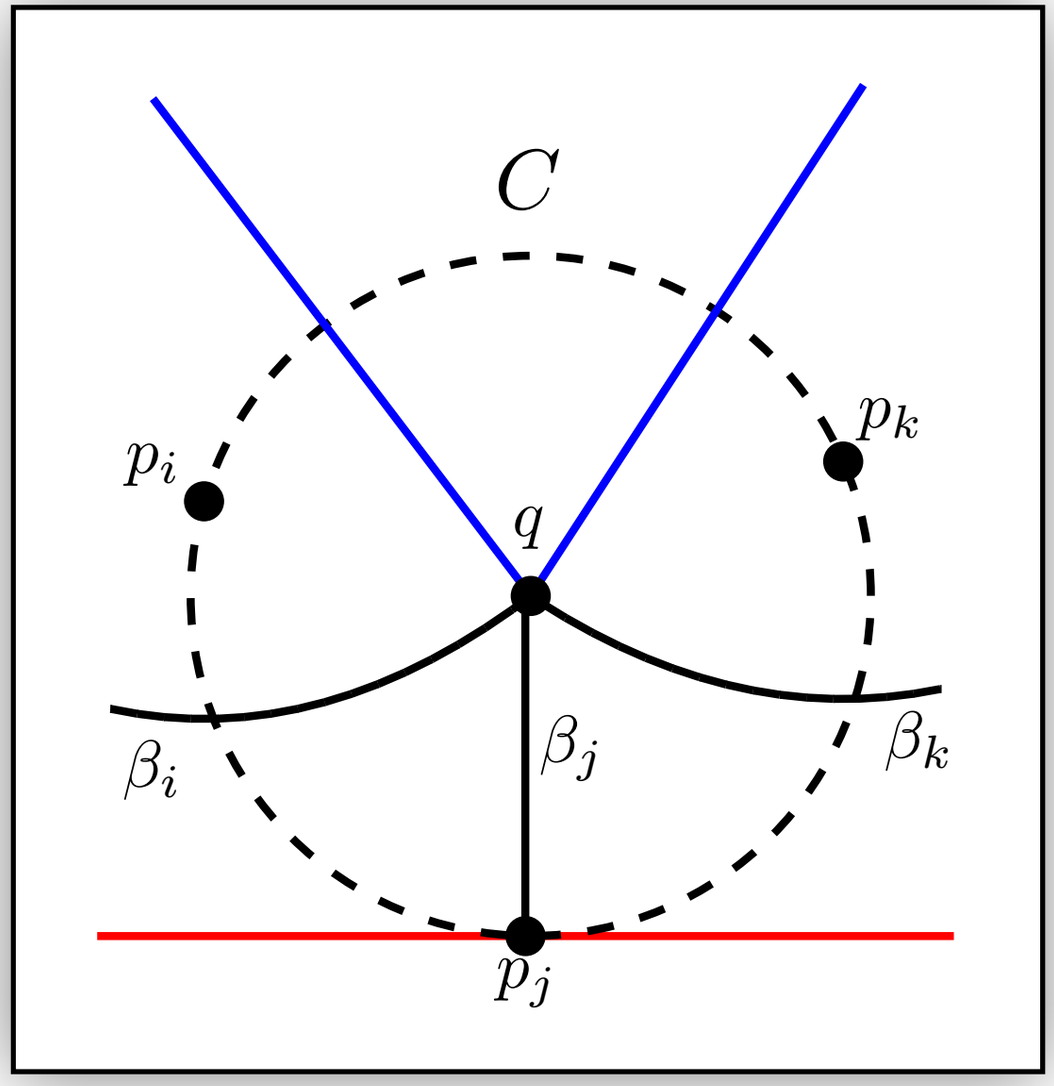
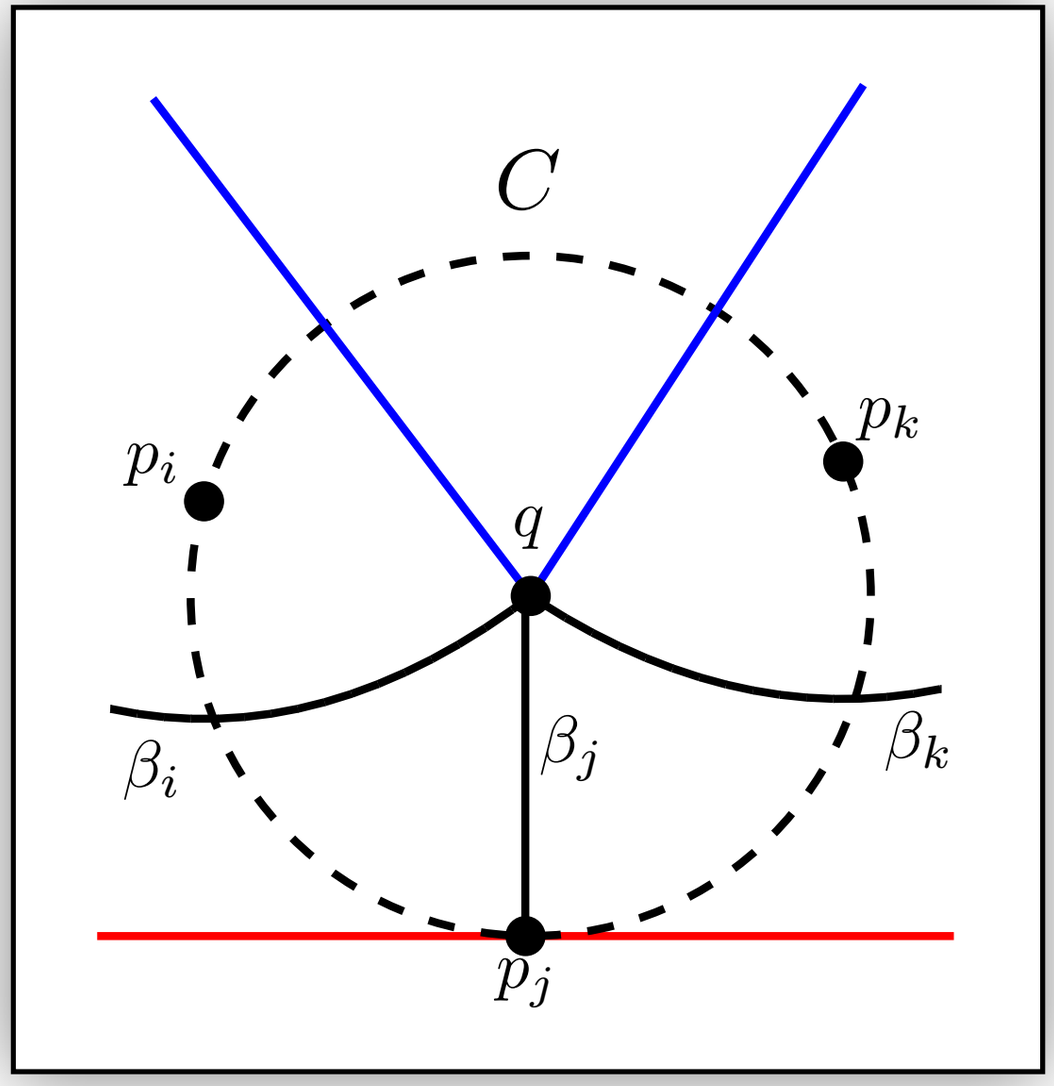


Degenerate situation:

- Point event p_j below q.



Degenerate situation:

- Point event p_j below q.
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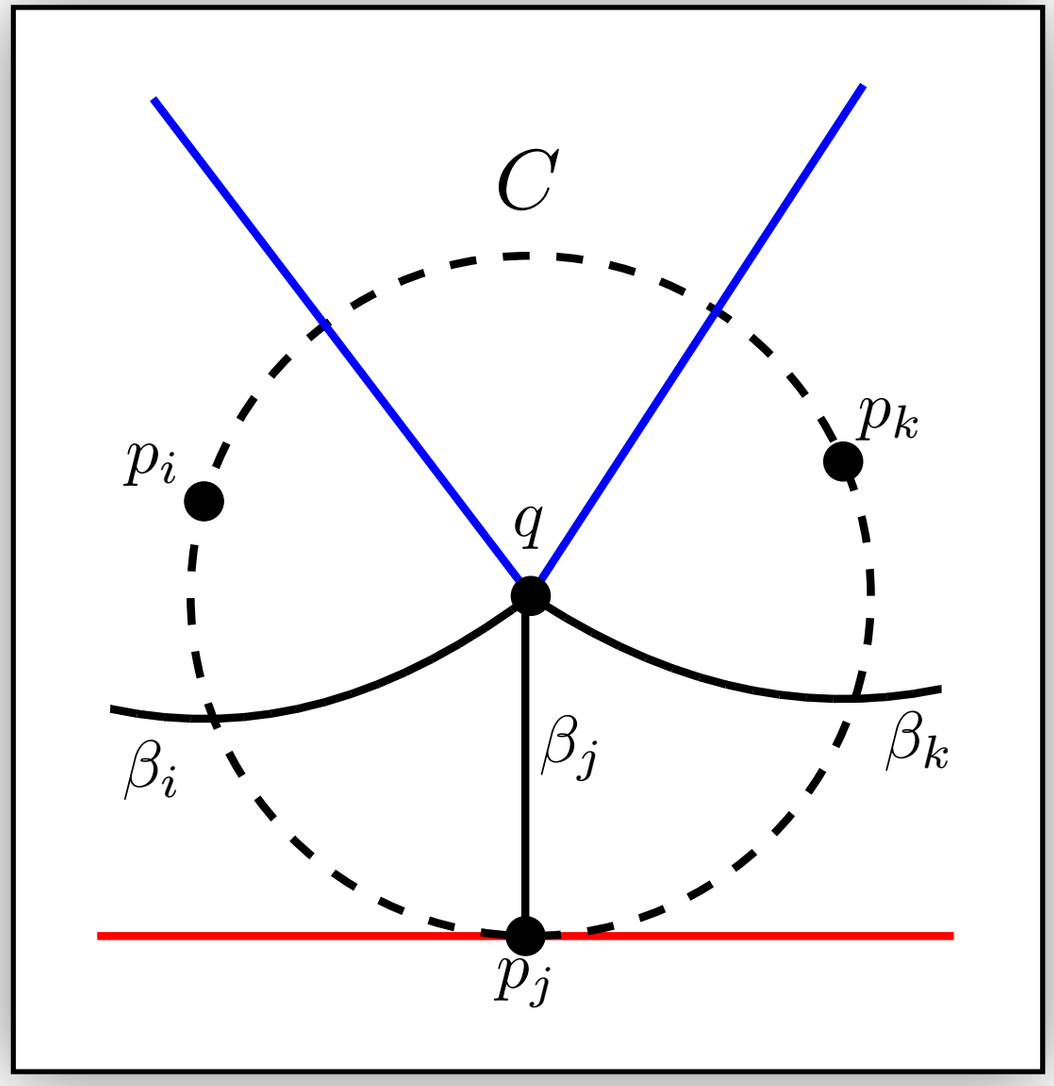

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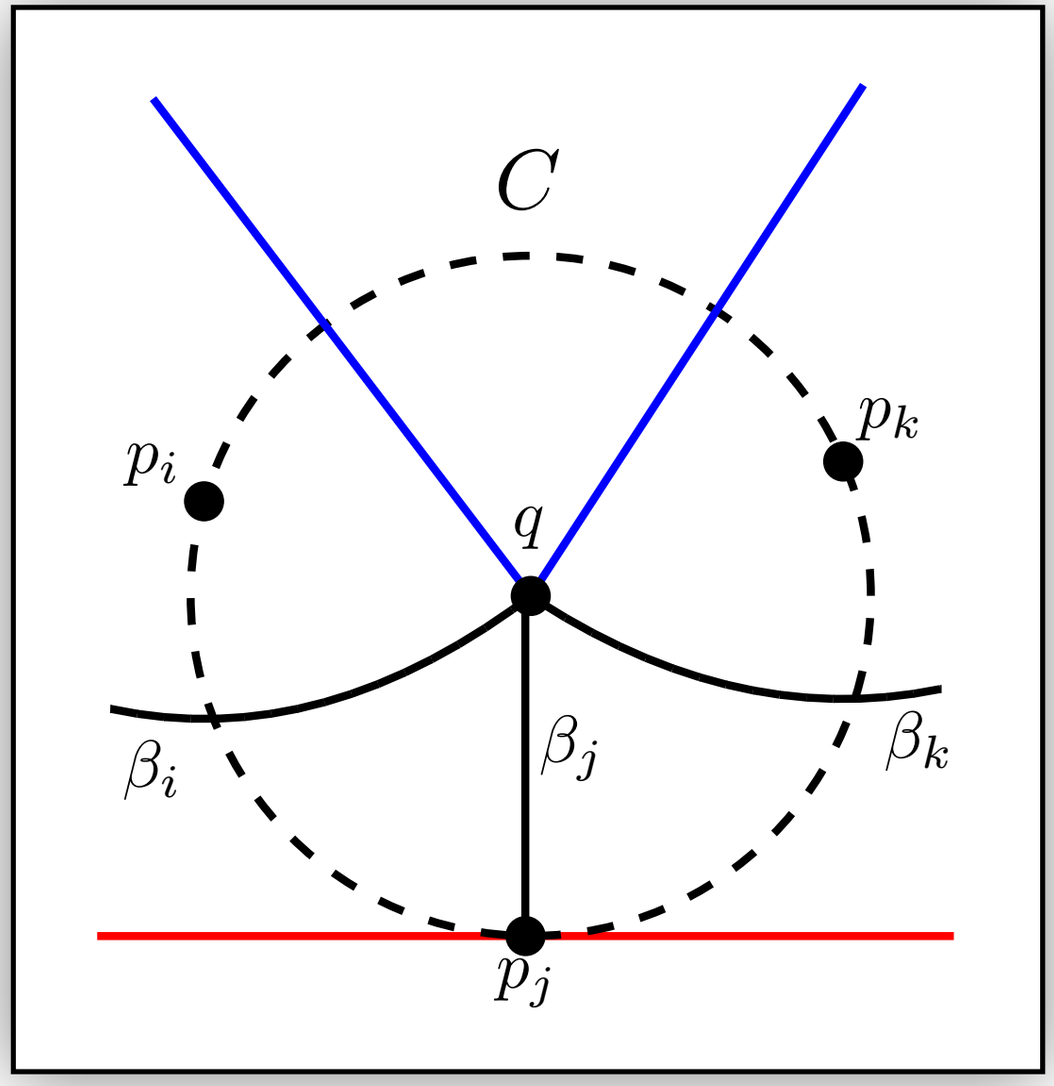

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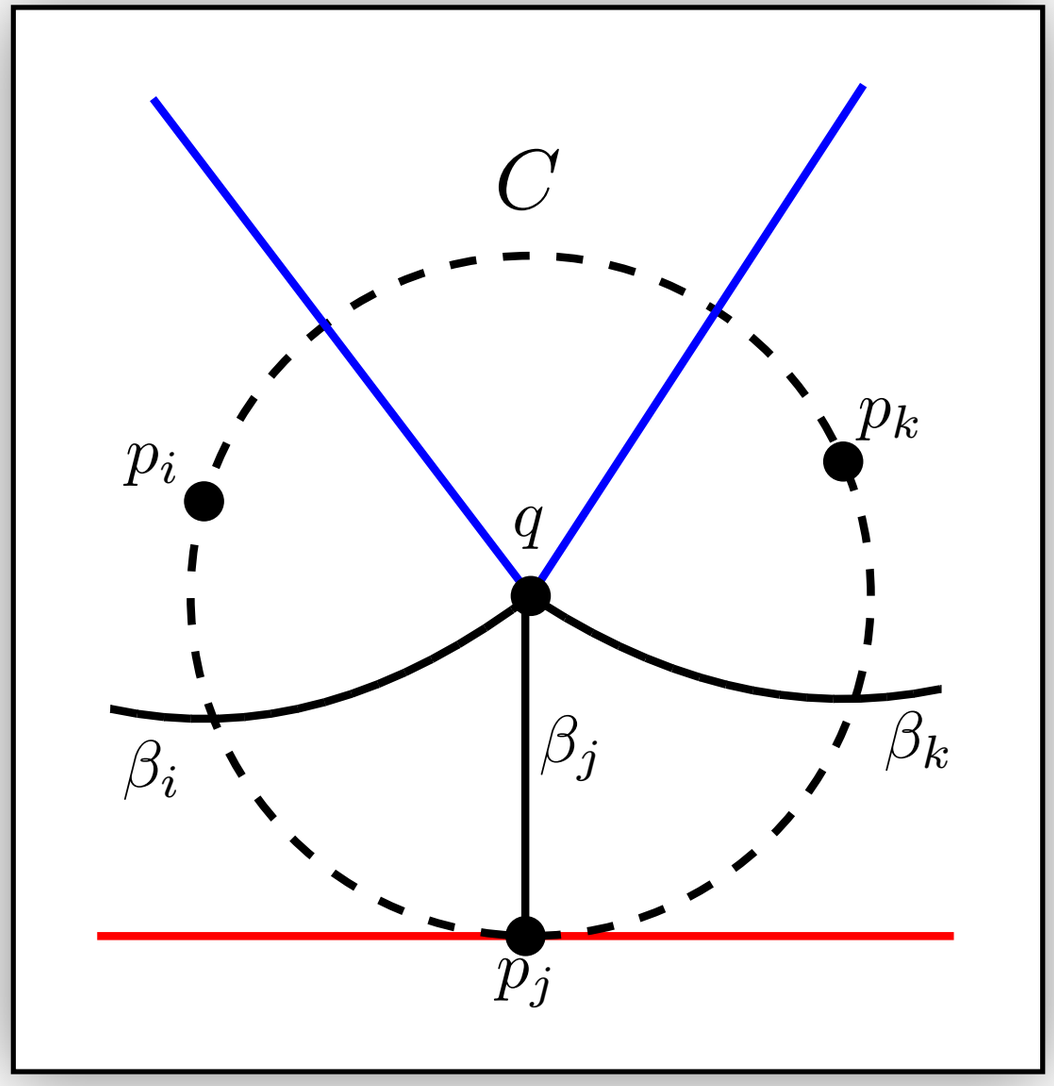
- Generate Voronoi vertex q.



Overview

Algorithm 1: Computation of $V^*(S)$.
Input: S is a set of $n \geq 1$ points with unique bottommost point.
Output: The bisectors and vertices of V^*.
Data structures: Q: a priority queue of points in the plane, ordered lexicographically. Each point is labeled as a site, or labeled as the intersection of a pair of boundaries of a single region. Q may contain duplicate instances of the same point with distinct labels; the ordering of duplicates is irrelevant.
L: a sequence $(r_1, e_1, r_2, \ldots, r_k)$ of regions (labeled by site) and boundaries (labeled by a pair of sites). Note that a region can appear many times on L.

1. initialize Q with all sites
2. $p \leftarrow$ extract_min(Q)
3. $L \leftarrow$ the list containing R_p.
4. **while** Q is not empty **begin**
5. $p \leftarrow$ extract_min(Q)
6. **case**
7. p is a site:
8. Find an occurrence of a region R_q^* on L containing p.
9. Create bisector B_{pq}^*.
10. Update list L so that it contains $\ldots, R_q^*, C_{pq}^-, R_p^*, C_{pq}^+, R_q^*, \ldots$ in place of R_q^*.
11. Delete from Q the intersection between the left and right boundary of R_q^*, if any.
12. Insert into Q the intersection between C_{pq}^- and its neighbor to the left on L, if any, and the intersection between C_{pq}^+ and its neighbor to the right, if any.
13. p is an intersection:
14. Let p be the intersection of boundaries C_{qr} and C_{rs}.
15. Create the bisector B_{qs}^*.
16. Update list L so it contains $C_{qs} = C_{qs}^-$ or C_{qs}^+, as appropriate, instead of C_{qr}, R_r^*, C_{rs}.
17. Delete from Q any intersection between C_{qr} and its neighbor to the left and between C_{rs} and its neighbor to the right.
18. Insert any intersections between C_{qs} and its neighbors to the left or right into Q.
19. Mark p as a vertex and as an endpoint of B_{qr}^*, B_{rs}^*, and B_{qs}^*.
20. **end**

Fig. 2.4. Algorithm 1: computation of $V^*(s)$.

Analysis

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- x-structure B, event queue Q: cost of $\mathcal{O}(\log n)$ per operation

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Theorem 4.23

Fortune's algorithm computes the Voronoi diagram of n points in time $\Theta(n \log n)$.

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  $\mathcal{O}(1)$ new circle events)

Theorem 4.23

Fortune's algorithm computes the Voronoi diagram of n points in time $\Theta(n \log n)$.

THEOREM 2.8. *Algorithm 1 can be implemented to run in time $O(n \log n)$ and space $O(n)$.*

Steve’s Questions

Steve's Questions

Steven Fortune 4. Dezember 2020 um 14:48

Aw: Computational Geometry - video message?

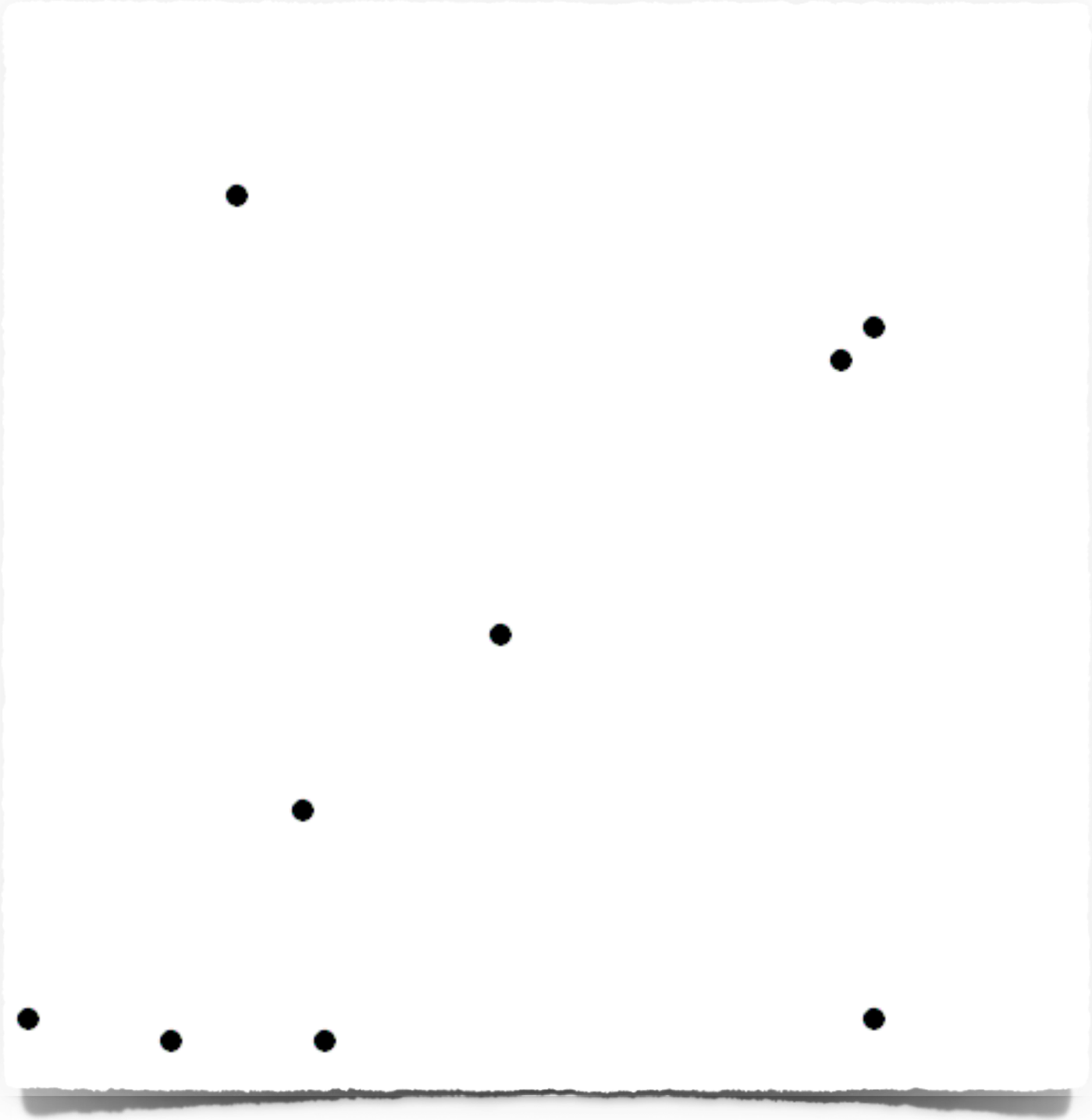
An: Sándor Fekete,

Umgeleitet von: fekete@tu-braunschweig.de

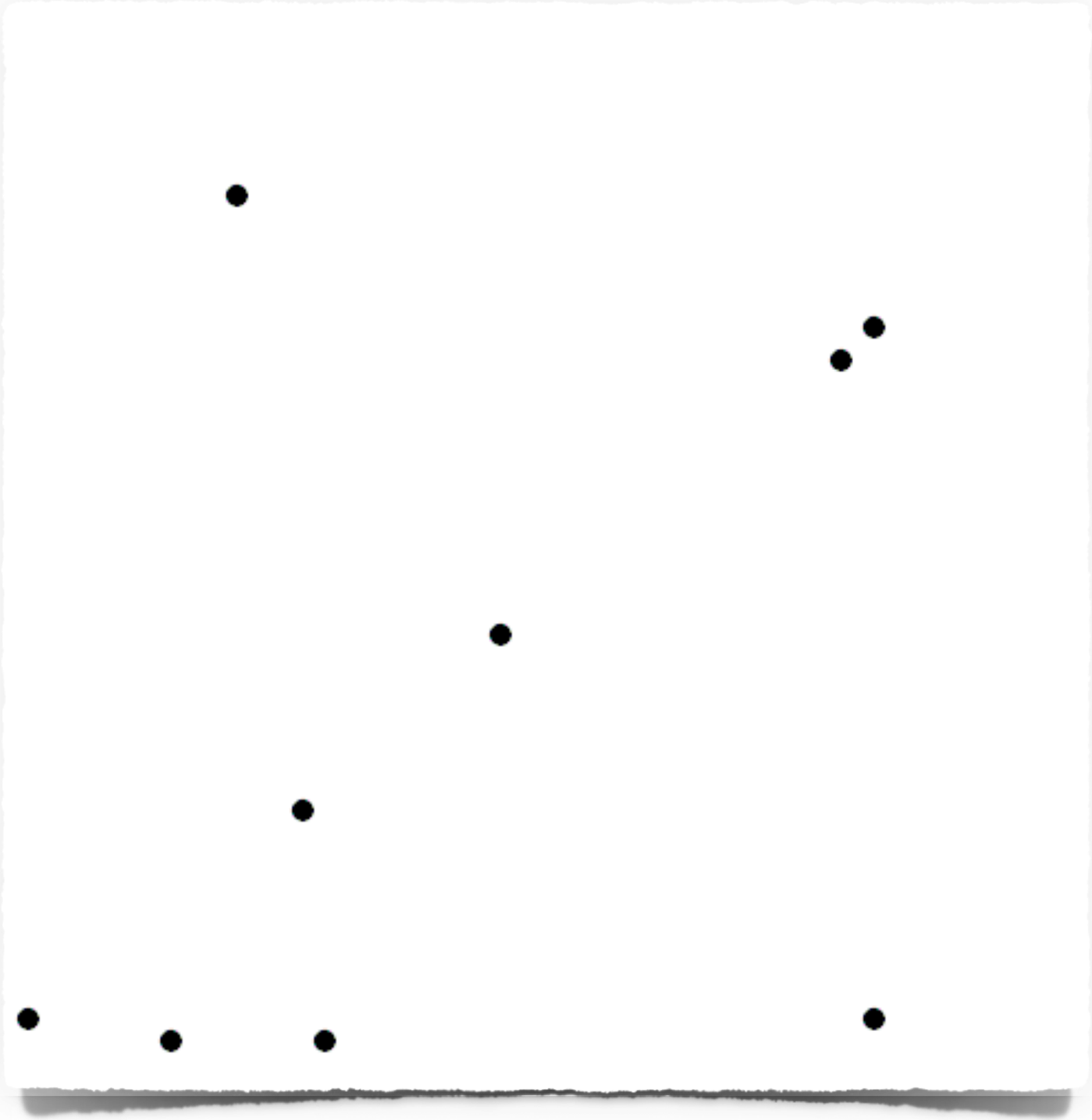
The ways I know:

1. original paper, transforming the plane
2. beach line (maintaining the "known" part of the VD)
3. sweep line maintains the top of circles
4. in 3d, sweep a plane at 45 degrees to xy, watch it intersect 45 degree cones centered at siteas

Steve’s Questions



Steve’s Questions



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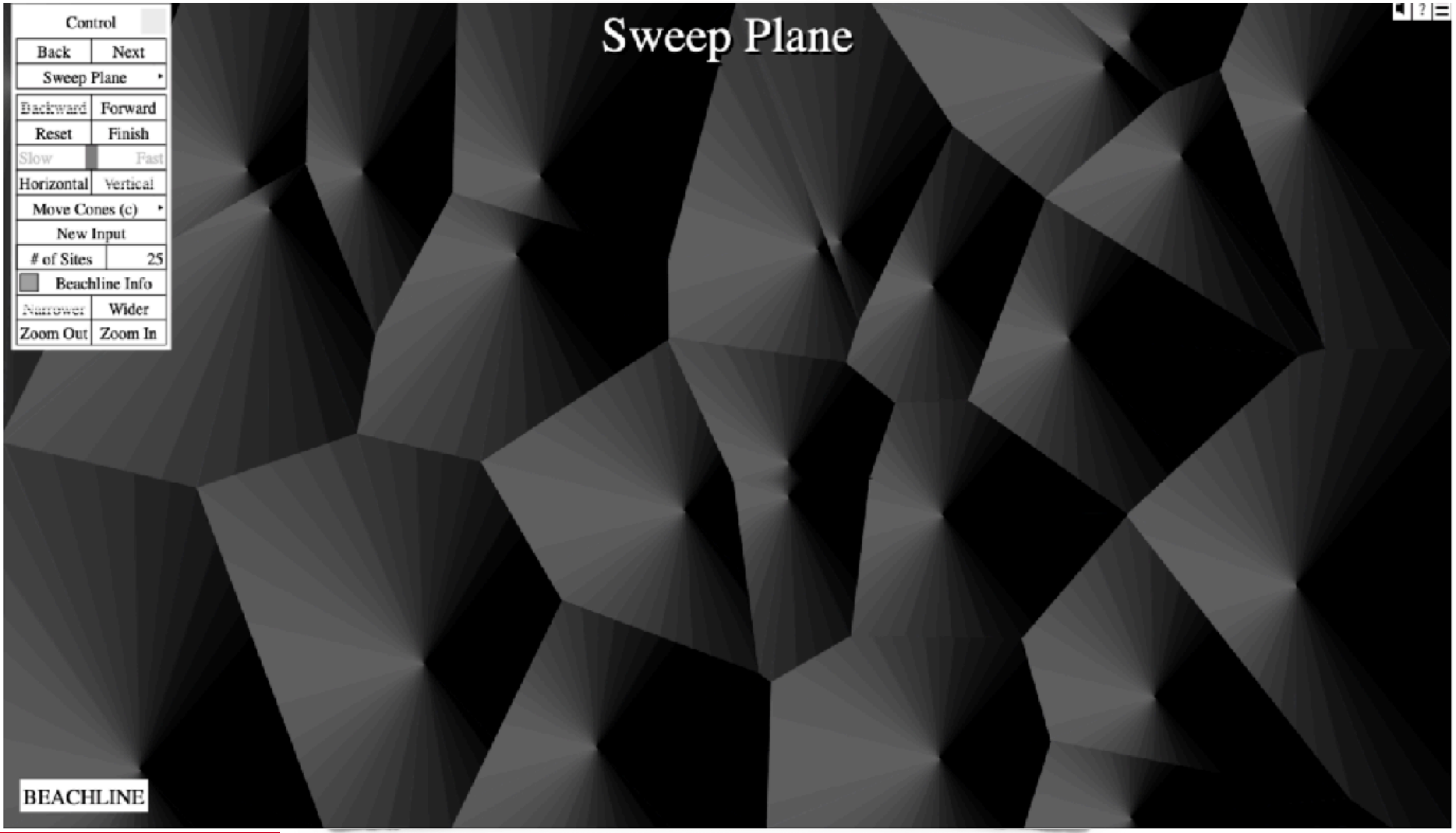

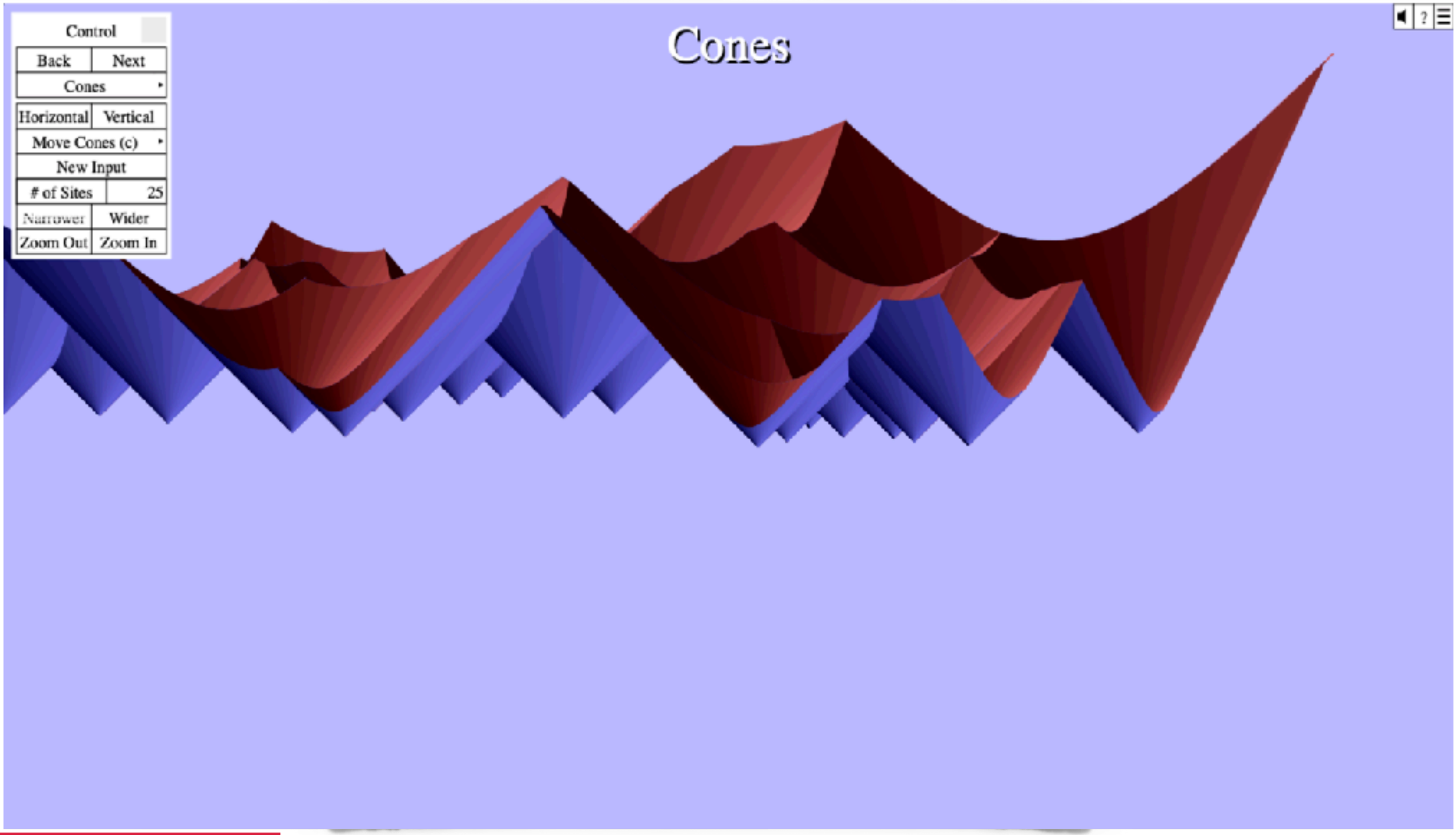
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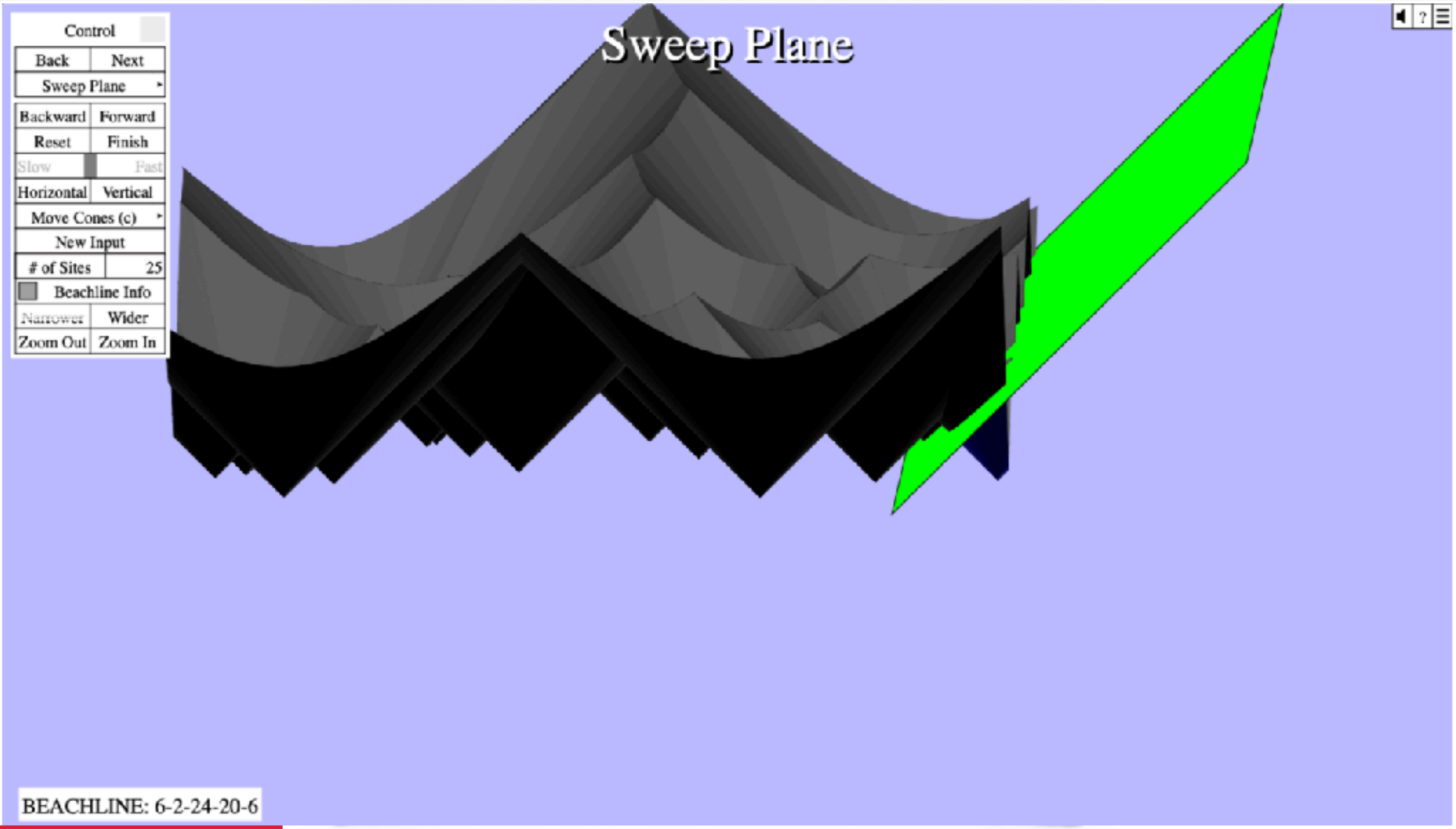
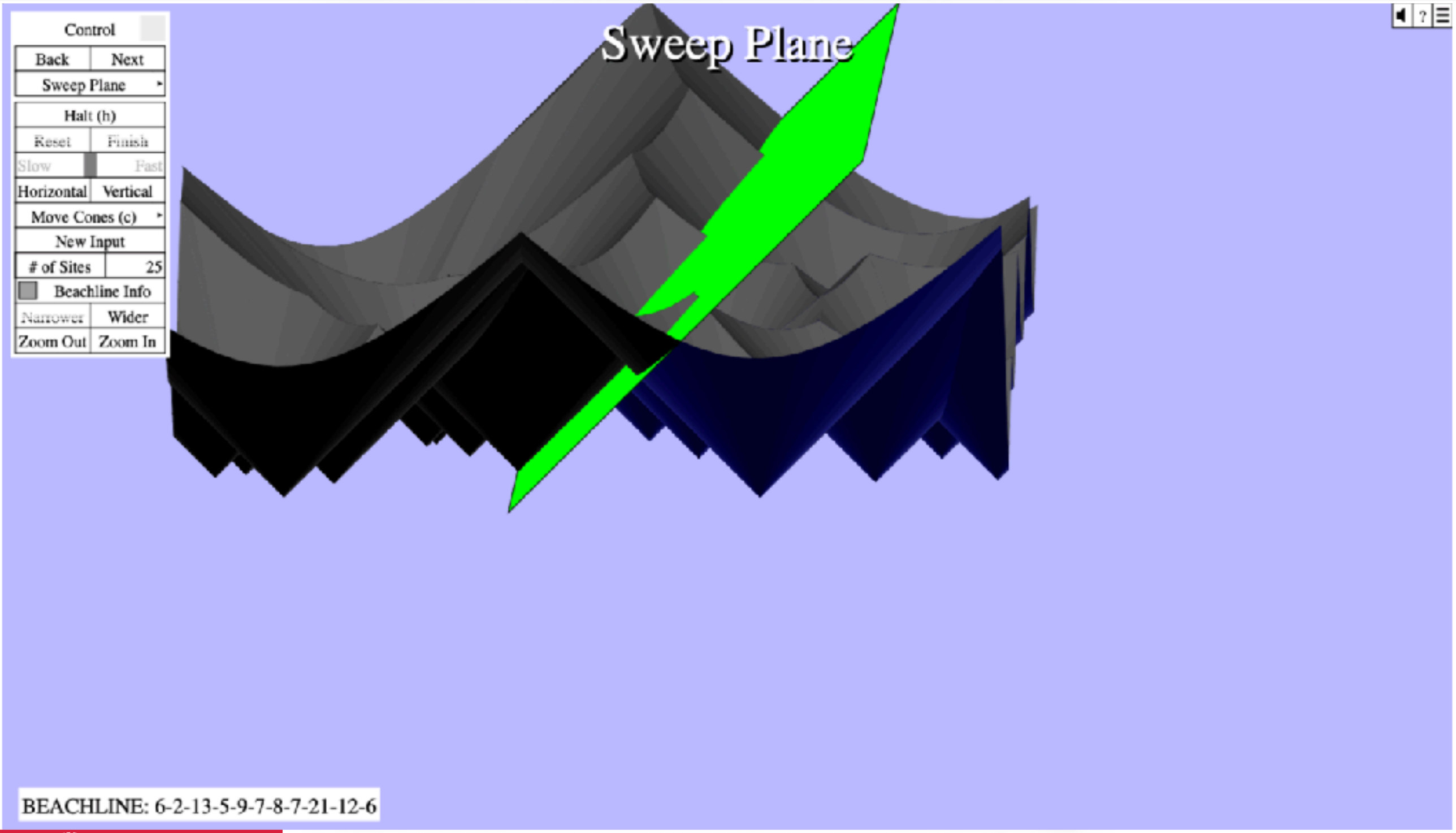
Steve's Questions

Steve's Questions



Steve's Questions



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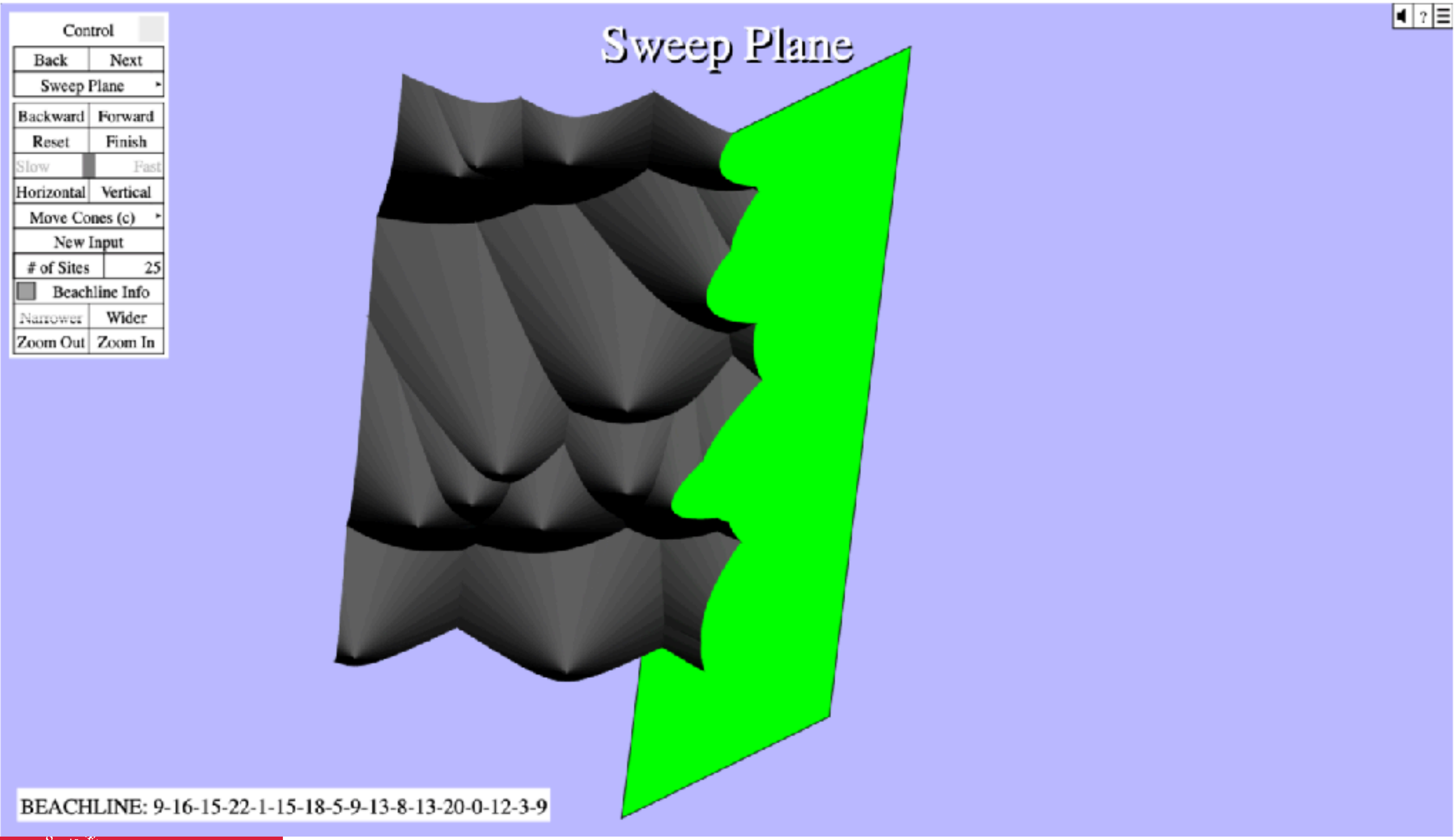

1. Introduction and Motivation
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Higher-order Voronoi diagrams:

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- Voronoi region := point set with same set of k nearest neighbors

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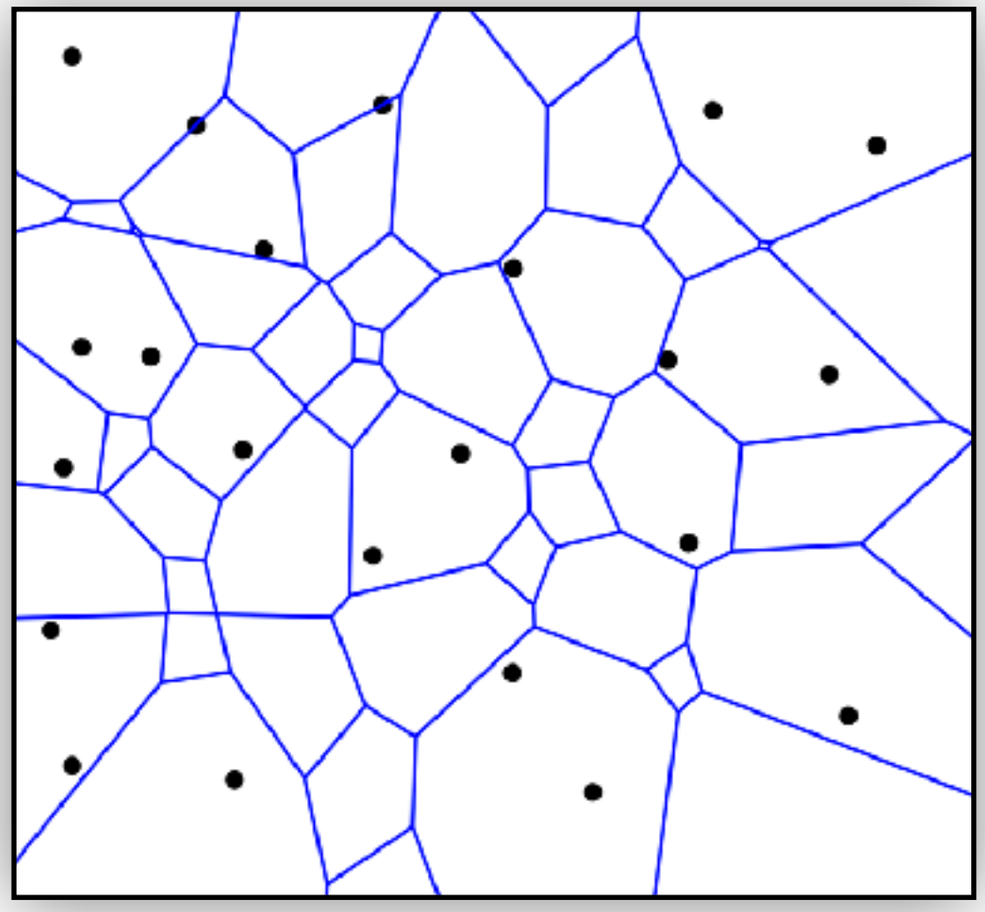

$k = 2$

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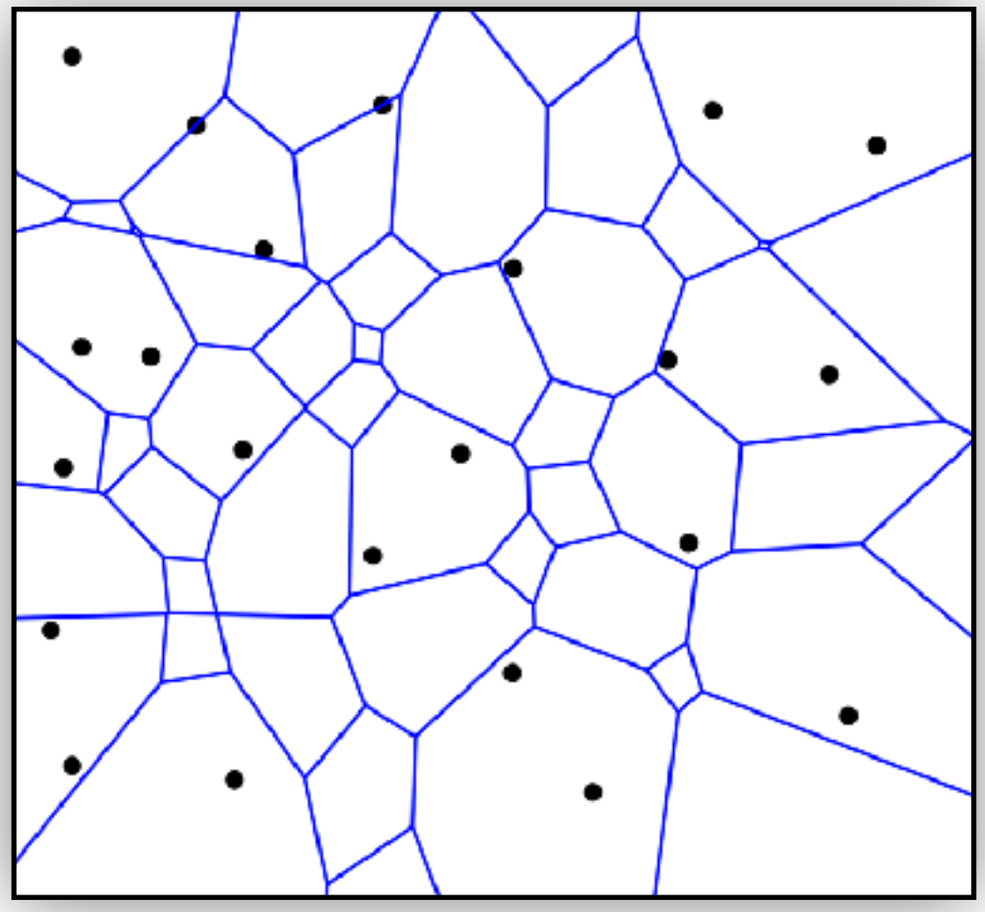

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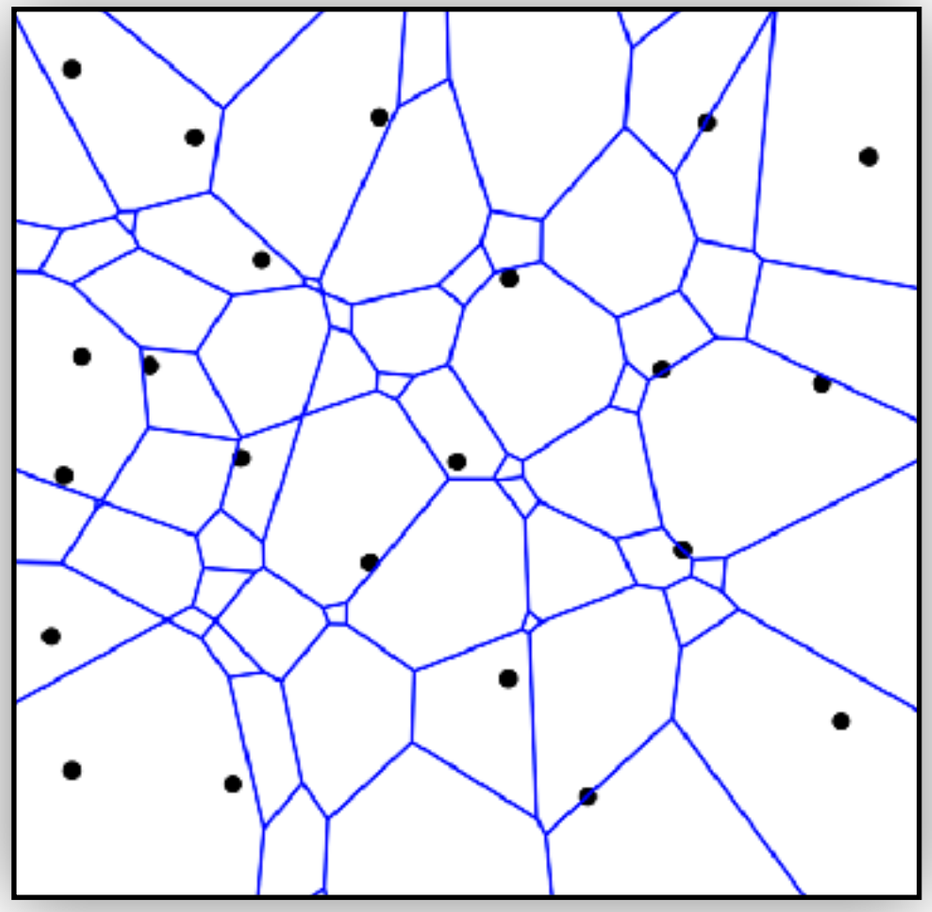

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Higher-order Voronoi diagrams:

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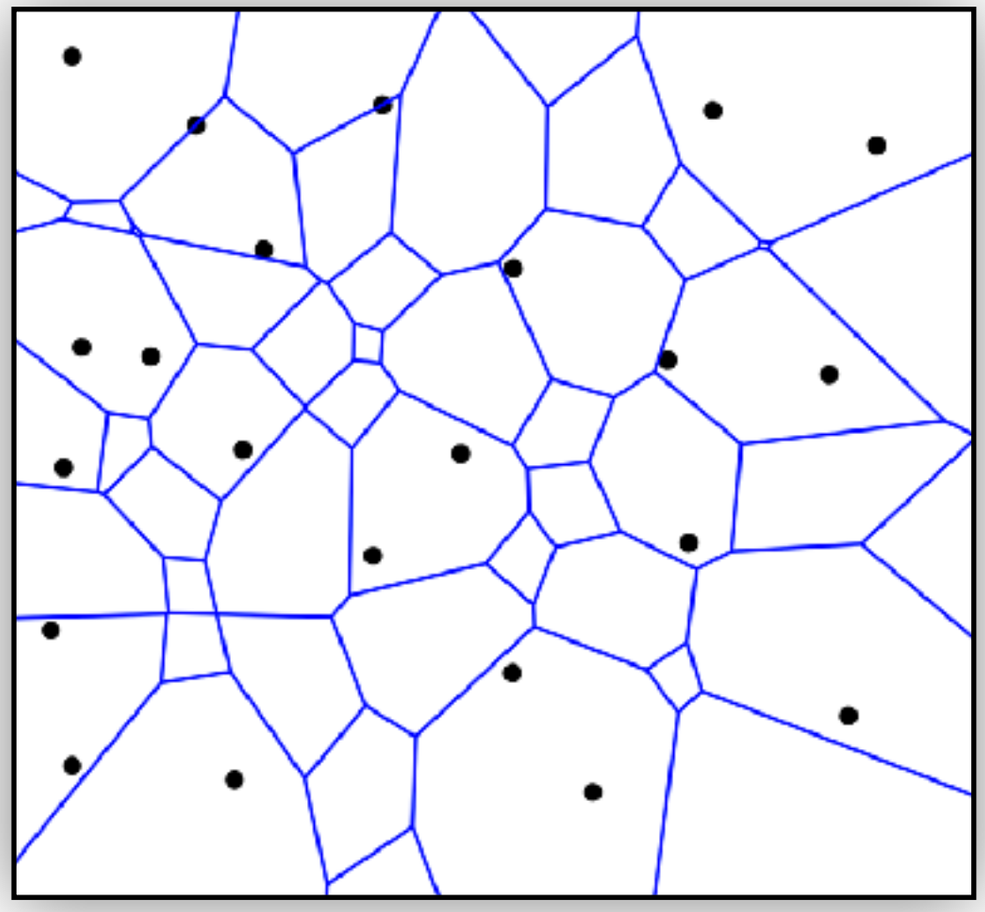

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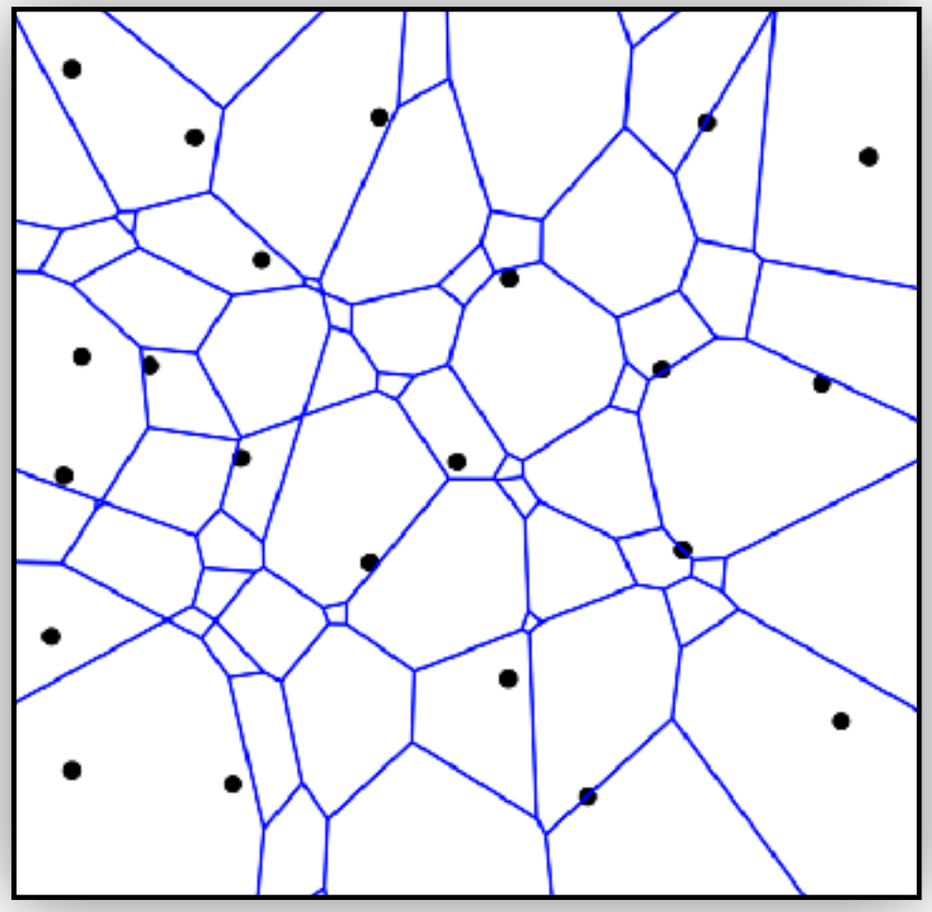
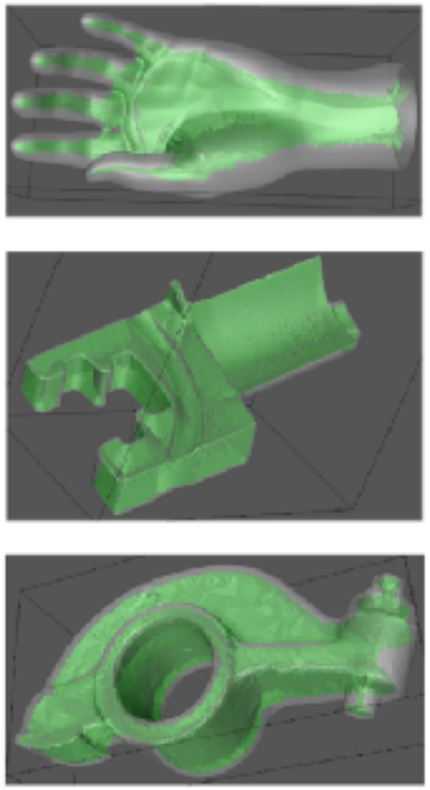


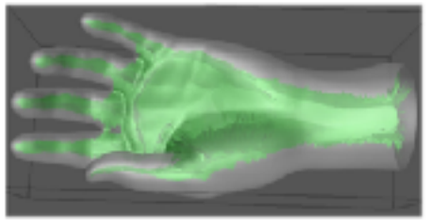
$k = 3$

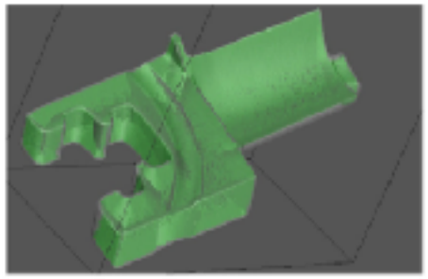
- *Worst-Case*-optimal: $\mathcal{O}(n^3)$ for arbitrary (but fixed) $k \geq 2$
  [Edelsbrunner and Seidel, 1986].

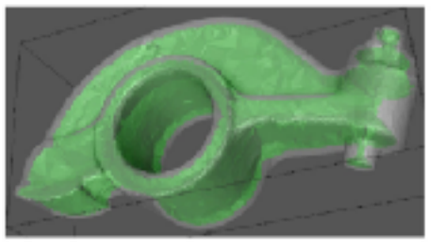
Medial Axis



Medial axis of a simple polygon







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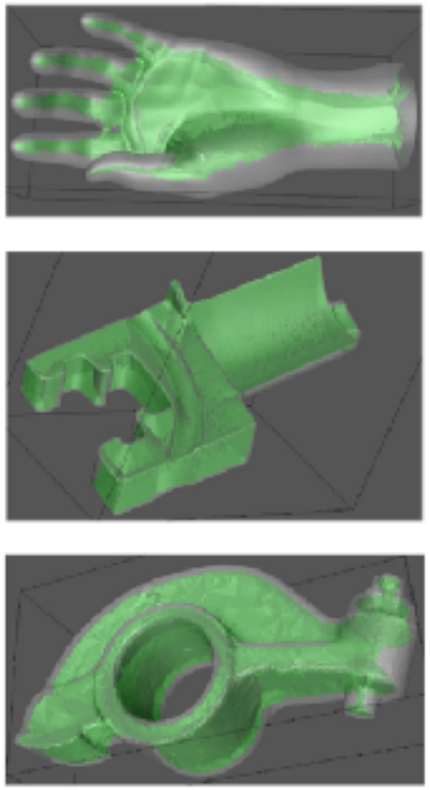

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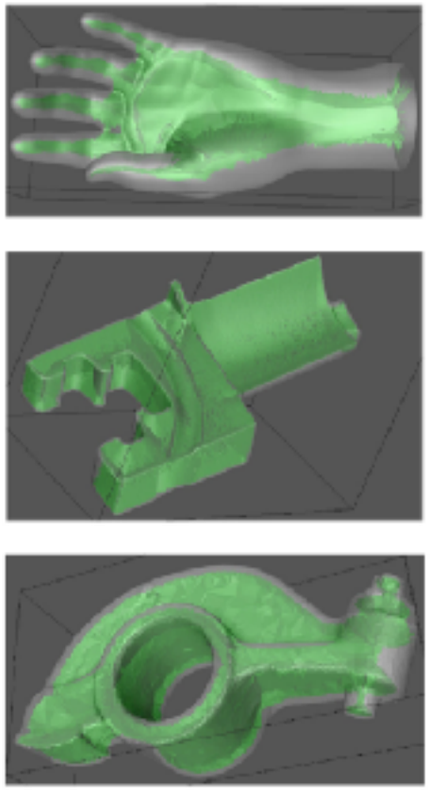

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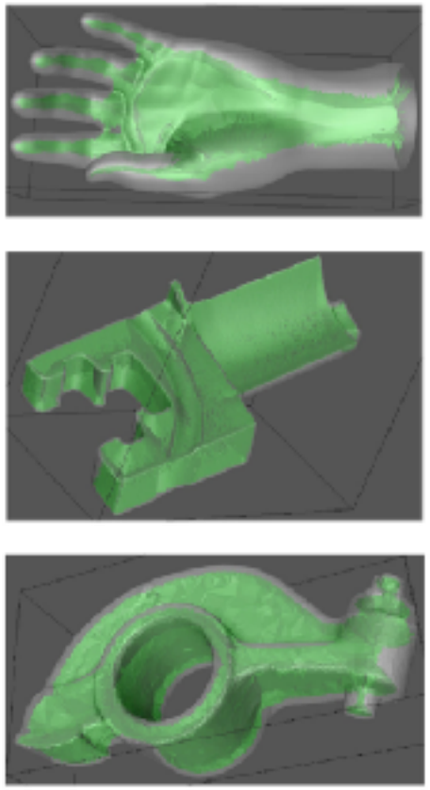

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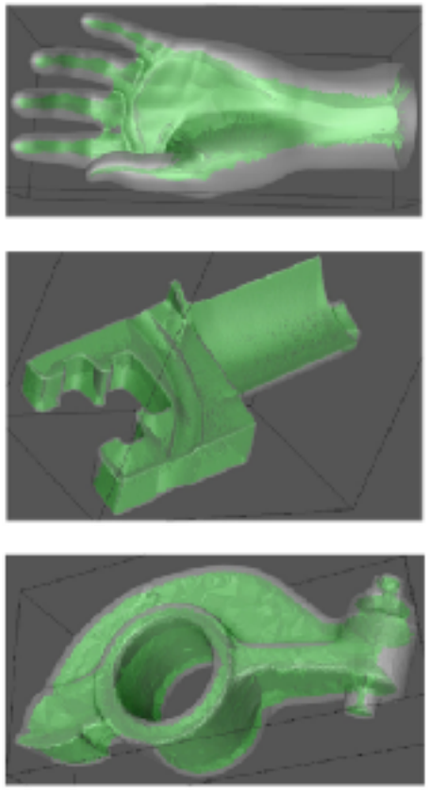

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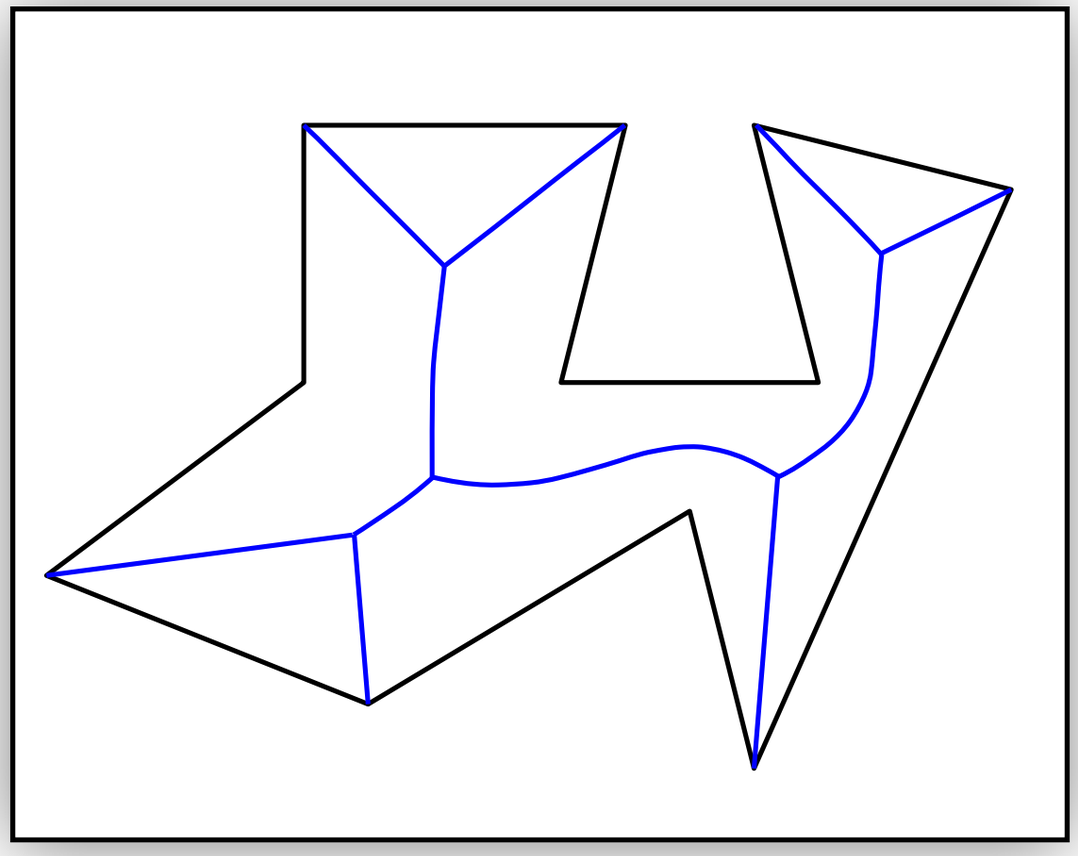

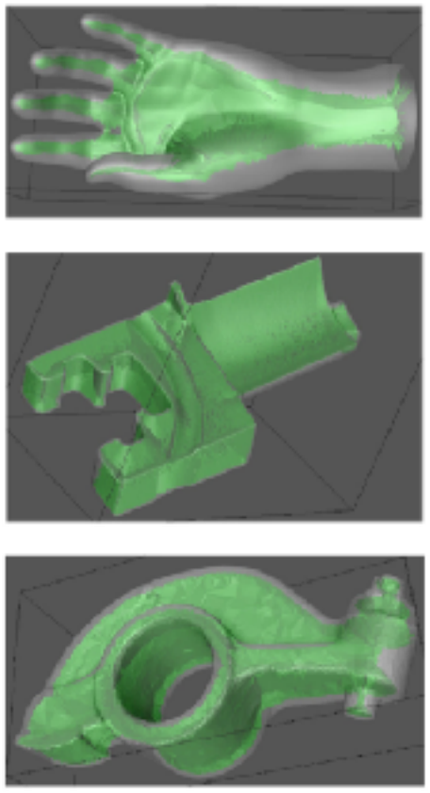

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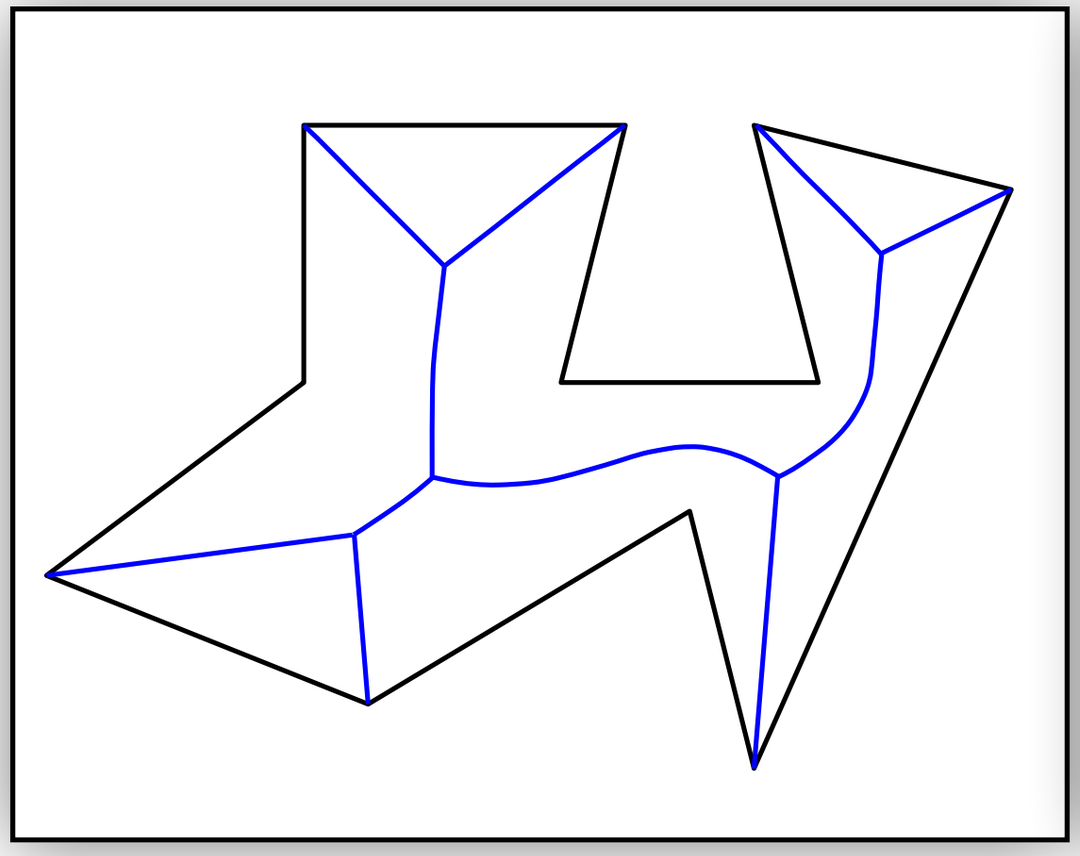

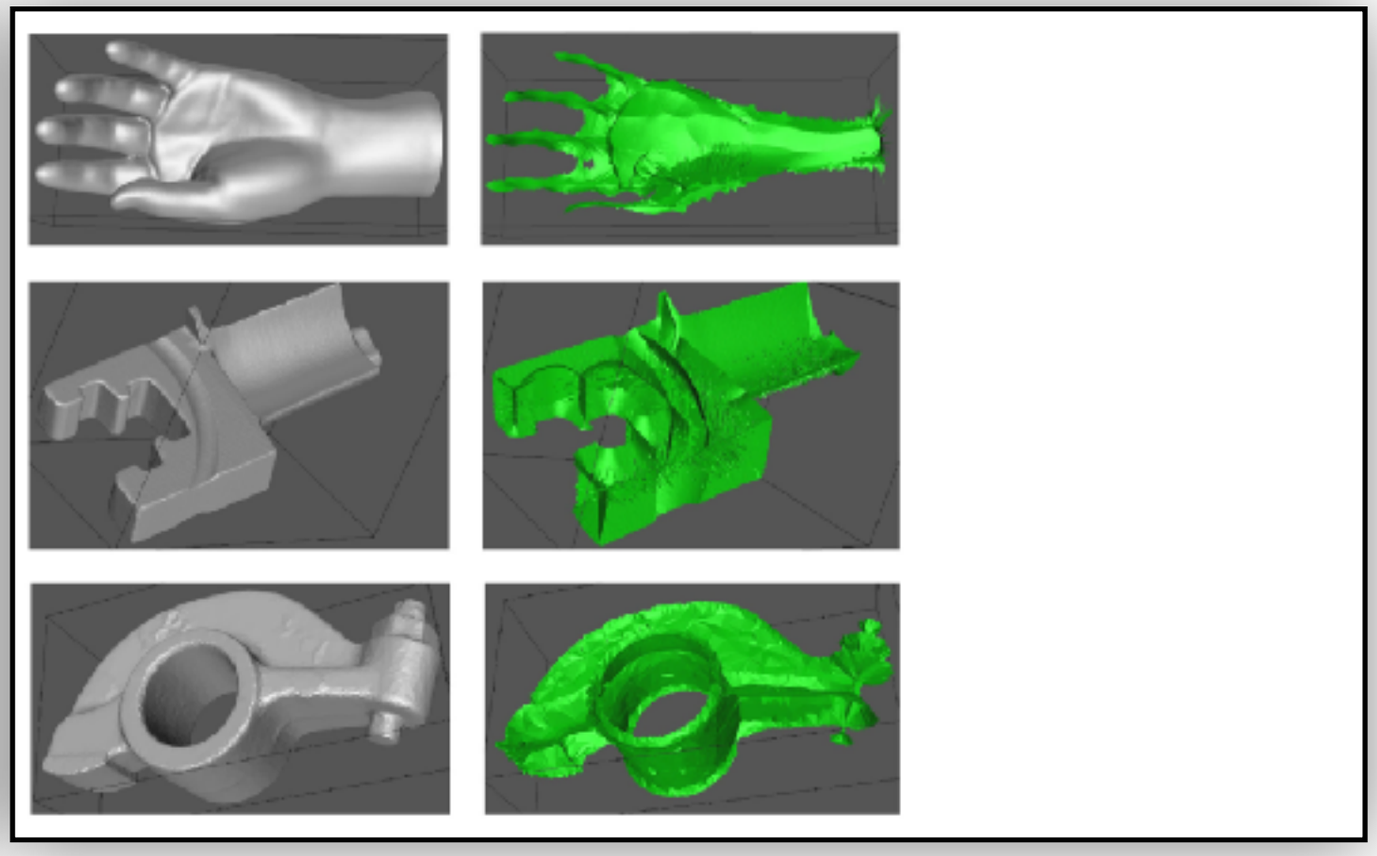

Straight Skeleton

Straight skeleton of a simple polygon:

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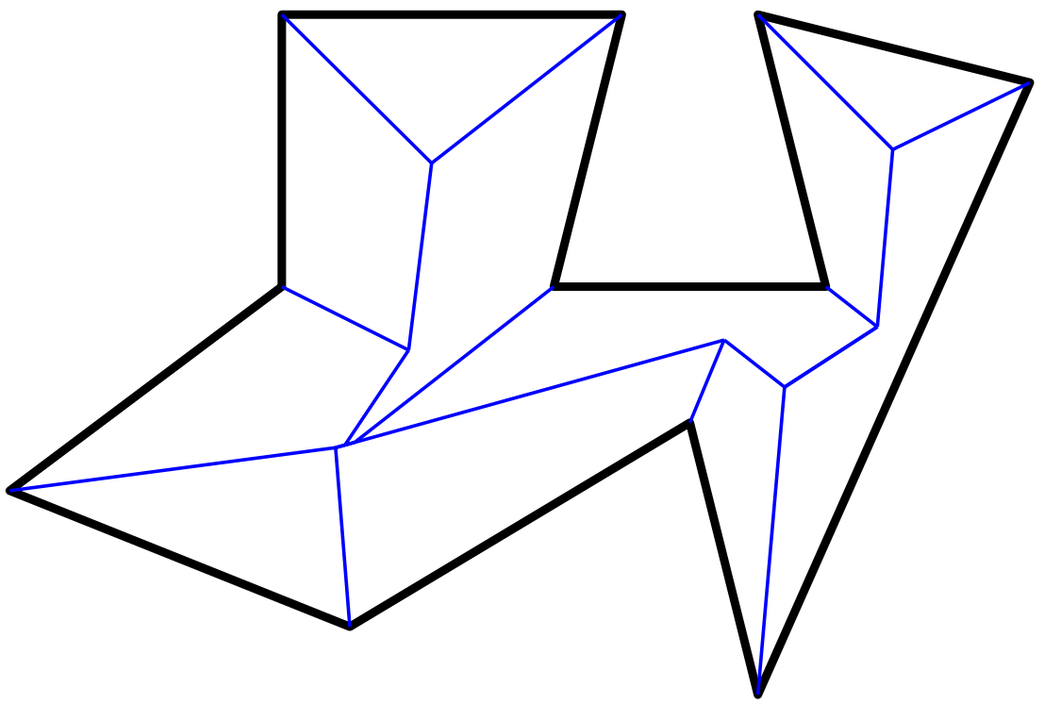

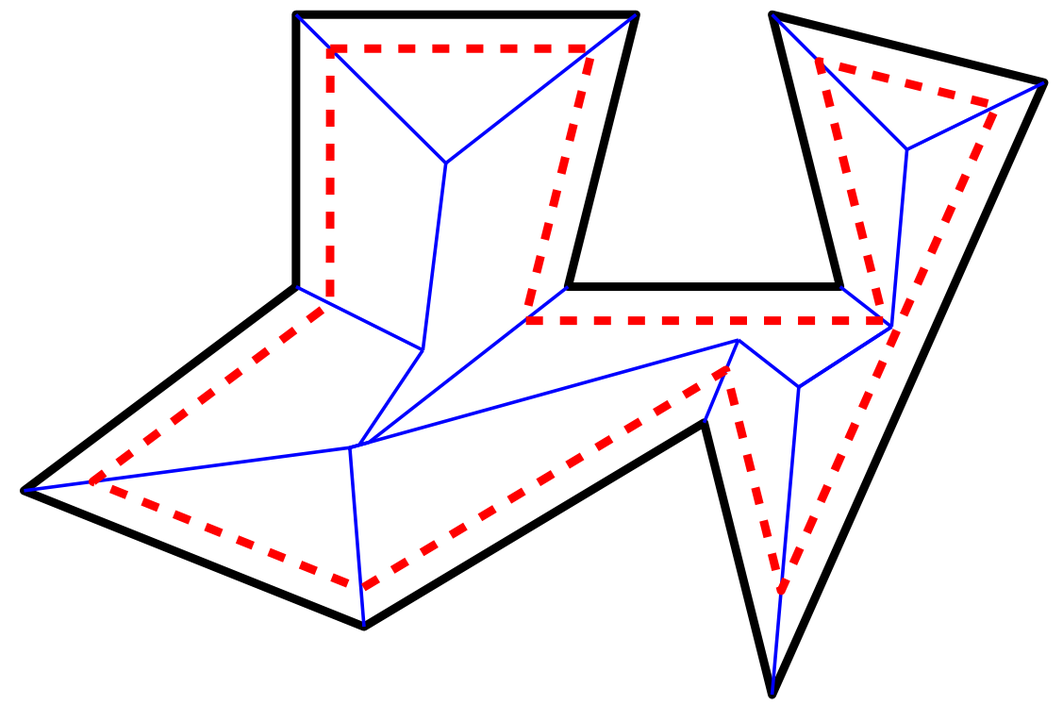

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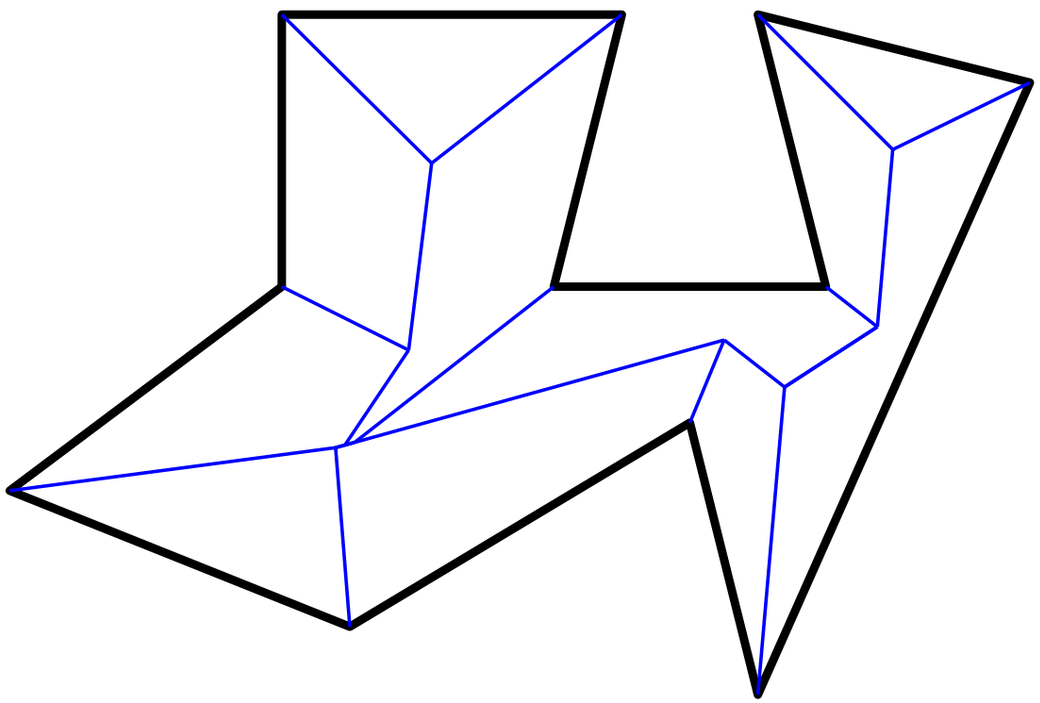

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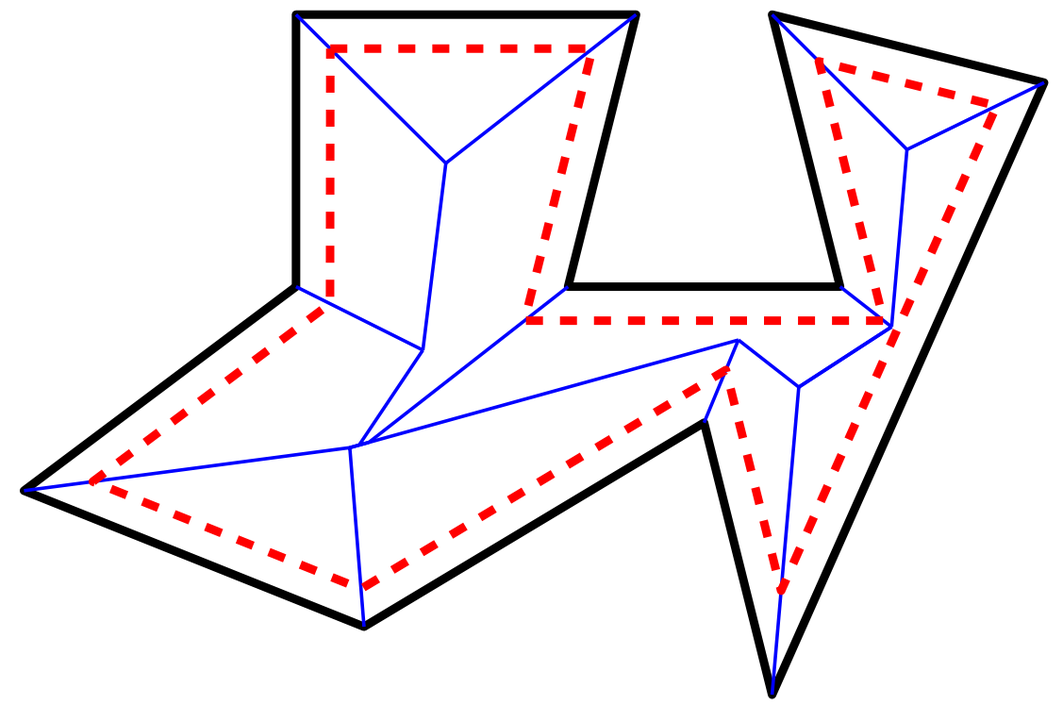

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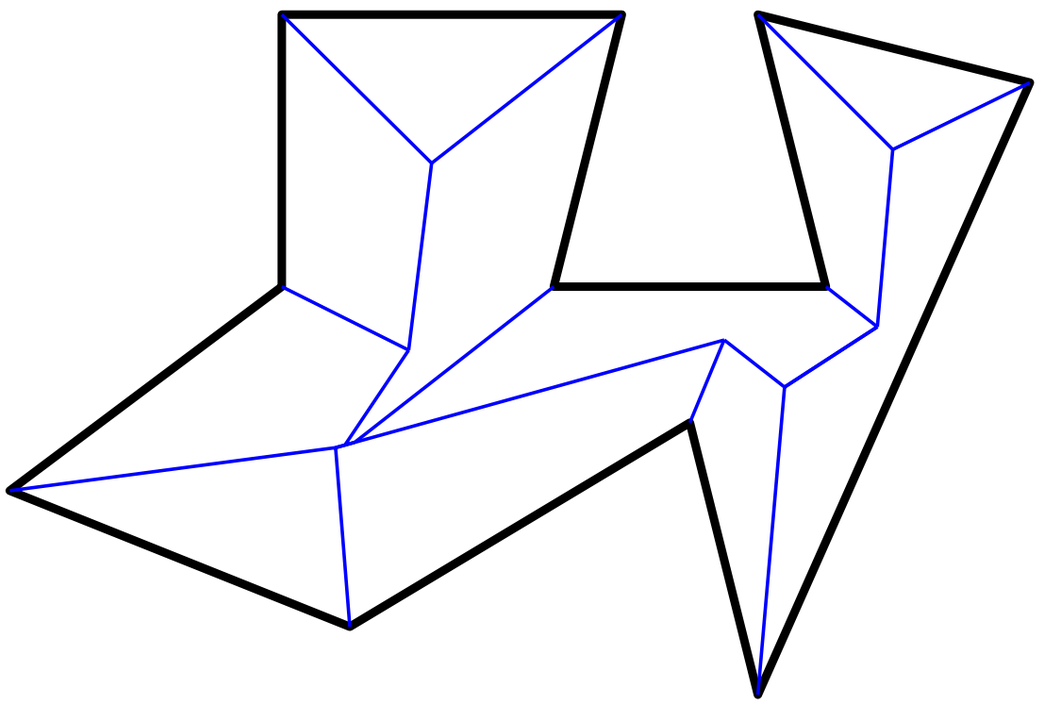

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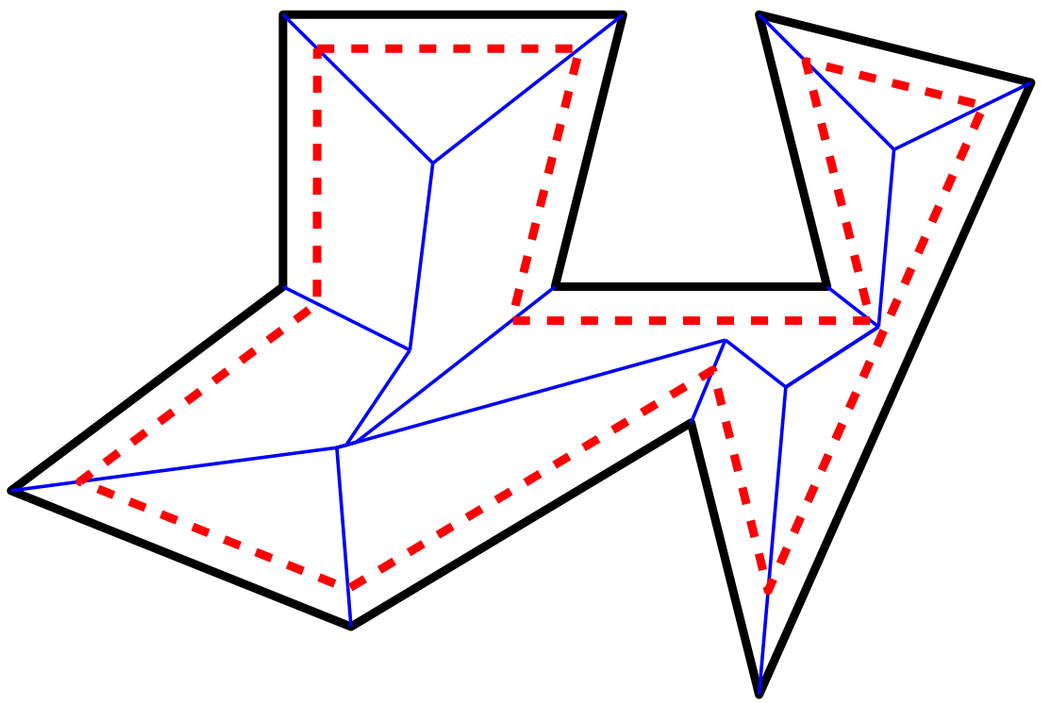

Parallel wave front

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Furthest-Point Voronoi Diagram

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- Voronoi region: Set of points with same furthest site $p \in \mathcal{P}$.

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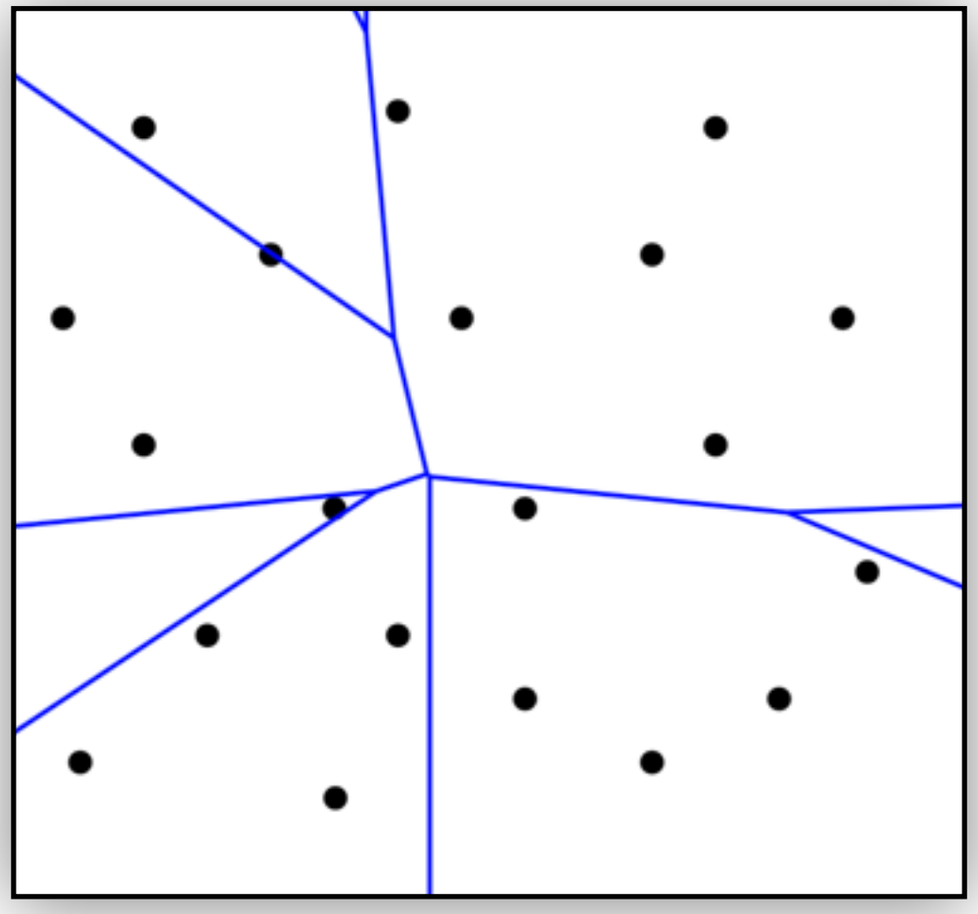

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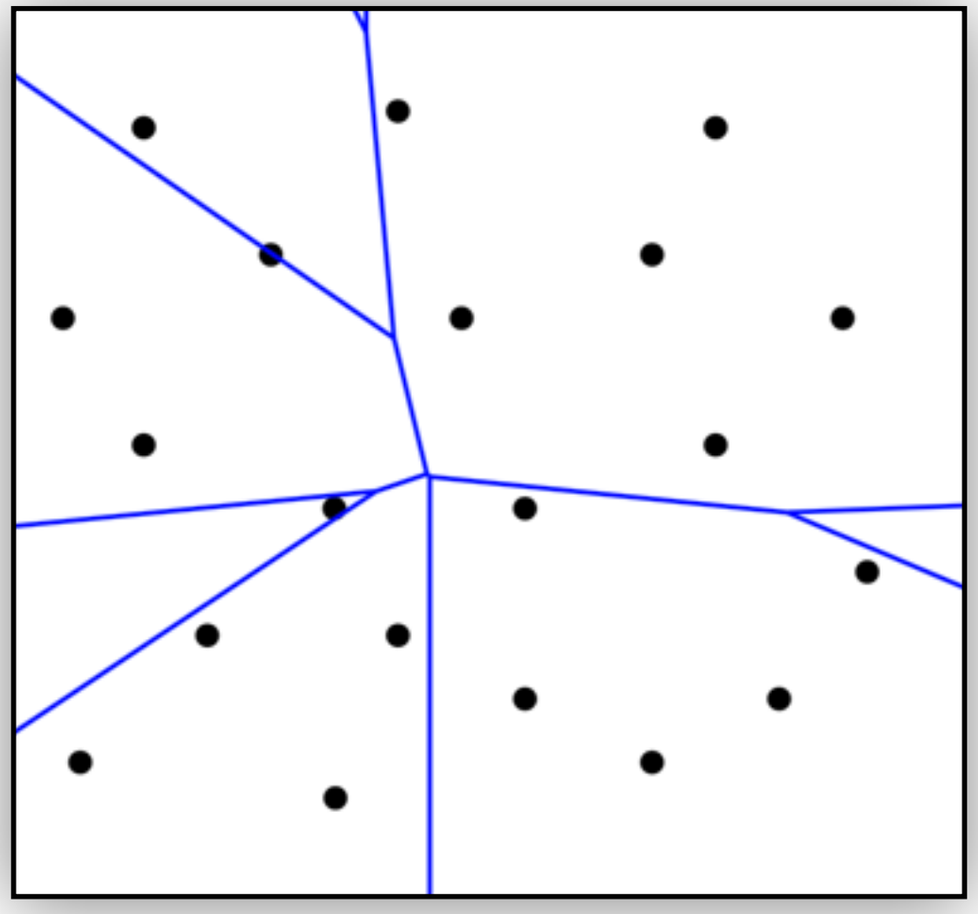

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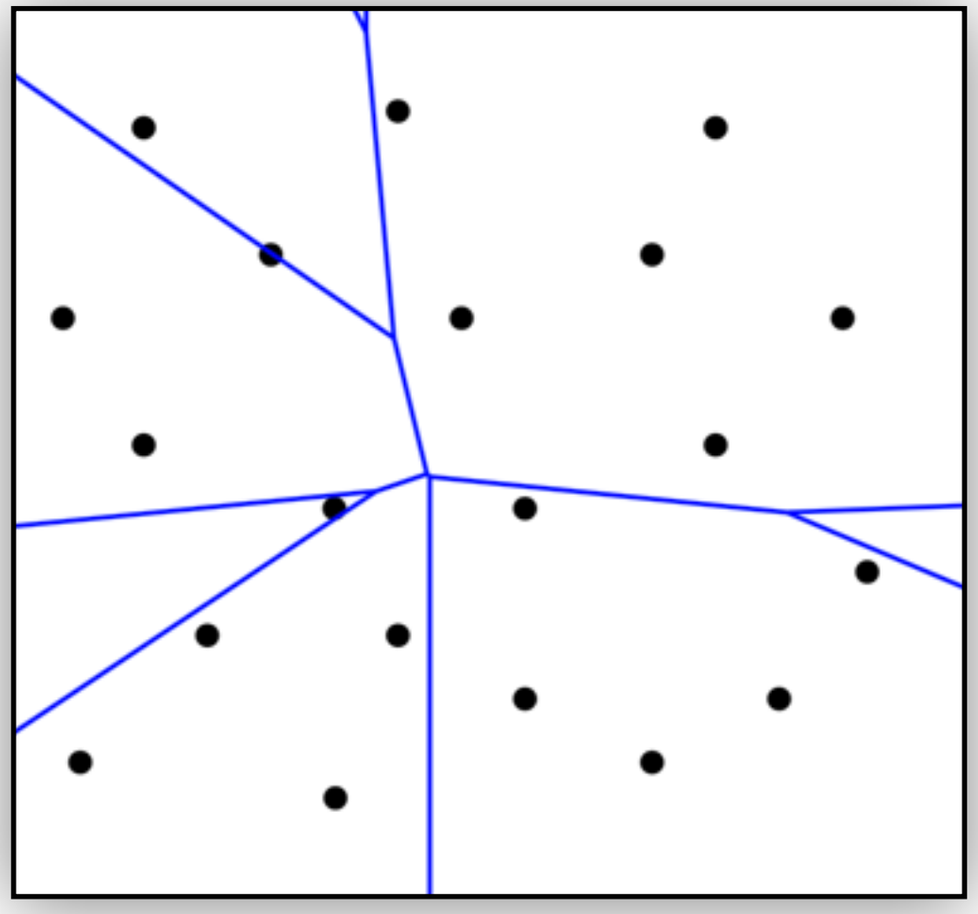

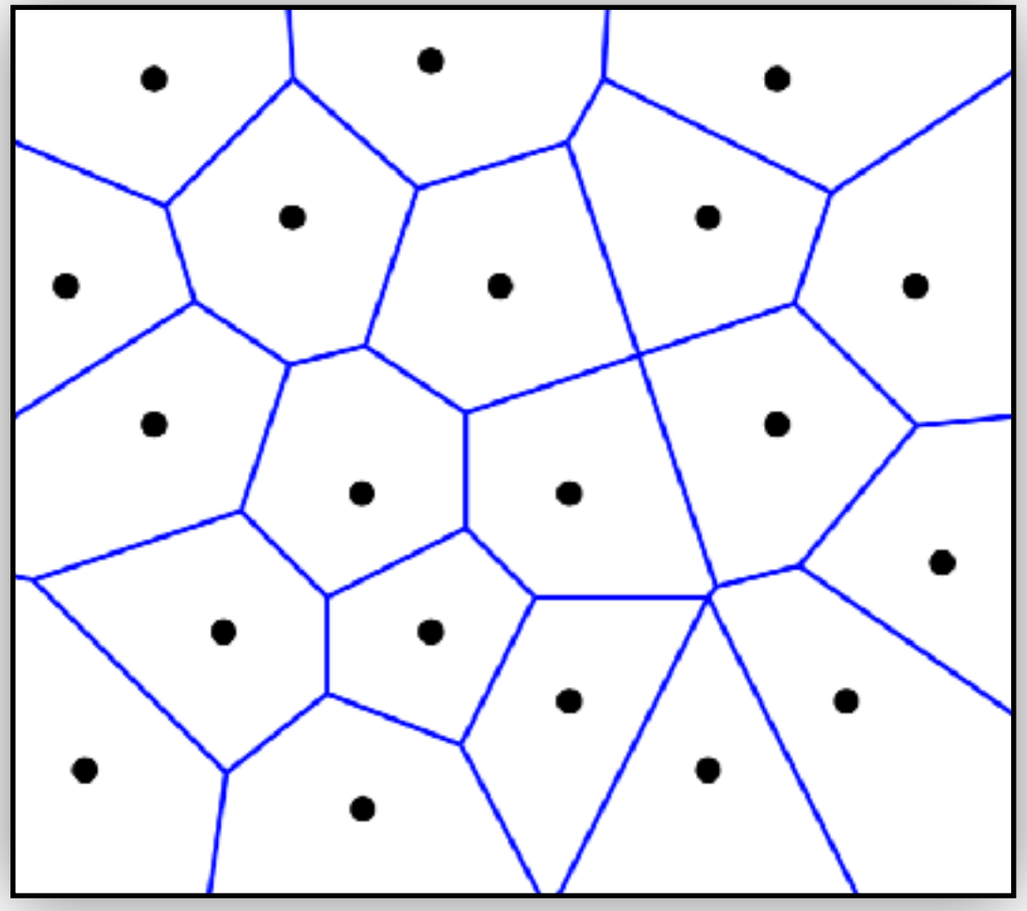

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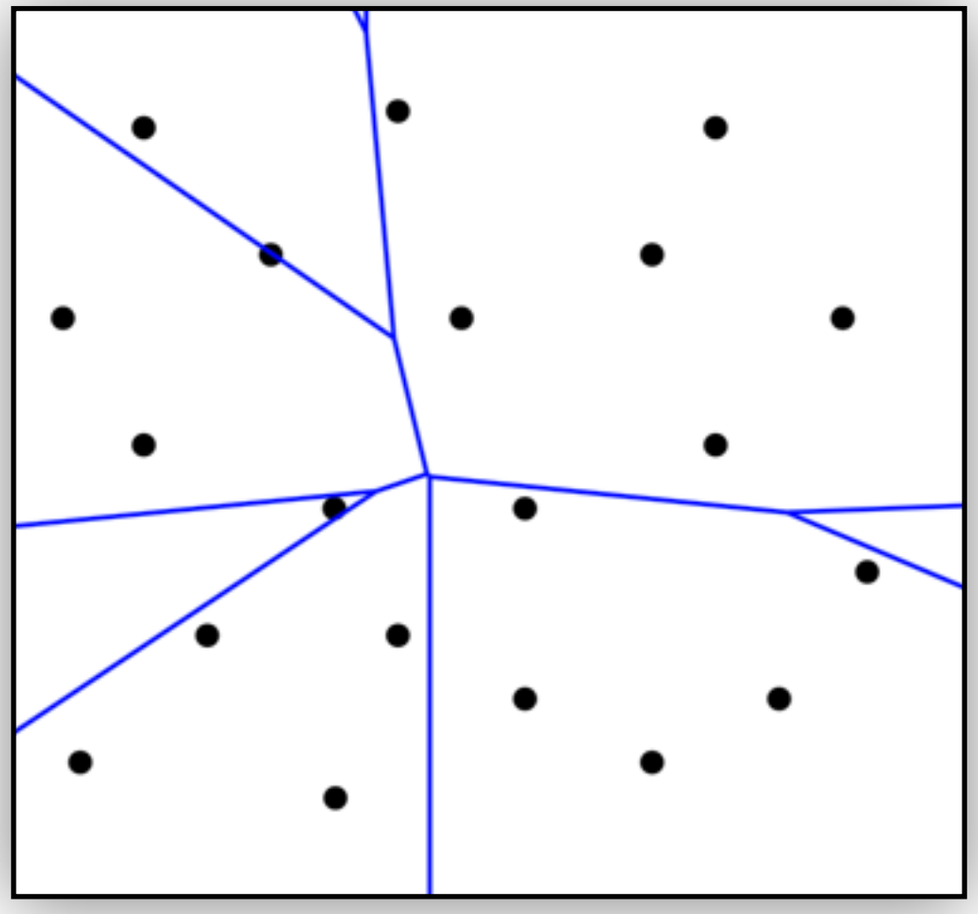

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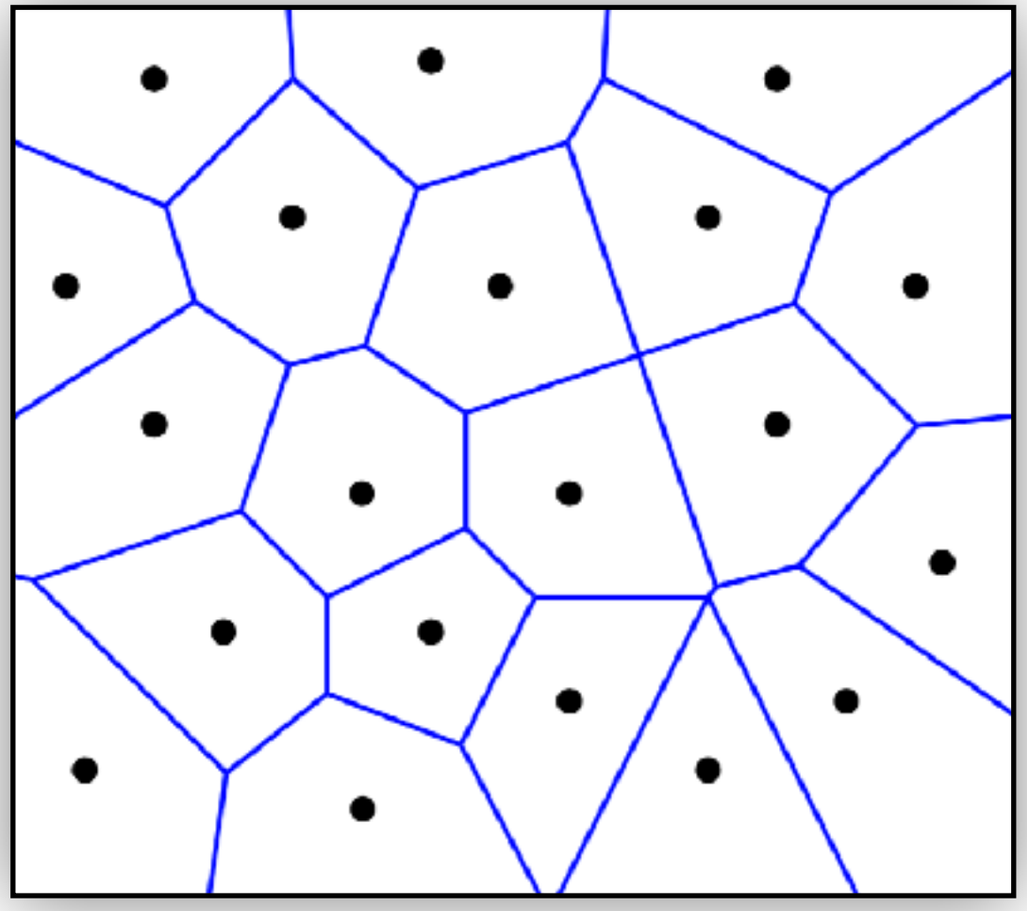

Voronoi diagram

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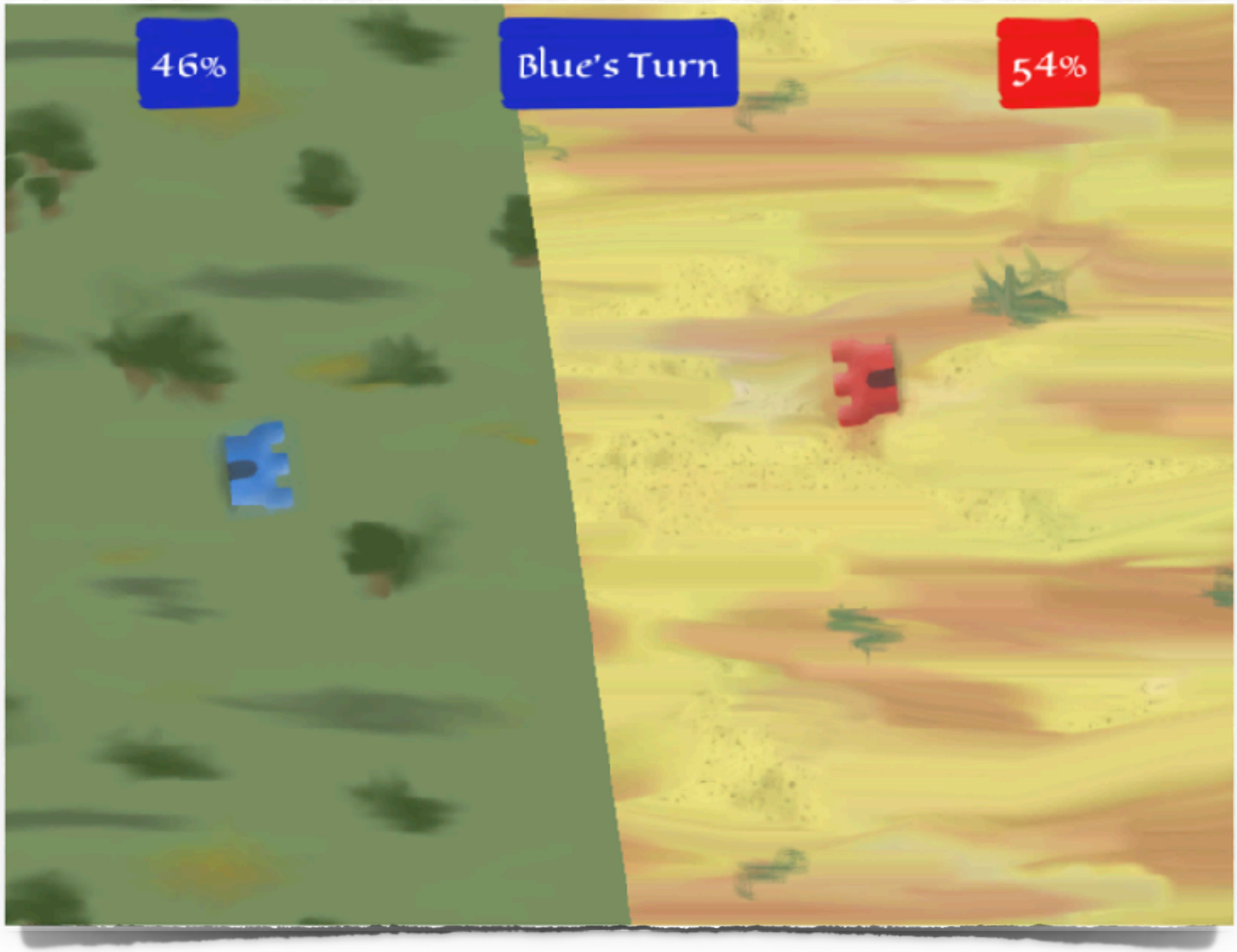
Ruler of the Plane

The Voronoi Game



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The one-round Voronoi game replayed ☆

Sándor P. Fekete [a,*], Henk Meijer [b,1]

[a] *Abteilung für Mathematische Optimierung, Braunschweig University of Technology, D-38106 Braunschweig, Germany*
[b] *School of Computing, Queen's University, Kingston, Ontario K7L 3N6, Canada*

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Available online 7 October 2004

Communicated by R. Klein

Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leqslant 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of the board closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: we show that Barney has a winning strategy for $n \geqslant 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geqslant 3$ and $\rho \leqslant \sqrt{2}/n$, for $n = 2$ and $\rho \leqslant \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.
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Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

☆ A previous extended abstract version appears in [Proceedings of the 8th Workshop on Algorithms and Data Structures, Springer, Berlin, 2003, pp. 150–161] [8].

* Corresponding author.
E-mail addresses: s.fekete@tu-bs.de (S.P. Fekete), henk@cs.queensu.ca (H. Meijer).
URLs: http://www.mathi.tu-bs.de/~fekete (S.P. Fekete), http://www.cs.queensu.ca/~henk/ (H. Meijer).

[1] Partially supported by NSERC. This work was done while visiting TU Braunschweig.

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doi:10.1016/j.comgeo.2004.05.005

The Voronoi Game

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We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: we show that Barney has a winning strategy for $n \geqslant 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geqslant 3$ and $\rho \leqslant \sqrt{2}/n$, for $n = 2$ and $\rho \leqslant \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.
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Keywords: Computational geometry; Voronoi diagram; Voronoi game; Competitive facility location; 2-person games; NP-hardness

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Competitive Location Problems: Balanced Facility Location and the One-Round Manhattan Voronoi Game *

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Abstract. We study competitive location problems in a continuous setting, in which facilities have to be placed in a rectangular domain R of normalized dimensions of 1 and $\rho \geq 1$, and distances are measured according to the Manhattan metric. We show that the family of *balanced* configurations (in which the Voronoi cells of individual facilities are equalized with respect to geometric properties) is richer in this metric than for Euclidean distances. Our main result considers the *One-Round Voronoi Game* with Manhattan distances, in which first player White and then player Black each place n points in R; each player scores the area for which one of its facilities is closer than the facilities of the opponent. We give a tight characterization: White has a winning strategy if and only if $\rho \geq n$; for all other cases, we present a winning strategy for Black.

Keywords: Facility location · competitive location · Manhattan distances · Voronoi game · geometric optimization.

1 Introduction

Problems of optimal location are arguably among the most important in a wide range of areas, such as economics, engineering, and biology, as well as in mathematics and computer science. In recent years, they have gained importance through clustering problems in artificial intelligence. In all scenarios, the task is to choose a set of positions from a given domain, such that some optimality criteria for the resulting distances to a set of demand points are satisfied; in a geometric setting, Euclidean or Manhattan distances are natural choices. Another challenge is that facility location problems often happen in a *competitive* setting, in which two or more players contend for the best locations. This change to competitive, multi-player versions can have a serious impact on the algorithmic difficulty of optimization problems: e.g., the classic Travelling Salesman Problem is NP-hard, while the competitive two-player variant is even PSPACE-complete [10].

* A full version can be found at arXiv: 2011.13275 [6].

1. Introduction and Motivation
2. Definitions
3. Representing planar partitions
4. Properties
5. Fortune's algorithm
6. Variations
7. The Voronoi game
8. Summary and conclusions

Summary

Voronoi diagrams:

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- Definition for point sets

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- Lower bound for runtime: $\Omega(n \log n)$

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- [Fortune, 2004] (Survey article)
- [Akabe *et al.,* 2000] (Survey book).

Thank you for today!