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Due June 26^{th}
Discussion July 2^{nd}

Please submit your handwritten answers in pairs*, using the box in front of* `IZ338` *before 1pm on the due date above. Make sure to include your* full names*,* matriculation numbers*, and the* study programmes *that you are enrolled in (e.g., computer science master).*

Exercise 1 (Maximum Cardinality Cut).

(2 + 9 + 4 points)

In the Maximum Cardinality Cut problem, we are given an undirected graph $G = (V, E)$. We want to partition $V = \{v_1, \ldots, v_n\}$ into two disjoint subsets S and T with $S \cup T = V$ such that the number of edges vw between a vertex $v \in S$ and a vertex $w \in T$ is maximized. We call these edges *crossing edges* because they *cross* the cut (S, T).

Consider the following algorithm A. Initially, ALG sets

$$S = \{v_1, \ldots, v_{\lceil n/2 \rceil}\} \quad \text{and} \quad T = \{v_{\lceil n/2 \rceil + 1}, \ldots, v_n\}.$$

In each step, ALG searches for a vertex v_i that can be moved from S to T (or vice versa) to increase the number of crossing edges between S and T by at least 1. If such a vertex v_i is found, ALG moves it to the other set and repeats. Otherwise, ALG terminates and outputs (S, T).

a) Argue that ALG has polynomial running time.

b) Prove that ALG is a $1/2$-approximation algorithm.

c) Give an example that shows that ALG is not better than a $1/2$-approximation.

Exercise 2 (Steiner Tree in Metric Graphs).

(15 points)

In this variant of the Steiner Tree problem, we are given an undirected graph $G = (V \cup S, E)$ with metric edge weights $c : E \to \mathbb{R}$. We call vertices in V *required* and those in S *Steiner*. Our task is to find a *Steiner tree* $T = (V \cup S', F)$ with $S' \subseteq S$ and $F \subseteq E$ that connects all required vertices, and (optionally) some Steiner vertices, while minimizing the total weight, see Fig. 1.

Prove that the minimum spanning tree of G gives a 2-approximation for this problem.

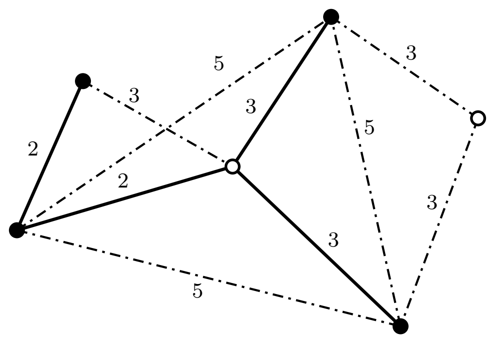

Figure 1: A graph G with required (black) and Steiner (white) vertices, and a Steiner tree.