Approximation Algorithms – Sheet 3

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Summer 2026

Due May 22^{nd}
Discussion June 3^{rd}

Please submit your handwritten answers in pairs, *using the box in front of* `IZ338` *before 1pm on the due date above. Make sure to include your* full names, matriculation numbers, *and the* study programmes *that you are enrolled in (e.g., computer science master).*

Exercise 1 (Set Packing). **(5 + 5 + (4+6) points)**

In the lecture, we saw the Set Cover problem, where we are given a universe $U = \{1, \ldots, m\}$ of elements and a family of n sets $S_i \subseteq U$ with associated weights c_i. The goal is then to find a cover $\mathcal{S} \subseteq \{S_1, \ldots, S_n\}$ of U, while minimizing the total weight of chosen sets. We showed that this problem is NP-hard and described a greedy algorithm.

There is an analogous *maximization* problem called Set Packing, where the goal is to find a subset $\mathcal{S}$ with maximum total weight that is pairwise disjoint, i.e., $S_i \cap S_j = \emptyset$ for $S_i \neq S_j$ in $\mathcal{S}$.

a) Show that Set Packing is NP-hard by reducing from Independent Set.

b) Prove that Set Packing and Set Cover are dual to one another by formulating their LP relaxations and arguing based on these.

c) We consider two greedy approaches to Set Packing. Show that neither approach works well, i.e., both approaches are at most $1/\Omega(m)$-approximation algorithms.

  (i) In each step, add a (feasible) set that maximizes c_i.

  (ii) In each step, add a (feasible) set that maximizes $c_i/|S_i|$.

Optional Exercise 2 (Primal-Dual for Set Cover). **(10 bonus points)**

In the tutorial, we formulated a primal-dual algorithm (Algorithm 1) for the Vertex Cover problem that constructs a solution to the instance by iteratively tightening constraints in the dual program (Matching) and assigning variables in the primal program.

Formulate a polynomial-time π-approximation algorithm for Set Cover using the primal-dual paradigm, where π is the maximal number of occurrences of any integer in the sets:

$$\pi = \max_{i=1\ldots n} |\{1 \leq j \leq m \mid i \in S_j\}|$$

Algorithm 1: Primal-Dual Vertex Cover. Let $G = (V, E)$.

$y \leftarrow (0, \ldots, 0)^\intercal \in \mathbb{Q}^m$ `(fractional, feasible solution for dual)`
$x \leftarrow (0, \ldots, 0)^\intercal \in \{0,1\}^n$ `(integral, infeasible solution for primal)`
while $\exists e \in E$ *with* $e = \{a, b\}$ *s.t.* $x_a + x_b \geq 1$ *is violated in primal* **do**
  increase the dual variable y_e until $\exists v \in \{a, b\}$ with $\sum_{e' \in E:\ v \in e'} y_e = 1$
  $x_v \leftarrow 1$
end
output x
