Approximation Algorithms – Sheet 1 | **Summer 2026**

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Due April 29^{th}
Discussion May 6^{th}

Please submit your handwritten answers in pairs*, using the box in front of* `IZ338` before *the exercise timeslot on the due date above. Make sure to include your* full names, matriculation numbers*, and the* study programmes *that you are enrolled in (e.g., computer science master).*

Exercise 1 (Complexity and Reductions: Independent Set). **(5+5 points)**

Let $G = (V, E)$ be a graph. A set of vertices $I \subseteq V$ is called *independent* if

$$\forall u, v \in I : \{u, v\} \notin E.$$

The Independent Set (IS) problem asks for an independent set of maximum cardinality.

a) Show that C is a vertex cover of G if and only if $I = V \setminus C$ is an independent set.

b) In Lecture 2, we showed that it is NP-complete to decide whether a given graph has a vertex cover of a given size ℓ. Prove that it is NP-complete to decide whether a given graph G has an independent set of a given size $k \in \mathbb{N}$.

Exercise 2 (Removing Directed Cycles). **(10 points)**

Consider the following optimization problem Π. We are given a directed graph $G = (V, A)$ without loops, meaning $\forall v \in V : (v, v) \notin A$. The goal is to find a *maximum-cardinality subset* of directed arcs $A' \subseteq A$ such that the (directed) subgraph $G' = (V, A')$ has no directed cycles.

Give a polynomial-time approximation algorithm ALG for Π such that $|\mathsf{ALG}(G)| \geq 1/2 \cdot |\mathsf{OPT}_\Pi(G)|$. That is, ALG finds a set of arcs A' that is at least half the size of an optimal solution for G.

(Hint: Assign a unique label from $\{1, \ldots, n\}$ to each vertex and consider what kinds of arcs must be present in any directed cycle.)

Exercise 3 (Diameter of Point Sets). **(7+3 points)**

Let P be a set of n points in the d-dimensional Euclidean space $\mathbb{R}^d$. Assume that d is a constant and the distance between points can be computed in $\mathcal{O}(1)$ time. The *diameter* Λ of P is the maximal pairwise distance between any two points of P. This can be computed in $\mathcal{O}(n^2)$ time.

a) Show that in $\mathcal{O}(n)$ time, we can compute a 2-approximation Λ' of the diameter, meaning

$$\Lambda' \leq \Lambda \leq 2 \cdot \Lambda'.$$

b) For the special case of $d = 2$, give a c-approximation for some factor $c < 2$ in $\mathcal{O}(n)$ time.