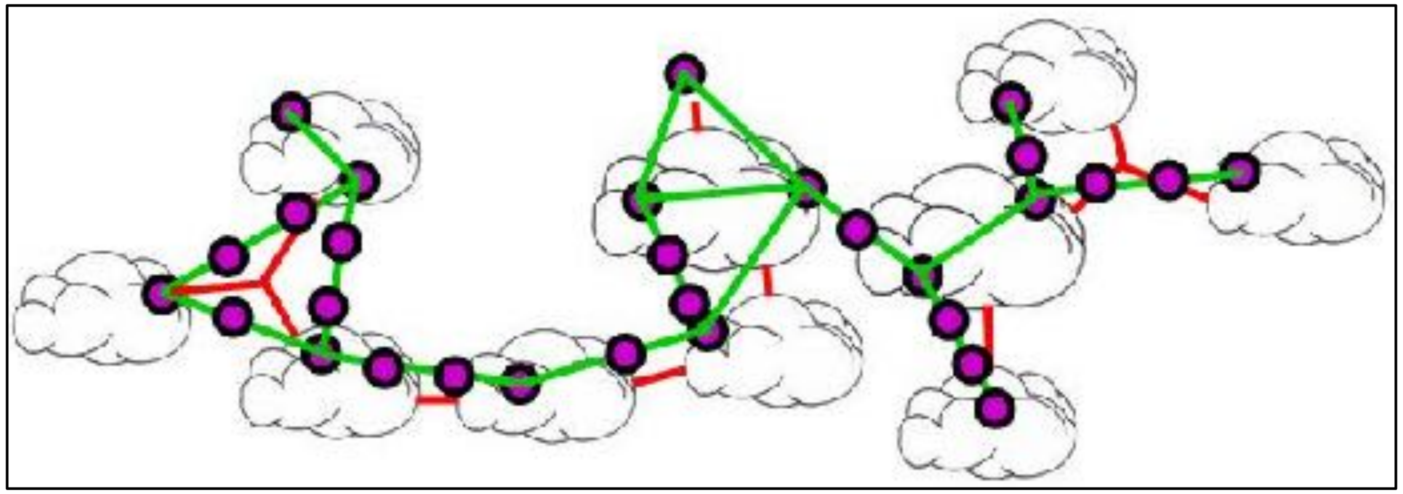
Approximation Algorithms for Relay Placement in the Plane



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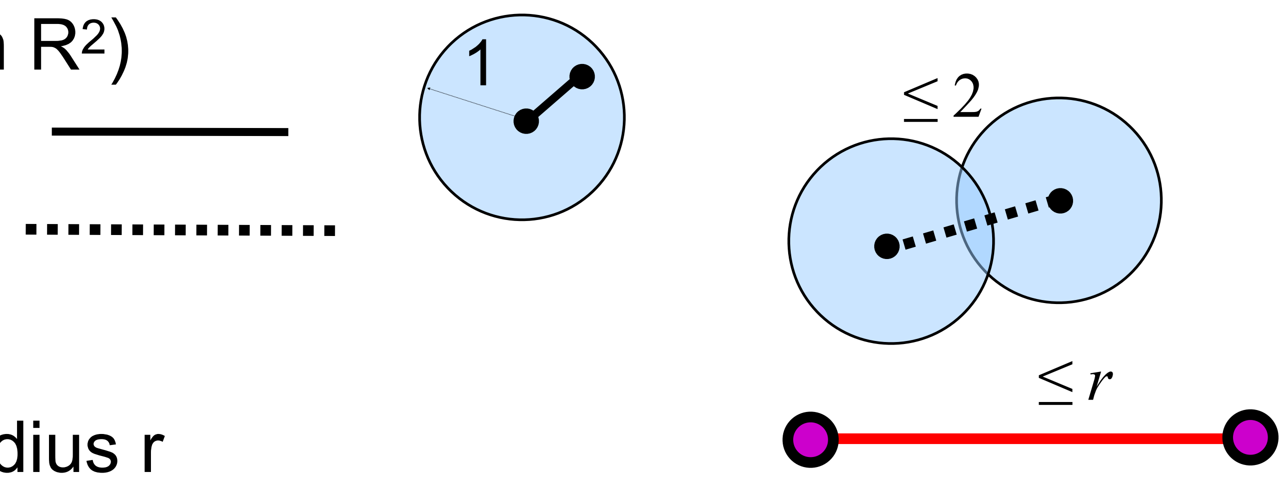
G. R. Poornananda, and Jukka Suomela

Relay Placement

Input: V = set of n sensors (points in R^2)

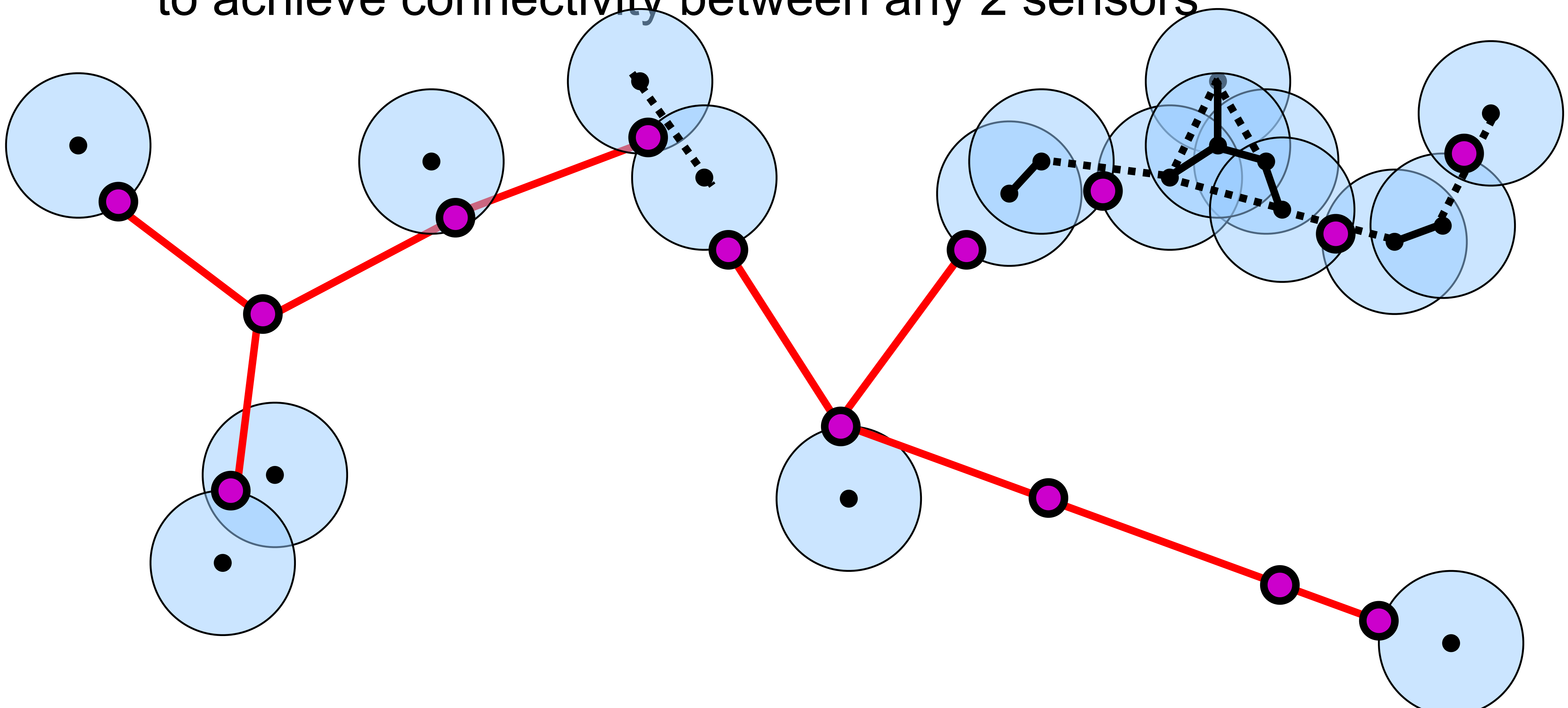
$G=(V,E)$ = unit disk graph

$F=(V,F)$ = radius-2 disk graph

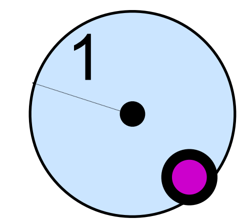


Add: relays with communication radius r

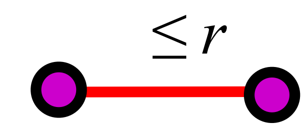
Goal: min # relays (communication radius *r*)
to achieve connectivity between any 2 sensors



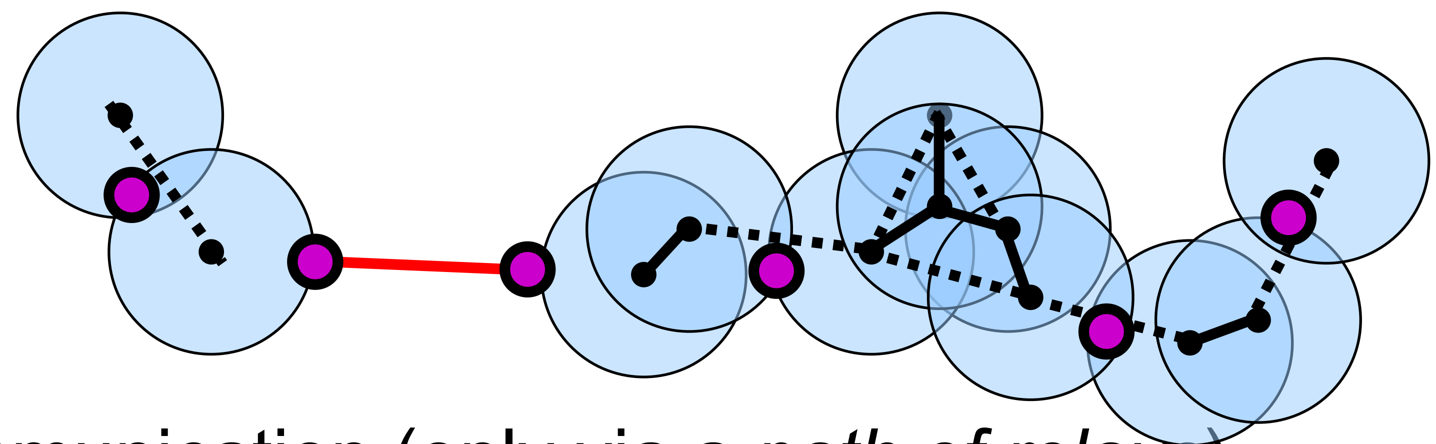
Model of Communication



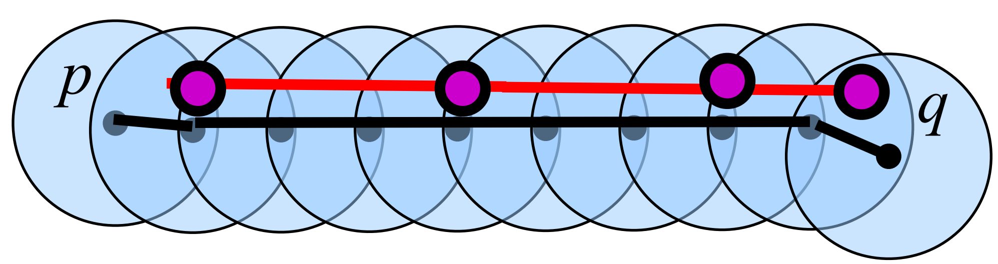
- Sensor and relay: communicate if dist ≤ 1
- Relay and relay: communicate if dist ≤ r



- Two sensors:
  - One-tier: if dist ≤ 1



  - Two-tier: no direct communication (only via a *path of relays*)



Prior Related Work

- **Steiner trees with min # Steiner points, edge length ≤ 1** (same as one-tier, $r=1$) :
  - NP-hard, 5-approx [Lin, Xue ’99]
  - [Lin,Xue] is actually a 4-approx; also give 3-approx [Chen et al ’00]
  - Faster 3-approx, randomized 2.5-approx [Cheng et al ’07]
- **7-approx for one-tier, arbitrary *r*** (time $O(n \log n)$)
  [Lloyd, Xue ’07]
- **($5+\varepsilon$)-approx for two-tier, any $r \geq 1$** [Lloyd, Xue ’07]
- **($4+\varepsilon$)-approx for two-tier, any $r \geq 2$**
  [Srinivas, Zussman, Modiano ’06]

Our Contributions

- Simple O(n log n)-time 6.73-approx for one-tier relay placement [vs. previous 7-approx]
- Poly-time 3.11-approx for one-tier, any $r \geq 1$
- No PTAS for one-tier (r is part of input), assuming P ≠ NP
- PTAS for two-tier relay placement

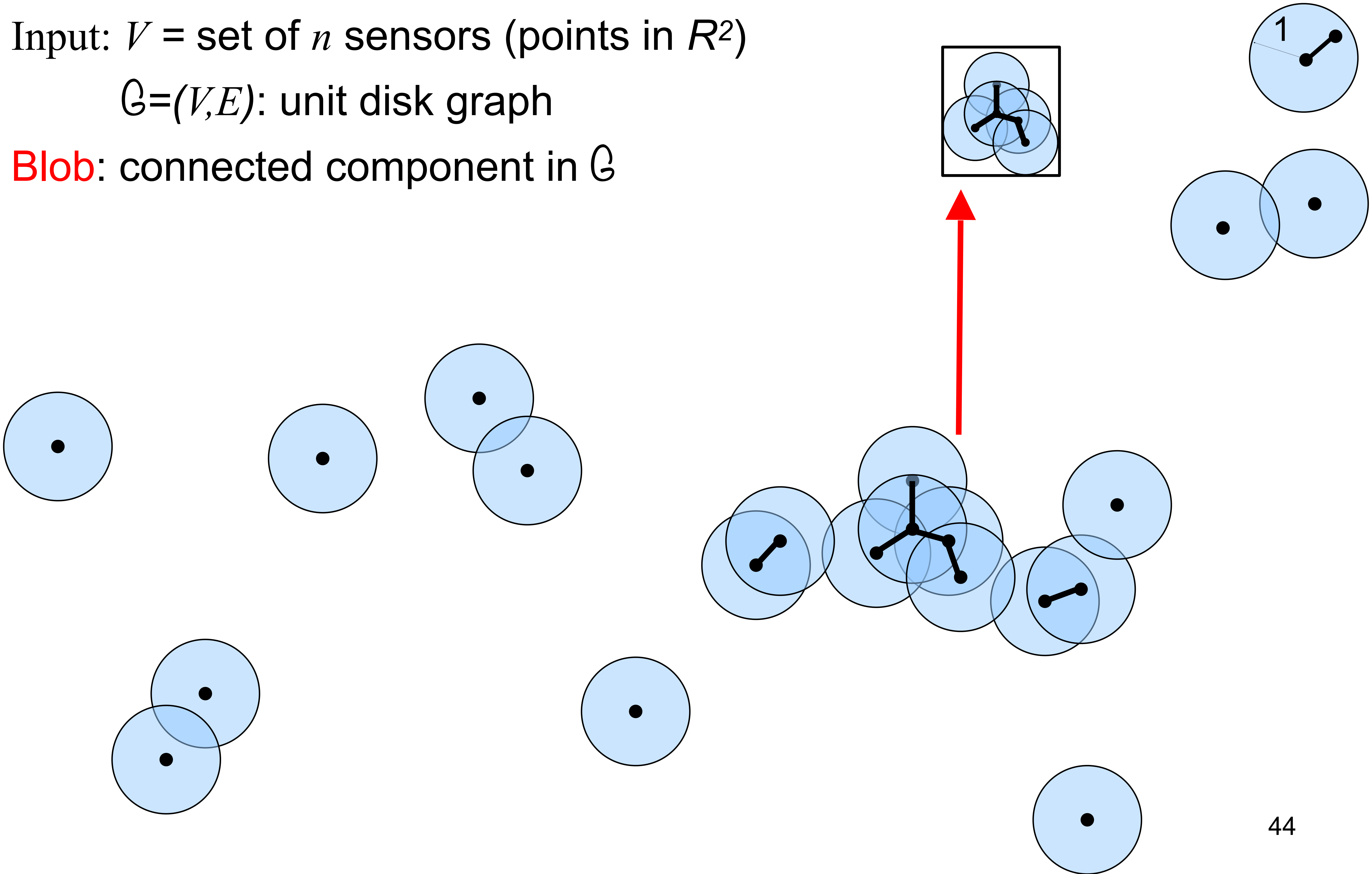
  [vs. previous (5+ε)-approx]

Terminology: Blobs and Clouds

Input: V = set of n sensors (points in R^2)

$\mathcal{G}=(V,E)$: unit disk graph

Blob: connected component in $\mathcal{G}$



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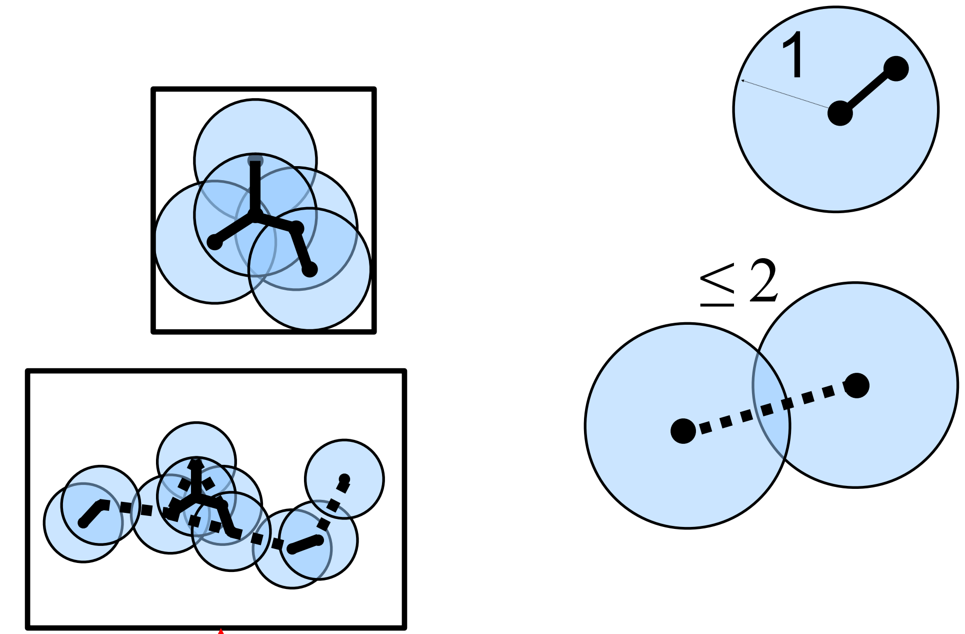

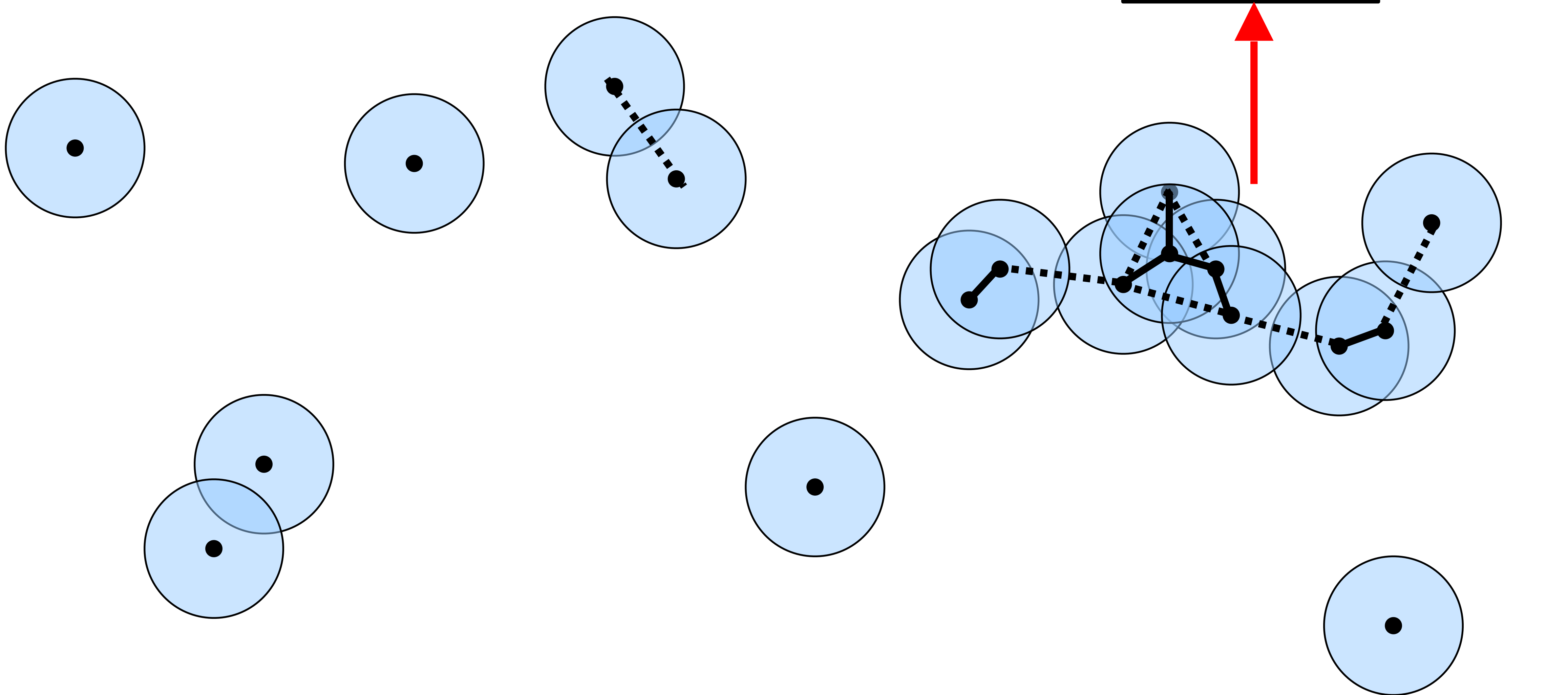

Terminology: Blobs and Clouds

Input: V = set of n sensors (points in R^2)

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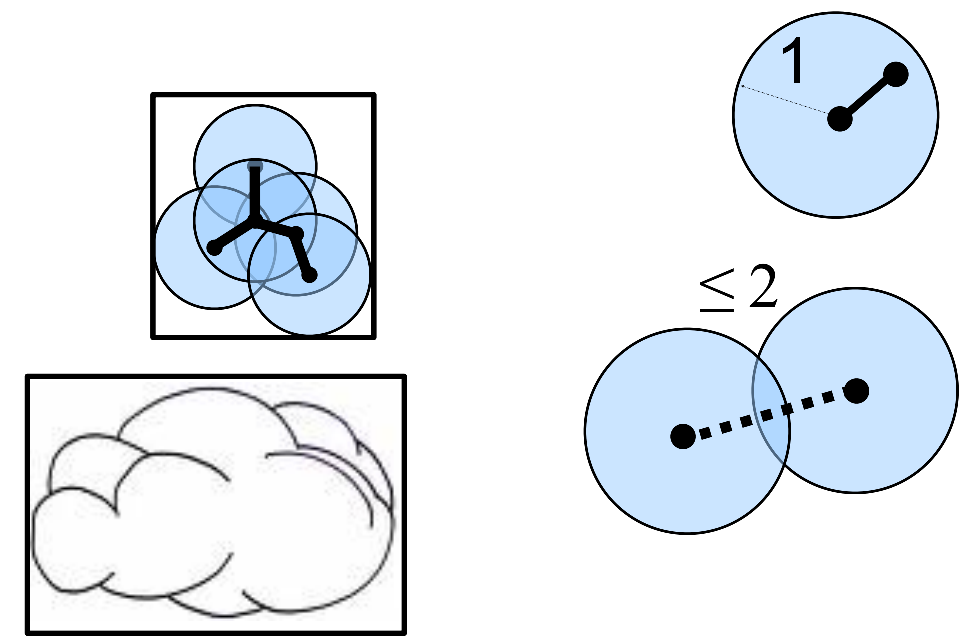
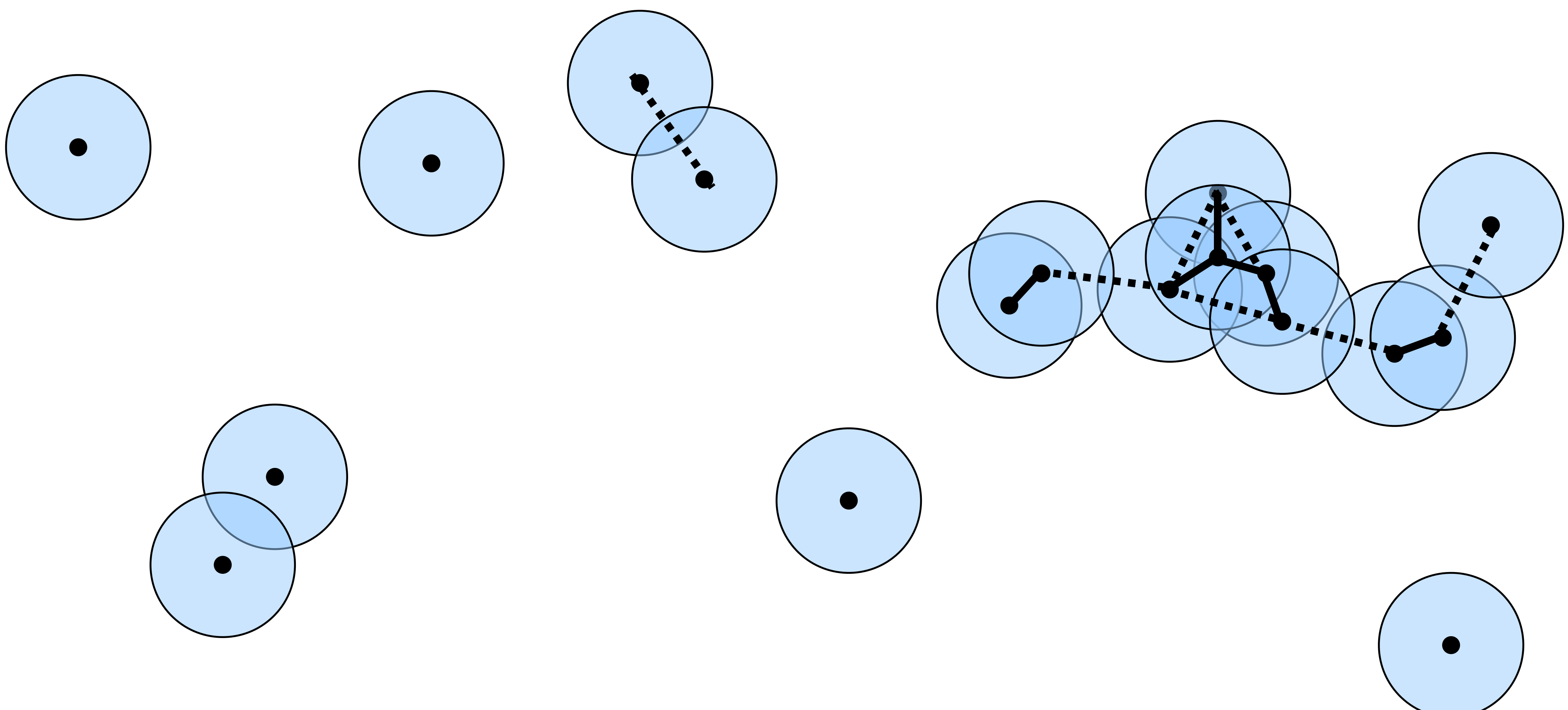
$\mathcal{F}=(V,F)$: radius-2 disk graph

Cloud: connected component in $\mathcal{F}$





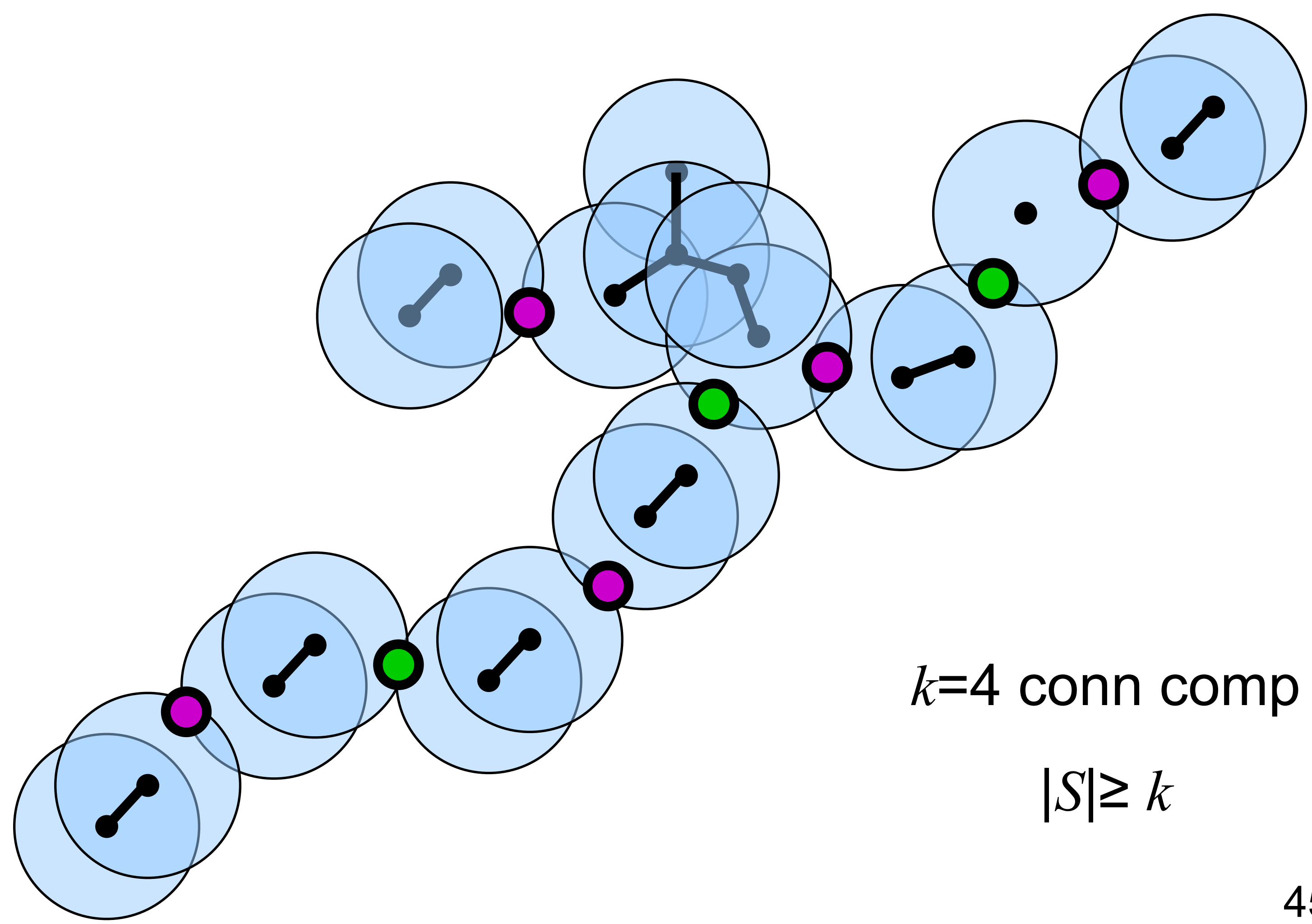
Lemma 1

Lemma 1: To connect all sensors within a cloud C, it suffices to take a set S that stabs all blobs, and add at most $|S|$-1 relays.

- S
- *Added relays*

Each added relay reduces the number of connected components by 1

Thus, at most $k\text{-}1 \leq |S|\text{-}1$ additional relays needed to connect the relays within C



k=4 conn comp

$|S| \geq k$

Steiner Forests and Spanning Trees with Neighborhoods

Let $\mathcal{P}$ = family of planar subsets (neighborhoods)

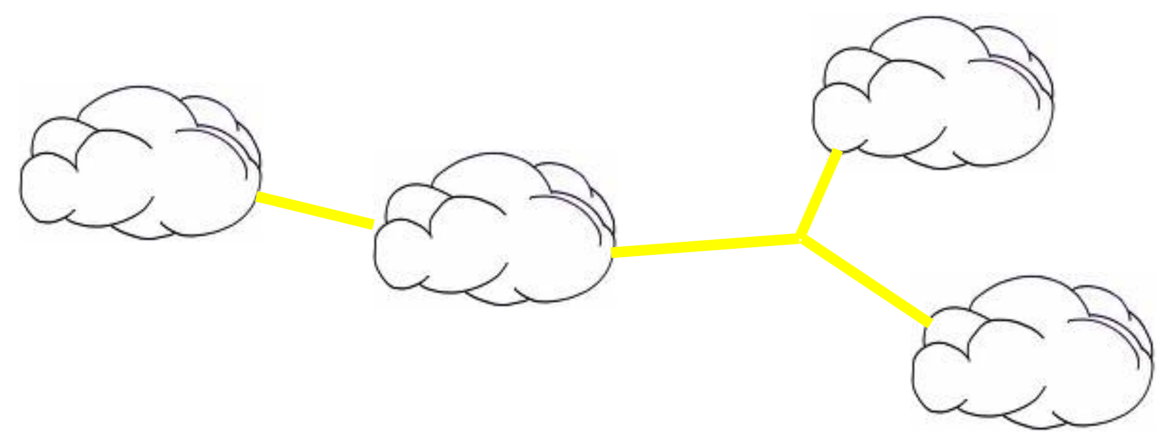
Minimum Steiner Forest for the Neighborhoods in P

MStFN($\mathcal{P}$) : min-length plane graph G s.t.

$G_{\mathcal{P}}=(\mathcal{P},E(G))$ is connected

Counts only length *outside* $\mathcal{P}$

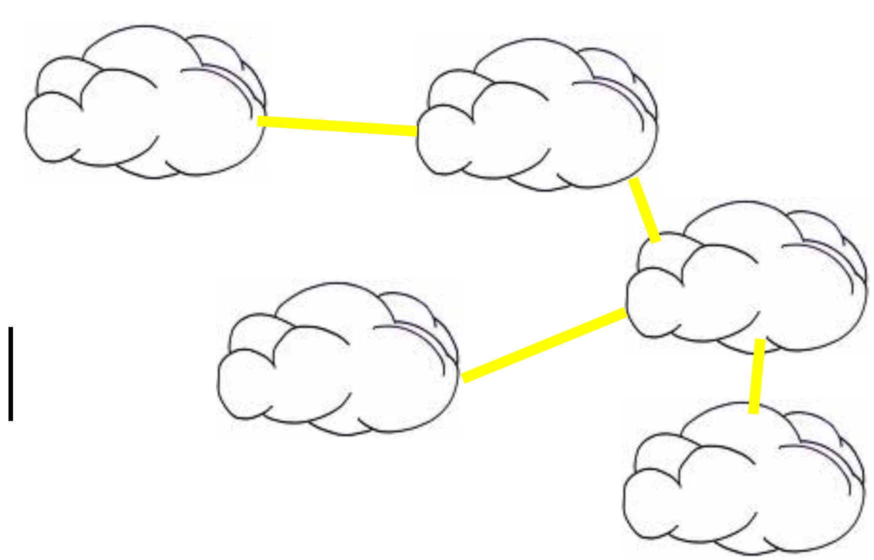
(vs. MStTN)



Minimum Spanning Forest for the Neighborhoods in $\mathcal{P}$

MSFN($\mathcal{P}$) : min spanning tree in graph on $\mathcal{P}$,

edge weights = distance between sets

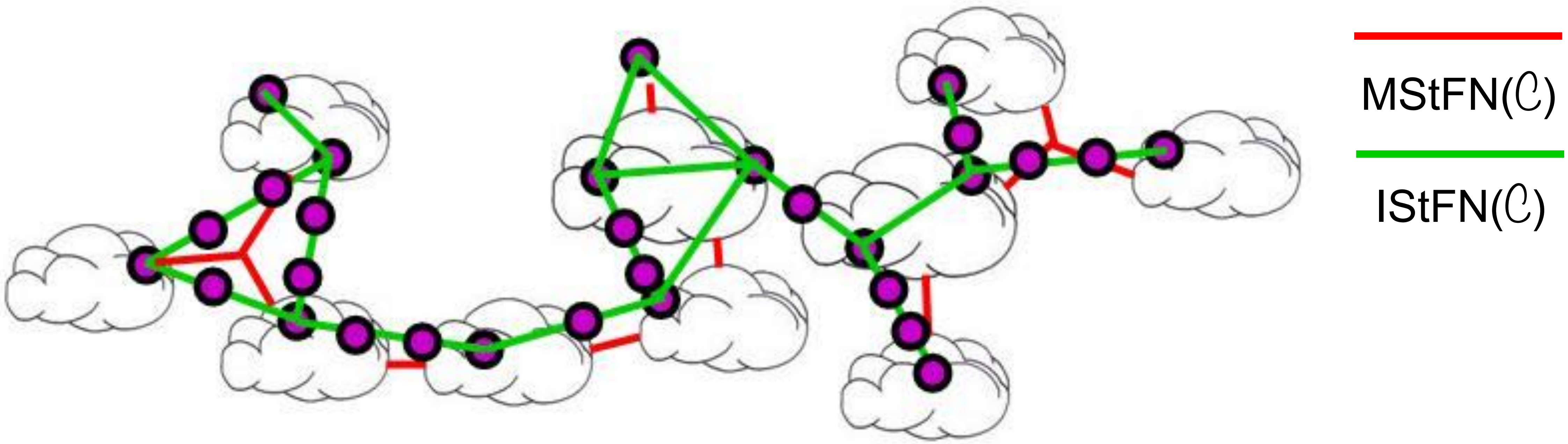


Lemma 2: For any $\mathcal{P}$,

$$|\text{MSFN}(\mathcal{P})| \leq \left(\frac{2}{\sqrt{3}}\right) |\text{MStFN}(\mathcal{P})|$$

Lower Bounds on OPT

(1) $|R^*| \geq |\mathrm{MStFN}(\mathcal{C})|/r$



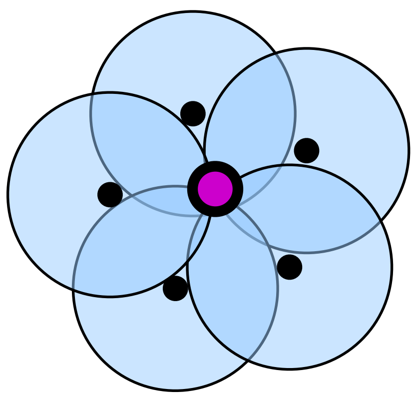
Optmal relays induce a Steiner Forest.

(2) $|R^*| \geq |\mathrm{Stab}(\mathcal{B})|$ There must be a relay in every blob.

(3) $|R^*| \geq |\mathcal{C}|$ There must be a relay in every cloud, and clouds are disjoint.

Simple 6.73-Approximation (I)

- For each cloud C:
  - Find relays R_C to hit (stab) all blobs $B \in \mathcal{B}_C$ using Greedy Set Cover



- A stab pierces ≤ 5 blobs
- Greedy gives approx factor:

$$|R_C| \le \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)|\mathrm{Stab}(\mathcal{B})| = \frac{137}{60}|\mathrm{Stab}(\mathcal{B})|$$

Simple 6.73-Approximation (II)

- (For each cloud C:)
  - Apply **Lemma 1**: Obtain relays R'_C connecting cloud C.

$$|R'_C| \leq 2|R_C| - 1 \leq \frac{137}{30}|\text{Stab}(\mathcal{B}_C)| - 1$$

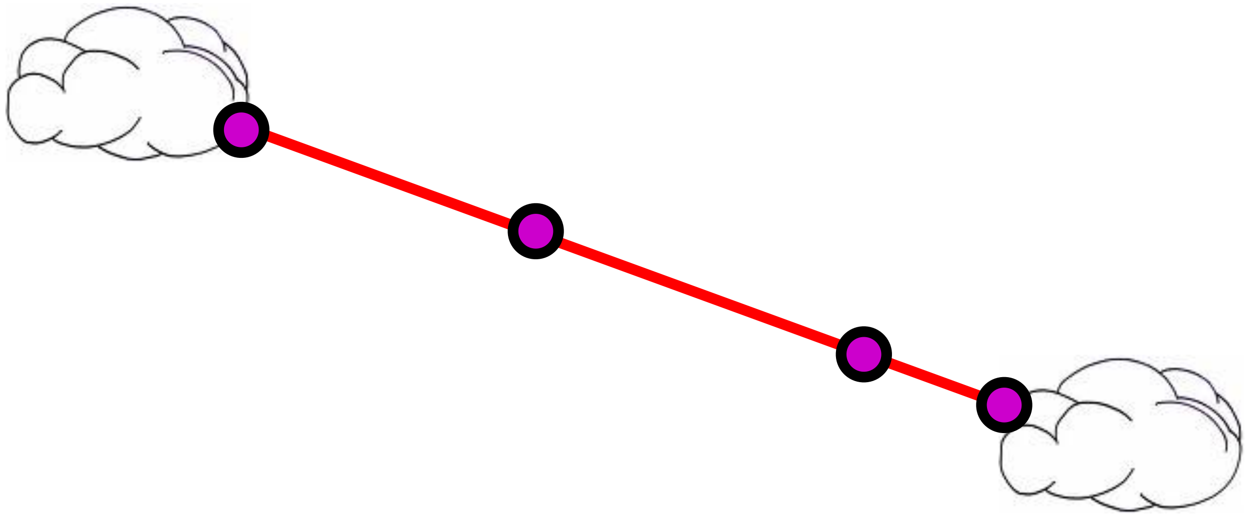
Total for all C:

$$\boxed{|R'|} = \sum_{C \in \mathcal{C}} |R'_C| \leq \sum_{C \in \mathcal{C}} \left(\frac{137}{30}|\text{Stab}(\mathcal{B}_C)| - 1 \right) = \boxed{\frac{137}{30}|\text{Stab}(\mathcal{B})| - |\mathcal{C}|}$$

(since blobs $\mathcal{B}_C$ for different C are disjoint)

Simple 6.73-Approximation (III)

- Compute MSFN($\mathcal{C}$).
- Put relays R'' along edges:
  - 2 relays at end of each edge
  - $\left\lfloor \frac{|e|}{r} \right\rfloor$ along each edge e



$$\boxed{|R''|} \leq 2(|\mathrm{C}|-1) + \sum_e \left\lfloor \frac{|e|}{r} \right\rfloor$$

$$\leq 2|\mathrm{C}| + \frac{|\mathrm{MSFN(C)}|}{r}$$

$$\leq \boxed{2|\mathrm{C}| + \frac{2}{\sqrt{3}} \cdot \frac{|\mathrm{MStFN(C)}|}{r}}$$

Simple 6.73-Approximation (IV)

$$|R'| + |R''| \le \frac{137}{30}|\text{Stab(B)}| - |\text{C}| + 2|\text{C}| + \frac{2}{\sqrt{3}} \cdot \frac{|\text{MStFN(C)}|}{r}$$

$$\le \left(\frac{137}{30} + 1 + \frac{2}{\sqrt{3}}\right)|R^*|$$

$$< 6.73|R^*|$$

Simple 6.73-Approximation (V)

- Running time:
  - Build Delaunay of n sensors $O(n \log n)$
  - Identify blobs and clouds $O(n)$
  - Greedy to stab blobs (≤5 per relay) $O(n)$
  - MSFN(C) is subgraph of Delaunay $O(n \log n)$
  - Total: $O(n \log n)$

A 3.11-Approximation

Overview:

- Compute *optimal* stabbings for clouds of blobs that can be stabbed with few (≤ k, constant) relays
  Complete connectivity of these clouds (Lemma 1)
- For other clouds, *greedily* connect blobs
- Greedily connect clouds into *clusters*, adding 2 additional relays per cloud
- Connect the clusters with spanning forest

Relays of 3 types: RED, GREEN, YELLOW : A_r , A_g , A_y

A 3.11-Approximation II

Approximation bound:

$$R^* = R_d^* \,\dot\cup\, R_l^*$$

dark: within clouds

light: outside of clouds

Will show:

$$\boxed{|A_r|} + \boxed{|A_g|} \leq (3.084 + 1/k)\,|R_d^*|$$

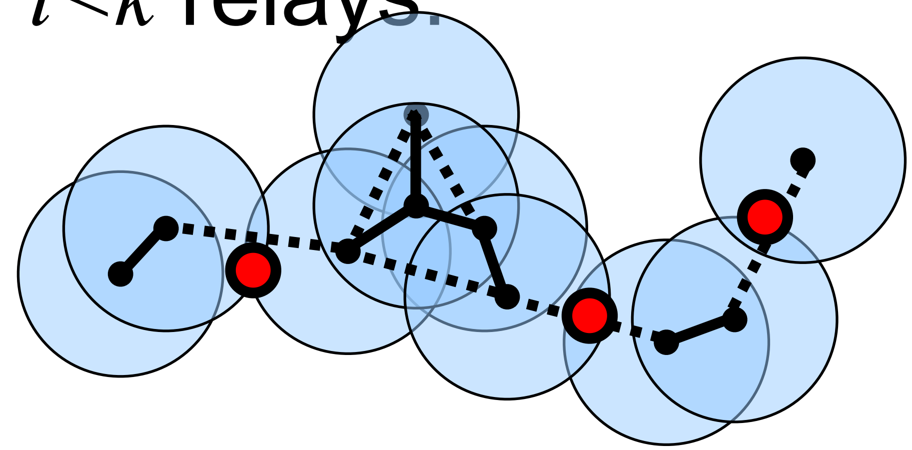
$$\boxed{|A_y|} \leq 3.11|R_l^*|$$

Consequence: Total # of relays we compute:

$$|A| = |A_r| + |A_g| + |A_y| \leq 3.11(|R_d^*| + |R_l^*|) = 3.11|R^*|$$

Clouds with Few Stabs

For any constant k, determine those clouds $C \in \mathcal{C}_i$ whose blobs can be stabbed with $i<k$ relays:



Then use Lemma 1 to complete connection:

$\leq 2i\text{-}1$ red relays to interconnect each $C \in \mathcal{C}_i$

Let $\mathcal{C}_{k+}$ be the set of clouds requiring $\geq k$ stabs

Stitching a Cloud From $\mathcal{C}_{k+}$

Consider clouds $C \in \mathcal{C}_{k+}$ that need more than k stabs.

Use greedy set cover for those clouds.

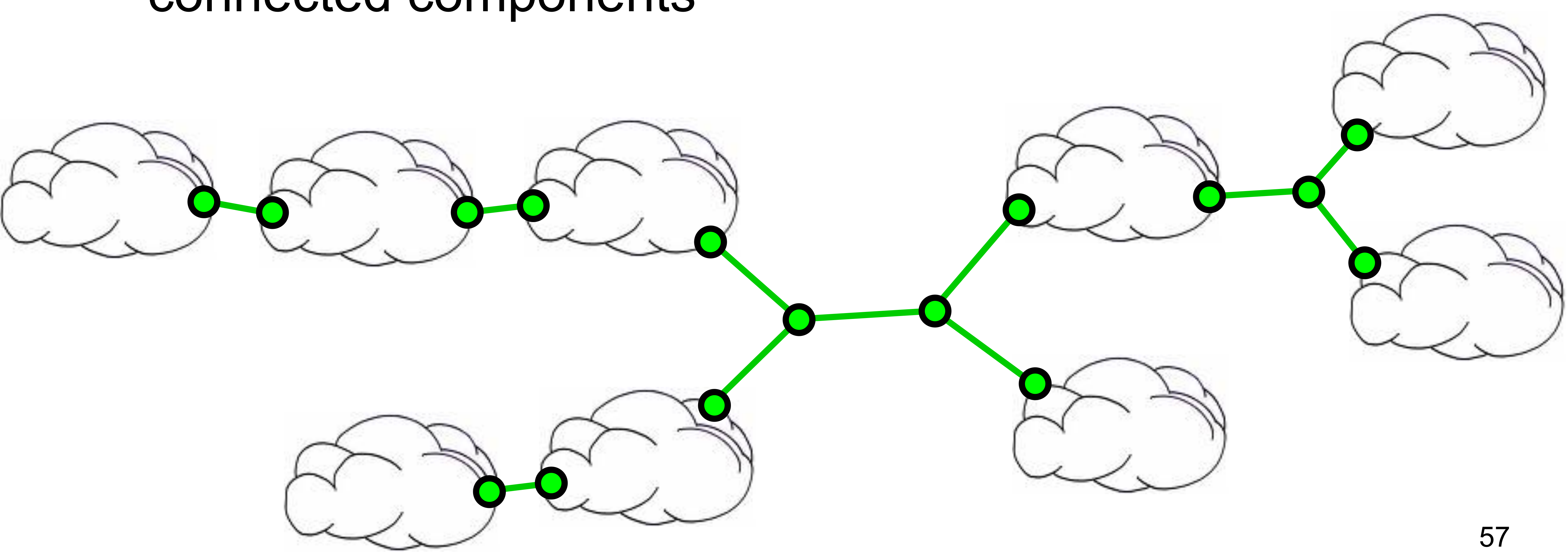
Lemma 3: For each cloud C,

$$|A_r^*| \le \frac{37}{12}|R_d^* \cap C| - 1$$

Proof: Omitted. Note that $1 + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = \frac{37}{12}.$

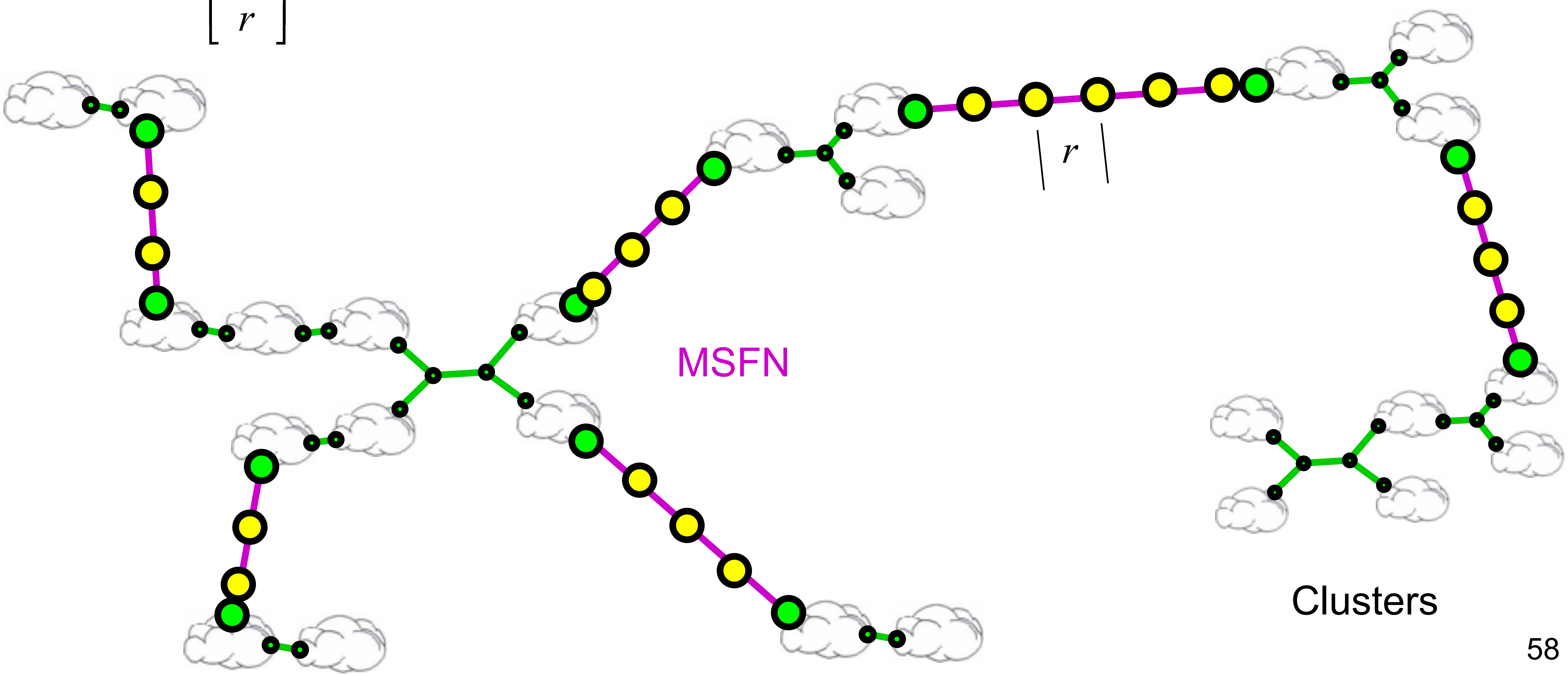
Cloud Clusters: Green Relays

- Add green relays to form cloud clusters:
  - Place 2 relays to connect 2 clusters; repeat
  - Then, place 4 relays to interconnect 3 clusters; repeat
  - Then, place 6 relays to interconnect 4 clusters; repeat
- On average: 2 green relays per decrease by 1 in # connected components



Interconnecting the Clusters

- Compute MSFN on set of *clusters*
- Put relays along edges:
  - 2 green relays at ends of each edge
  - $\left\lfloor \frac{|e|}{r} \right\rfloor$ yellow relays along each edge e



Analysis: Red and Green Relays

- An opt solution R* has at least
$$|R_d^*| \geq k\,|\mathcal{C}^{k+}| + \sum_{i=1}^{k-1} i\,|\mathcal{C}^i|$$
dark relays (inside clouds)
- Also, $|R_d^* \cap C| \geq 1, \quad \forall C$
- We place ≤ $2i-1$ red relays per cloud of $\mathcal{C}^i$,

and ≤ 37/12$|R^*_d \cap C|$-1 red relays per cloud of $\mathcal{C}^{k+}$

- We place ≤ 2$|\mathcal{C}|$ green relays for cloud interconn

$$|A_r| + |A_g| \leq \sum_{C \in \mathcal{C}^{k+}} \left(\frac{37}{12} |R_d^* \cap C| - 1 \right) + \sum_{i=1}^{k-1} (2i-1)\,|\mathcal{C}^i| + 2|\mathcal{C}|$$

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$$\le \frac{37}{12} \left(|R_d^*| - \sum_{i=1}^{k-1} i |\mathsf{C}^i| \right) - |\mathsf{C}^{k+}| + \sum_{i=1}^{k-1} 2i |\mathsf{C}^i| - \sum_{i=1}^{k-1} |\mathsf{C}^i| + 2|\mathsf{C}|$$

Analysis: Red and Green Relays

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$$= \frac{37}{12} \left(|R_d^*| - \sum_{i=1}^{k-1} i |\mathsf{C}^i| \right) + |\mathsf{C}^{k+}| + \sum_{i=1}^{k-1} (2i+1) |\mathsf{C}^i|$$

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$$\boxed{\frac{37}{12} i > (2i+1)}$$

$$= \boxed{\frac{37}{12}} \boxed{\left(|R_d^*| - \boxed{\sum_{i=1}^{k-1} i|\mathsf{C}^i|} \right)} + \boxed{|\mathsf{C}^{k+}|} + \boxed{\sum_{i=1}^{k-1} (2i+1)|\mathsf{C}^i|}$$

$$\le \frac{37}{12}|R_d^*| + |\mathsf{C}^{k+}| < \left(3.084 + \frac{1}{k}\right)|R_d^*|$$

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$$= \boxed{\frac{37}{12} \left(|R_d^*| \right.} \boxed{} + \boxed{|\mathcal{C}^{k+}|} + \boxed{}$$

$$\le \frac{37}{12} |R_d^*| + |\mathcal{C}^{k+}| < \left(3.084 + \frac{1}{k} \right) |R_d^*|$$

Analysis: Red and Green Relays

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$$\le \frac{37}{12} |R_d^*| + |\mathrm{C}^{k+}| < \left(\boxed{3.084} + \boxed{\frac{1}{k}} \right) |R_d^*|$$

Analysis: Yellow Relays

Lemma 4: ∃ spanning forest with neighborhoods on clusters requiring

$$|A_y| \le \left(\frac{4}{\sqrt{3}} + \frac{4}{5}\right)|R_w^*| < 3.11\,|R_w^*|$$

yellow relays.

Proof: Omitted.

(Combines Steiner tree ratio with degree arguments.).

Summary: 3.11-Approximation

Approximation bound:

$$R^* = R_d^* \cup R_l^*$$

dark — Within clouds

light — Outside clouds

We showed:

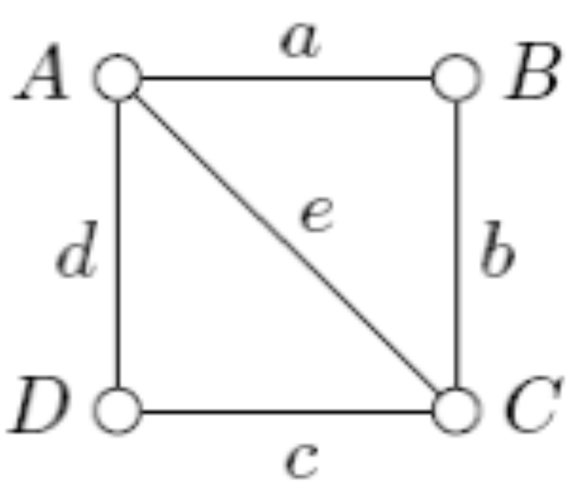
$$|A_r| + |A_g| \le \boxed{(3.084} + \boxed{1/k)} |R_d^*|$$

$$|A_y| \le 3.11 |R_l^*|$$

Theorem: Total # of relays we compute in poly time

$$|A| = |A_r| + |A_g| + |A_y| \le 3.11(|R_d^*| + |R_l^*|) = 3.11 |R^*|$$

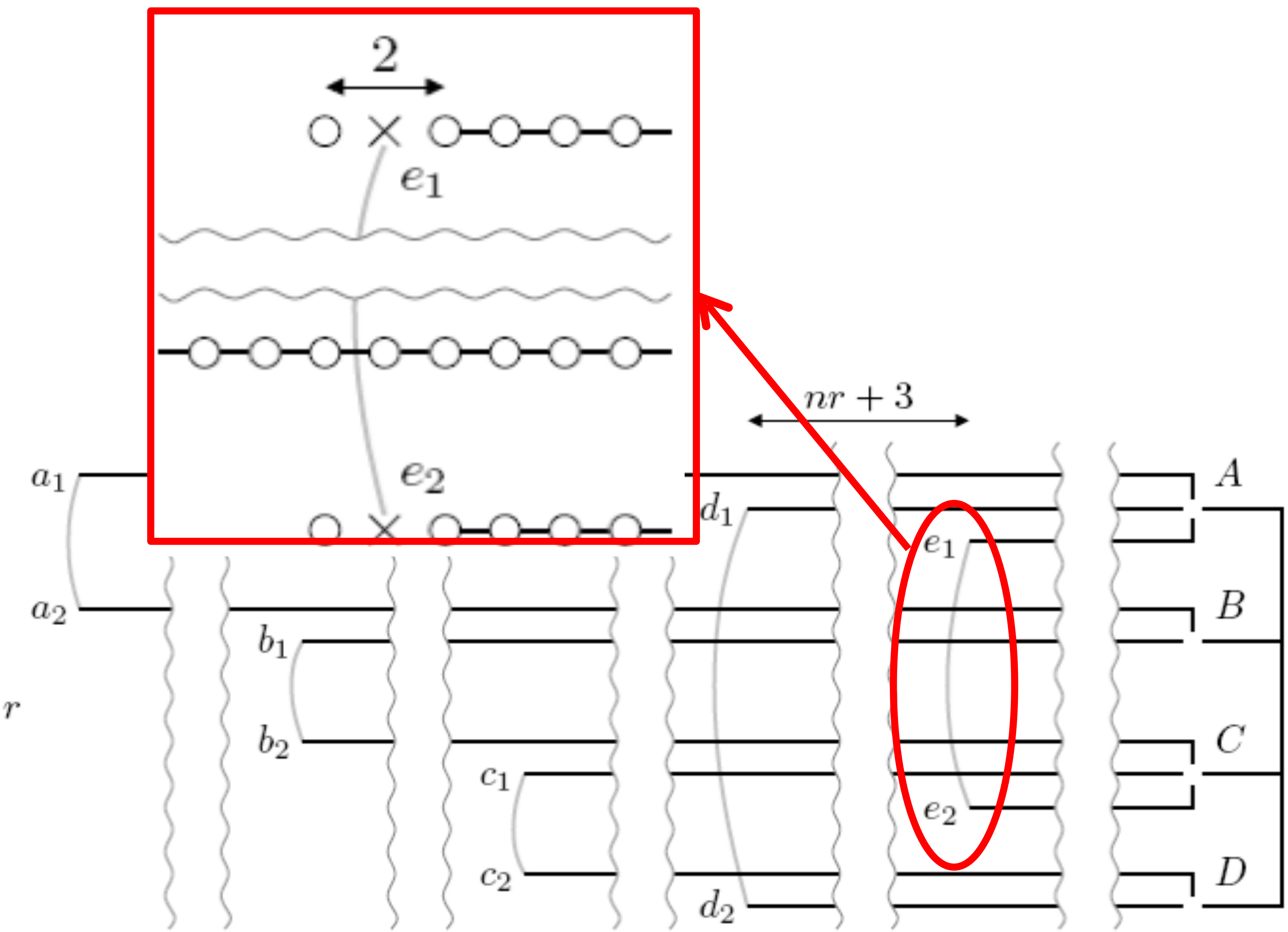
Inapproximability



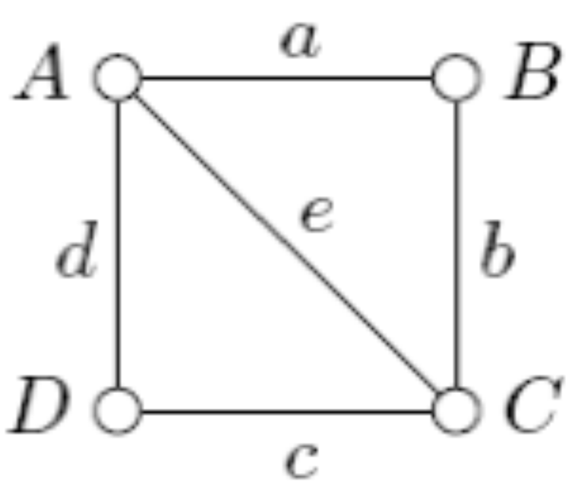
$n = 4$ vertices

$m = 5$ edges

(b)



Inapproximability



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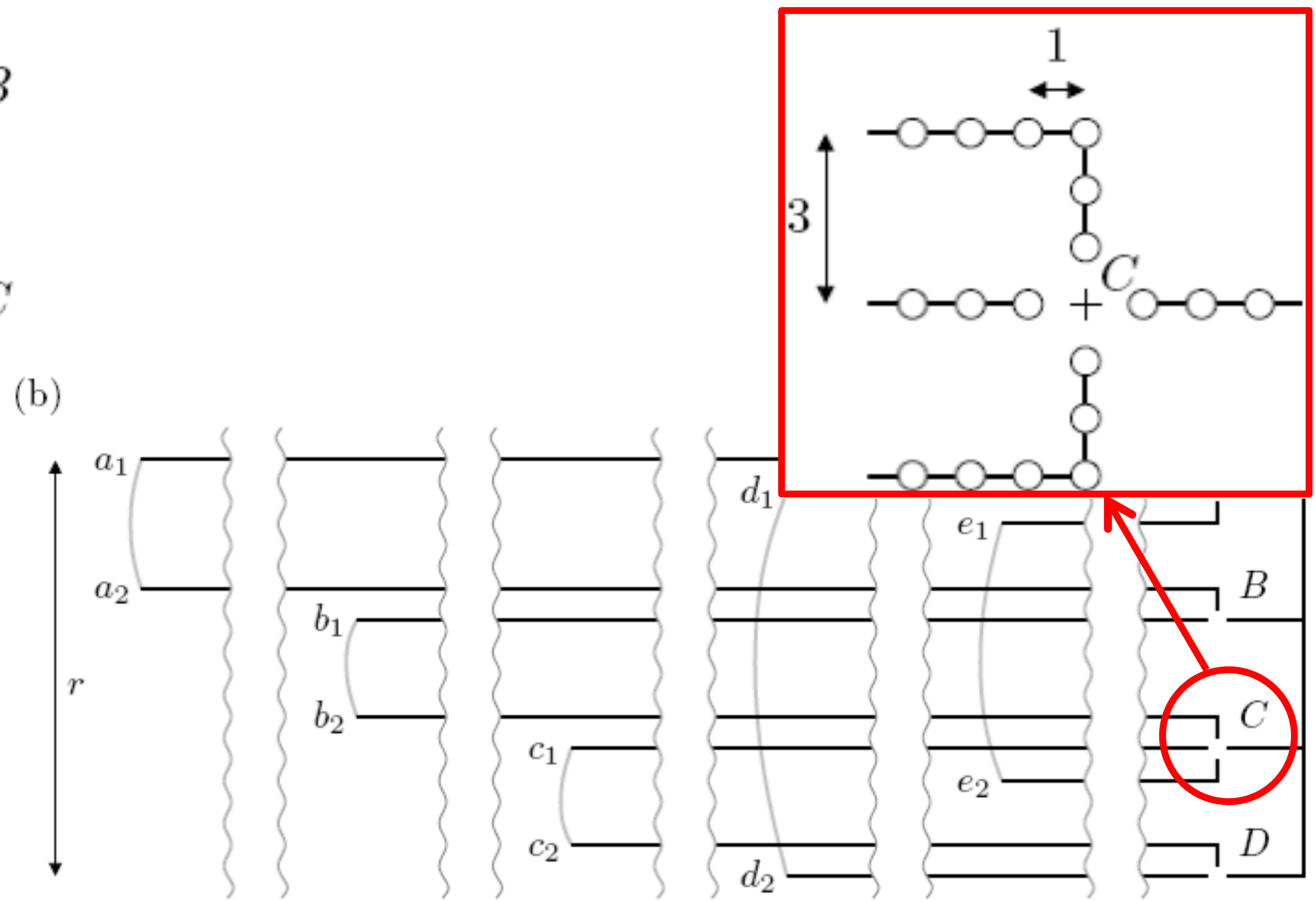
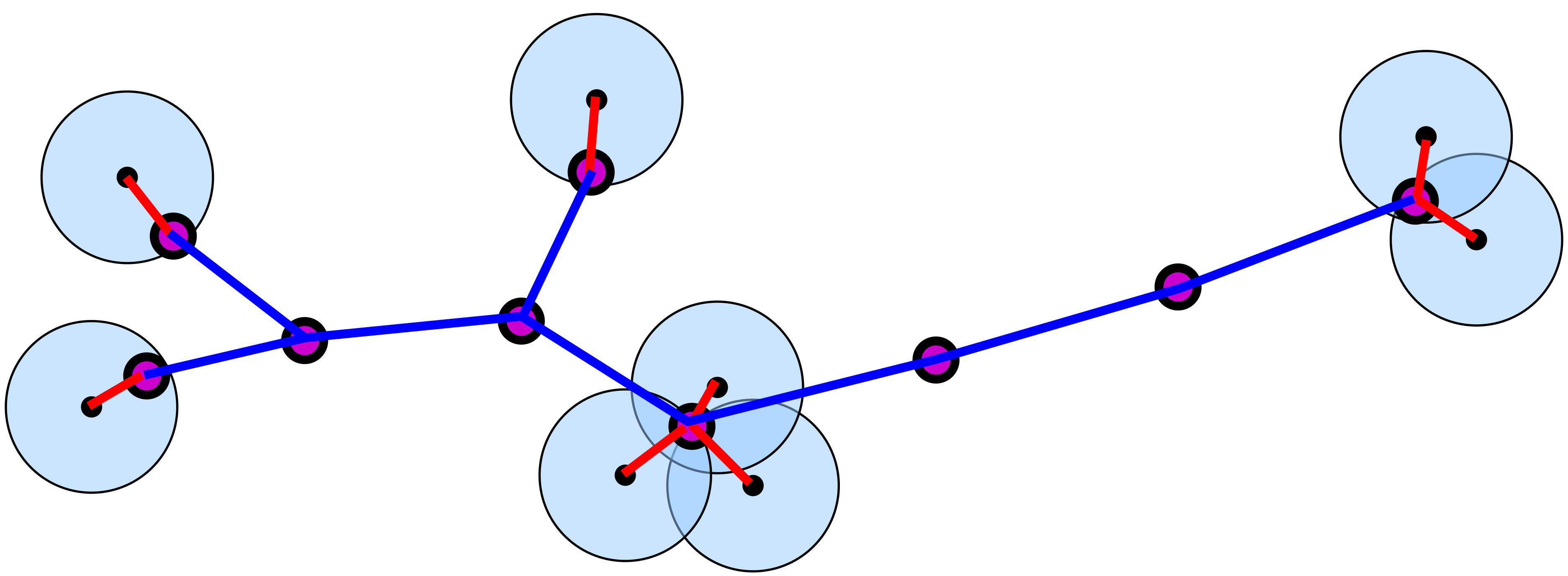


PTAS: Two-Tier Relay Placement

- Steiner spanning tree
- Edges of length ≤ r between relays, ≤ 1 between a relay and a sensor
- A sensor has degree 1 (cannot relay data)
- Goal: Min # Steiner points (relays)



PTAS: Outline

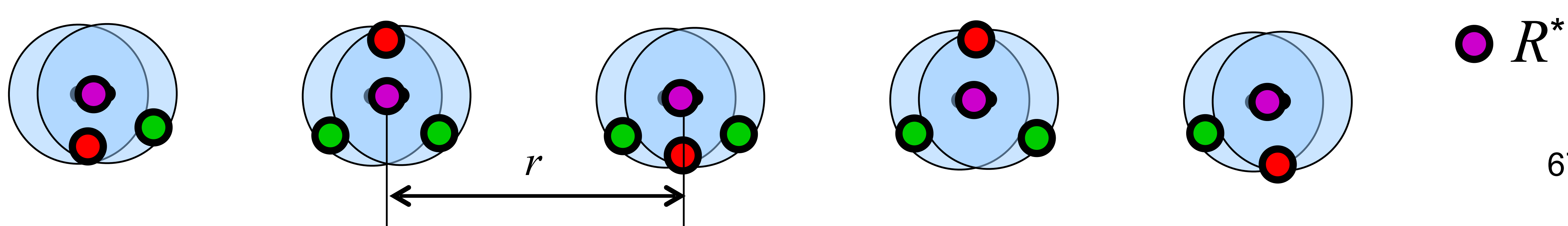
- Spanning tree has red edges (incident on sensors) and blue edges (between relays)
- Round OPT to a poly-size grid, G, of candidate locations for relays
- Set of blue edges of OPT can be made m-guillotine, increasing length by only factor $(1+\varepsilon)$ by the added bridges
- Bridges can be replaced by a set of relays
- Optimize over all m-guillotine spanning trees using dynamic programming

Summary

- Simple O($n \log n$)-time 6.73-approx for one-tier relay placement
- Poly-time 3.11-approx for one-tier, any $r \geq 1$
- No PTAS for one-tier (r is part of input), assuming P $\neq$ NP
- PTAS for two-tier relay placement

Additional Remarks

- We believe we can give a PTAS for PRBDSP (Planar Red/Blue Dominating Set Problem)
  - Then 6.73 becomes 4.16
- More involved geometric argument: Lemma 4 improves (3 instead of 3.11), resulting in 3.09-approx
  - With a PTAS for PRBDSP we get 3+ε.
  - 3-approx is best possible analysis for our method:



Concluding Remarks

- Is there a PTAS for *constant* values of *r* ?
  - Our hardness of approximation relies on large *r.*
  - Problem is related to TSP/MST with Neighborhoods.
- Fault tolerance: *k*-connectivity
  - [Bredin, Demaine, Hajiaghayi, Rus, MobiHoc'05, Zhang, Xue, Misra]

Thank you!

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