Solution 5
Approximation Algorithms

30.6.2021

1) Let $K = \sum_{e \in E} c(e) + 1$.

To transform $I = (G, c)$ with $G = (V, E)$ to

$J = ((V, \binom{V}{2}), c')$, an instance on a complete graph,

$$\binom{V}{2} = \{\{v,w\} \mid v,w \in V, v \neq w\}$$

we set $c'(e) = c(e)$, if $e \in E$ and $c'(e) = K$ otherwise.

If an optimal solution to J has value $OPT_J \geq K$,

then I does not have a tour / G is non-Hamiltonian.
Otherwise, OPT_J only uses edges from E and
is an optimal solution to I.

2) a) Let $G = (V, \binom{V}{2})$ be a complete graph,
let $c(e)$ be a cost function with $c(e) \in \{1,2\}$ for
all e. Let $u, v, w \in V$.

$$c(uv) + c(vw) \geq 1 + 1 \geq c(uw).$$

b) Let $G = (V, E)$ be an instance of Hamiltonian Cycle;
E does not have to be complete.
Construct an instance of Metric TSP as follows:
$$G' = (V, \binom{V}{2}), \quad c'(e) = \begin{cases} 1, & e \in E \\ 2, & e \notin E \end{cases}$$

This only uses edge weights 1 and 2 and thus
satisfies Δ-inequality by (a).
G has a Hamiltonian cycle iff (G', c') has a tour
of length $n = |V|$.

c) Assume towards a contradiction that for some $f(n)$, A is an approximation algorithm with approx.-factor $f(n)$.

We construct a polynomial-time algorithm for Hamiltonian Cycle, contradicting $P \neq NP$.
Given a graph $G=(V,E)$ with n vertices, we construct an instance of TSP by setting $c(e)=1$ for $e \in E$ and $c(e) = n \cdot f(n)+1$.
We then pass this instance to A.
If A returns a tour T with length ℓ, we check $\ell \geq n \cdot f(n)+1$.
If $\ell \geq n \cdot f(n)+1$, then the tour from A uses an edge that is not in E.
If $\ell < n \cdot f(n)+1$, then $\ell \leq n$ $(\ell = n)$, because it only uses edges from E.

$$\leq n \xleftrightarrow{\text{large gap}} \geq n \cdot f(n)+1$$

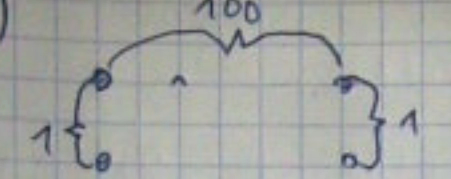
$$\ell \leq f(n) \cdot OPT \iff \frac{1}{f(n)} \cdot \ell \leq OPT$$

$$\ell \geq n \cdot f(n)+1 \Rightarrow OPT \geq \frac{n \cdot f(n)+1}{f(n)} = n + \frac{1}{f(n)} > n.$$

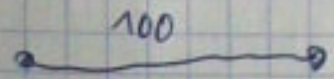
$\Rightarrow$ G is non-Hamiltonian.

Otherwise, A gives a Hamiltonian Cycle
$\Rightarrow$ Our algorithm solves HC in polynomial time.
$\Rightarrow$ $\lightning$ $P \neq NP$.

3) a)

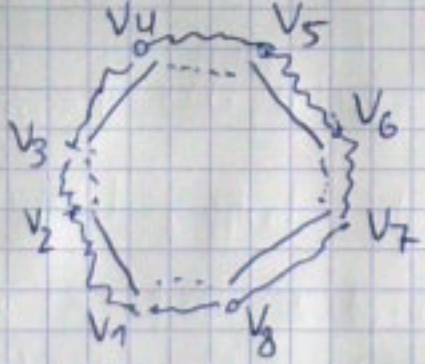
V: 

Perfektes Matching: 2

V' 

Perfektes Matching: 100

b) Let $V' \subseteq V$, $|V'| = 2k$. $V' = \{v_1, \dots, v_{2k}\}$, where $v_1, v_2, \dots, v_{2k}$ is the order in which the vertices of V' appear in the tour.



Consider the pieces $v_1 \rightsquigarrow v_2$, $v_2 \rightsquigarrow v_3$, $\dots$, $v_{2k} \rightsquigarrow v_1$.

Triangle inequality: $\underbrace{v_1 \rightsquigarrow v_2}_{\geq c(v_1 v_2)}$, $\underbrace{v_2 \rightsquigarrow v_3}_{\geq c(v_2 v_3)}$, $\dots$, $\underbrace{v_{2k} \rightsquigarrow v_1}_{\geq c(v_1 v_{2k})}$

Observe that $v_1v_2, v_3v_4, v_5v_6, \dots, v_{2k-1}v_{2k}$ (M_1) and

M_2: $v_2v_3, v_4v_5, v_6v_7, \dots, v_{2k}v_1$ are perfect matchings on V'.

Thus, $OPT \geq c(v_1v_2) + \dots + c(v_1v_{2k})$

$= M_1 + M_2$, and the Min-Cost Perfect Matching on V' (M_{min}) is at most

$M_{min} \leq M_1, M_2$

$\rightsquigarrow OPT \geq 2 M_{min} \rightsquigarrow M_{min} \leq \frac{1}{2} OPT$. □