$L \in NP \Leftrightarrow$ There is a polynomial-time algorithm V and a polynomial $q(n)$ such that an instance I is in L iff $\exists x \in \{0,1\}^{q(n)}: V(I,x)$.

No Vertex Cover?

$$\exists x \in \{0,1\}^{q(n)}: \neg V(I,x) \not\equiv \underbrace{\forall x \in \{0,1\}^{q(n)}: \neg V(I,x)}_{\text{No Vertex Cover}}$$

No Vertex Cover $\in$ co-NP (co-NP-complete)

$\rightsquigarrow$ (probably) No Vertex Cover $\notin NP$

Exercise 1: (a) Show that C is a vertex cover of G iff $I = V \setminus C$ is an independent set.

$$\begin{aligned}
C \text{ is VC} &\Leftrightarrow \forall \overset{\{u,v\}}{e} \in E: (u \in C \vee v \in C) \\
&\overset{I = V\setminus C}{\Leftrightarrow} \forall e \in E: (u \notin I \vee v \notin I) \\
&\Leftrightarrow \forall e \in E: \neg(u \in I \wedge v \in I) \\
&\Leftrightarrow \neg \exists \{u,v\} \in E: (u \in I \wedge v \in I) \\
&\Leftrightarrow I \text{ is an independent set.}
\end{aligned}$$

(b) NP-hardness

- Reduction f: Instances of VC $\to$ Instances of IS, $(G,k) \mapsto (G, |V|-k)$.
- Obviously polynomial-time.
- Prove that (G,k) is a (VC) yes-instance iff $(G,|V|-k)$ is a IS yes-instance: Using (a), G has VC of size k iff G has an IS of size $|V|-k$.

NP-membership: NTM guesses $C \subseteq V$, $|C| = k$.
Check if every edge of G has an endpoint in C. $\square$

Exercise 2: Given a directed graph $G=(V,A)$, find a subset $A^* \subseteq A$ without directed cycles. Chose an arbitrary labeling $\ell: V \to \{1,\dots,n\}$.

F: Forward Edges: $F \subseteq A$, $(v,w) \in F$ iff $\ell(v) < \ell(w)$.

B: Backward Edges: $B \subseteq A$, $(v,w) \in B$, iff $\ell(v) > \ell(w)$.

F and B are disjoint, $A = F \cup B$.

$|A| = |F| + |B| \rightsquigarrow \frac{|A|}{2} \leq \max\{|F|, |B|\} =: A'$.

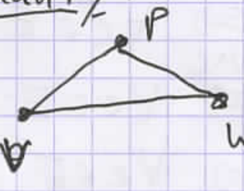
Why are F and B cycle-free?

Starting from any vertex v, all reachable vertices (via F) have labels $\ell(w) > \ell(v)$, so we cannot reach v starting from v. (Analogously for B).

Exercise 3: Euclidean space $\mathbb{R}^d$ (with Euclidean distances $\|.\|_2$)

$d \geq 2$ $(c=2)$: We use the triangle inequality:

$\|v-w\|_2 \leq \|v-p\|_2 + \|p-w\|_2$.



Choose an arbitrary point p.

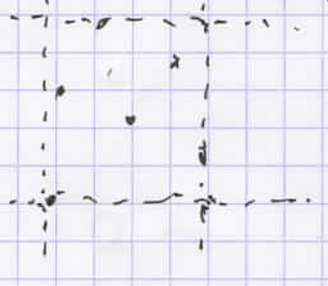
Compute the point q that maximizes $\|p-q\|_2$, i.e., the distance to p. $\Delta' := \|p-q\|_2$.

The diameter is between two points v, w: $\Delta = \|v-w\|_2 \geq \|p-q\|_2$. ✓

$\|v-w\|_2 \leq \underbrace{\|v-p\|_2}_{\leq \|p-q\|_2} + \underbrace{\|p-w\|_2}_{\|p-q\|_2} \leq 2\|p-q\|_2 = 2\Delta'$. ✓

$d=2$, $c<2$:

2 points? ↯



Compute x_{min}, x_{max}, y_{min}, y_{max}: the min./max. x- and y-coordinates of all points.

$\Delta' := \max\{x_{max} - x_{min}, y_{max} - y_{min}\}.$

$$\Delta = \sqrt{\underbrace{(x_v - x_w)^2}_{\leq (x_{max}-x_{min})^2 \leq \Delta'^2} + \underbrace{(y_v - y_w)^2}_{\leq (y_{max}-y_{min})^2 \leq \Delta'^2}} \leq \sqrt{\Delta'^2 + \Delta'^2} = \sqrt{2\Delta'^2} = \sqrt{2} \cdot \Delta'. \qquad \square$$