Algorithms Department **Summer 2021**
Department of Computer Science - IBR
TU Braunschweig

Prof. Dr. Sándor P. Fekete
Dr. Phillip Keldenich
Dominik Krupke

Approximation Algorithms
Exercise 5
June 16, 2021

Please hand in your solutions until June 30, 11:30 am by e-mail to `keldenich@ibr.cs.tu-bs.de`.

Exercise 1 (TSP on Graphs):
In the lecture, we generally considered the TSP on complete graphs. In other words, in the context of the lecture, it was always possible to directly move from any city u to any other city v without visiting any other city in between. Thus, it was always clear that a tour exists.
In this exercise, we consider the Traveling Salesman Problem on weighted graphs (Graph TSP), where tours do not necessarily exist. We are given an n-vertex graph $G = (V, E)$ with non-negative edge costs $c(e) : E \to \mathbb{R}_{\geq 0}$ and have to compute a tour, i.e., a sequence $v_1 v_2 \dots v_n v_{n+1}$ of vertices that contains all n vertices, such that $v_{n+1} = v_1$ and $v_i v_{i+1} \in E$ for all i and the sum of all edge weights $c(v_i v_{i+1})$ is minimized.
Reduce Graph TSP to the TSP on complete graphs shown in the lecture, i.e., efficiently transform an instance I of Graph TSP to an instance J of the TSP such that you can transform the optimal solution of J to an optimal solution of I or decide that no such solution exists in polynomial time. **(5 P.)**

Exercise 2 (TSP: Hardness and Inapproximability):
In the previous exercise, we have seen that the TSP is still hard even if the existence of a tour is guaranteed and finding a valid tour is trivial. Graph TSP is also still hard if all edge weights are set to $c(e) = 1$; the problem is then also known as Hamiltonian Cycle.
In this exercise, we consider the other assumption that we made in the lecture: we assumed that the edge weights satisfy the triangle inequality, i.e., that for any three vertices u, v, w, we have $c(uv) + c(vw) \geq c(uw)$. TSP on complete graphs with this additional restriction is also called Metric TSP.

(a) Show that the edge weights of any complete graph that only uses edge weights 1 and 2 satisfy the triangle inequality.

(b) By a reduction from Hamiltonian Cycle, show that the TSP is still NP-hard on complete graphs with edge weights that satisfy the triangle inequality.

(c) Let $f(n) : \mathbb{N} \to \mathbb{Q}$ be any polynomial-time computable function (this also implies that the output of $f(n)$ can be encoded in $\text{poly}(n)$ bits). Assuming $\mathsf{P} \neq \mathsf{NP}$, show that there is no polynomial-time $f(n)$-approximation algorithm for the TSP on complete graphs that do not have to satisfy the triangle inequality.

(3+4+8 P.)

Exercise 3 (TSP: Christofides' Algorithm):
In the lecture, we sketched a $\frac{3}{2}$-approximation algorithm for METRIC TSP due to Christofides. The algorithm computes an MST on V and a minimum-cost perfect matching M on the (even number of) vertices with odd degree in that MST. It then adds the matching edges to the MST, resulting in a Eulerian graph U, computes a Eulerian walk on U and constructs a tour by skipping repeated vertices in that walk using the triangle inequality. We claimed that the total edge weight of U was at most $\frac{3}{2}\,\text{OPT}$, where OPT denotes the weight of an optimal tour. In particular, we claimed that the weight of M is at most $\frac{1}{2}\,\text{OPT}$.

(a) Let $G = (V, E)$ be a graph with non-negative edge weights satisfying the triangle inequality. Prove or disprove: For any $V' \subseteq V$ with evenly many vertices, the cost of a minimum-cost perfect matching on V' is upper-bounded by the cost of a minimum-cost perfect matching on V.

(b) Prove: For any $V' \subseteq V$ with evenly many vertices, the cost of a minimum-cost perfect matching on V' is at most half the cost of an optimal tour on V.

(3+7 P.)