Approximation Algorithms
Chapter 4: Tours

Sándor P. Fekete

Algorithms Division
Department of Computer Science
TU Braunschweig

1. Introduction
2. Longest Tours
3. Stars and Matchings
4. Nonsimple Polygons
5. Optimal Area
6. Turn Cost

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Several Centuries before Euclid

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Several Centuries before Euclid

Ανδρα μοι εννεπε, Μουσα, πολυτροπον, οσ μαλα πολλα πλανχθε...

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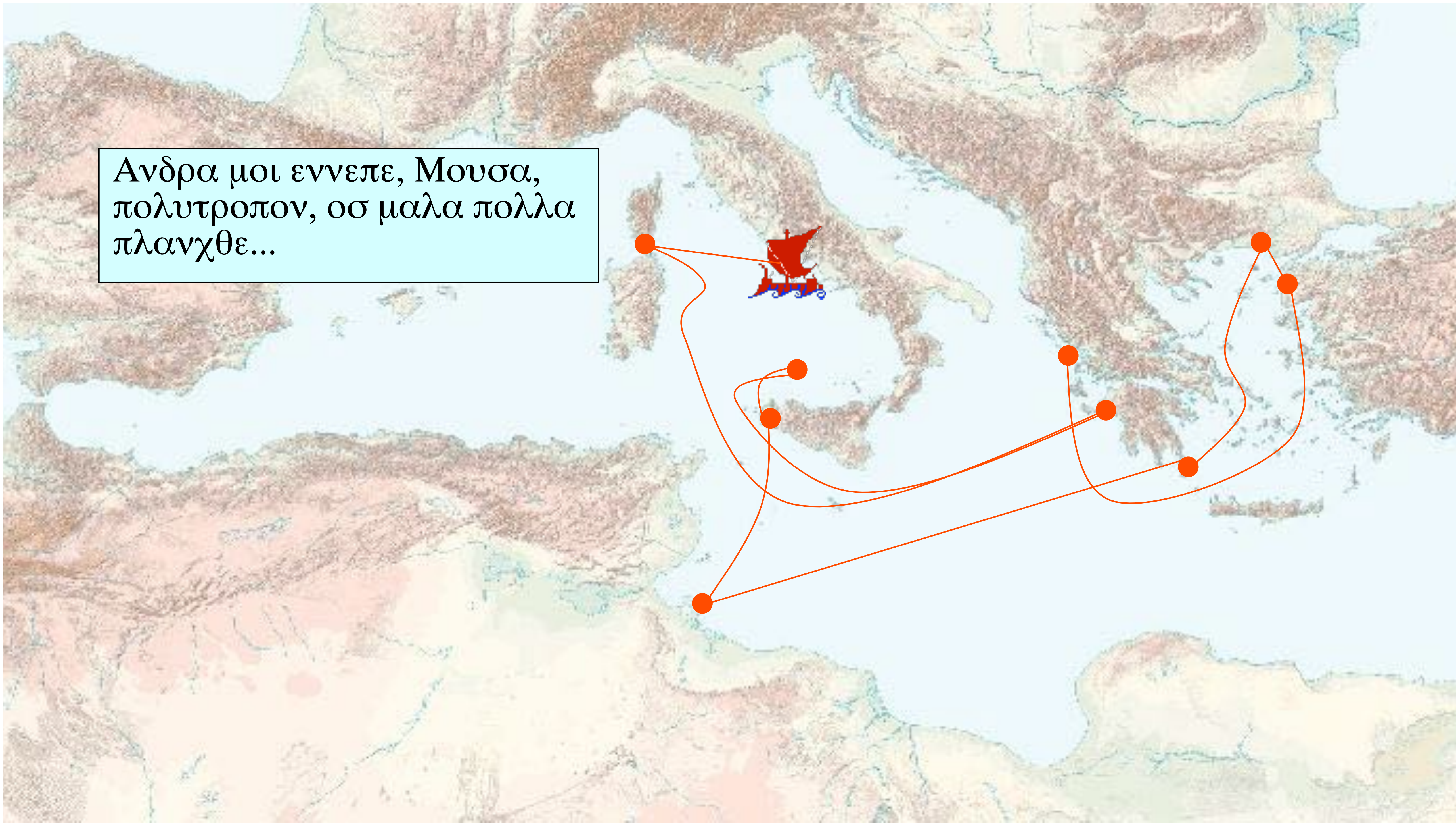

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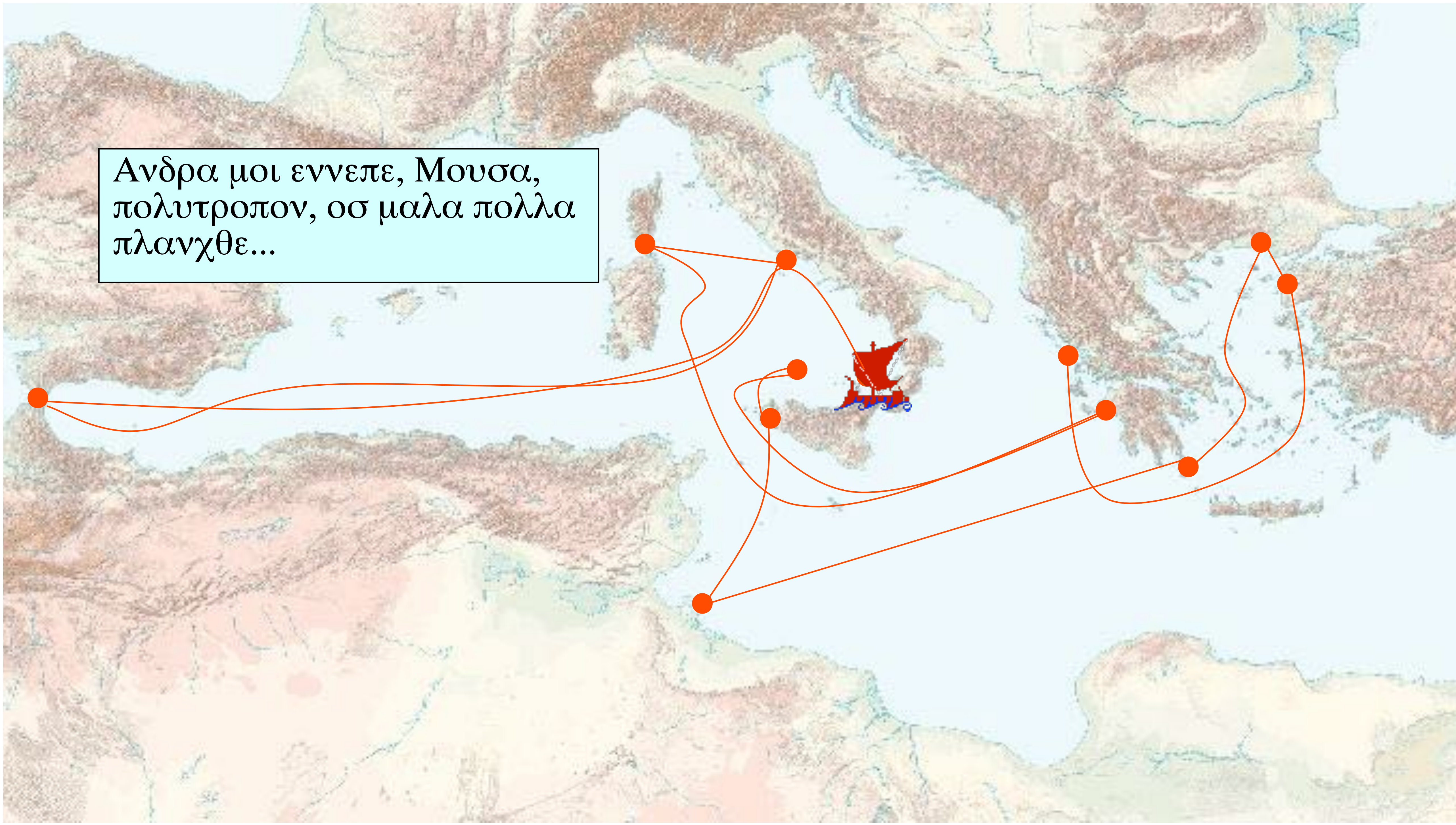

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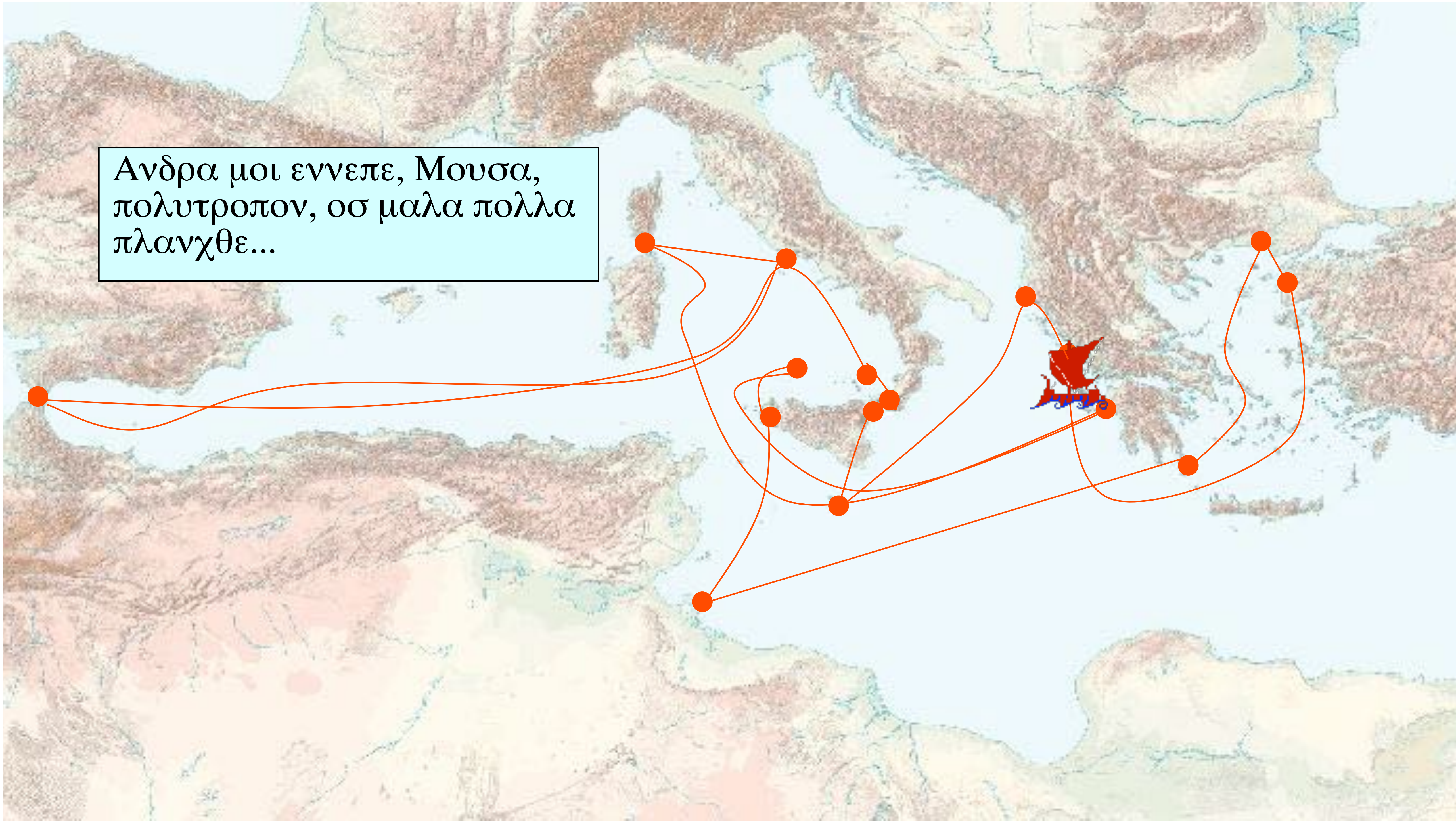

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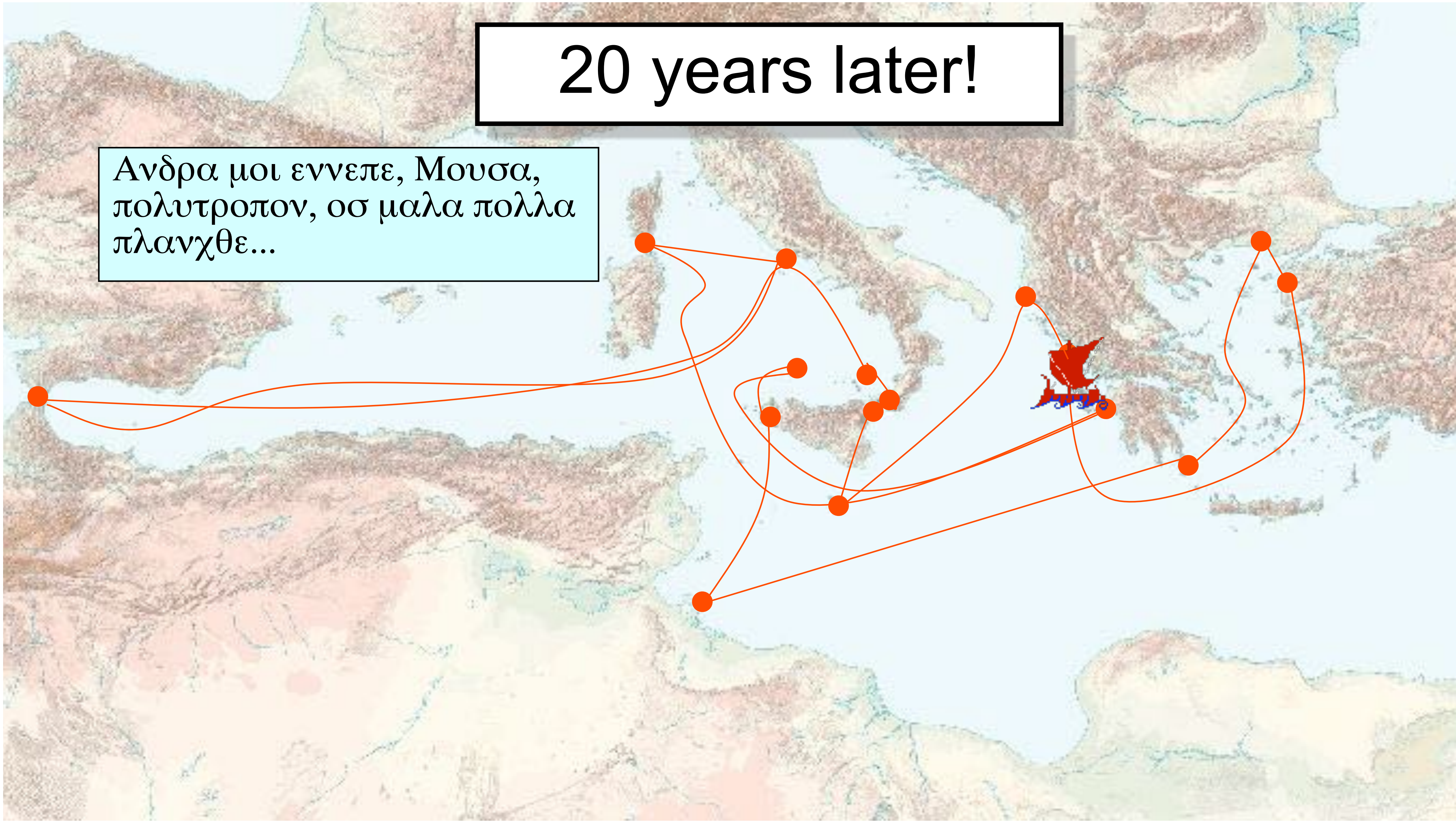

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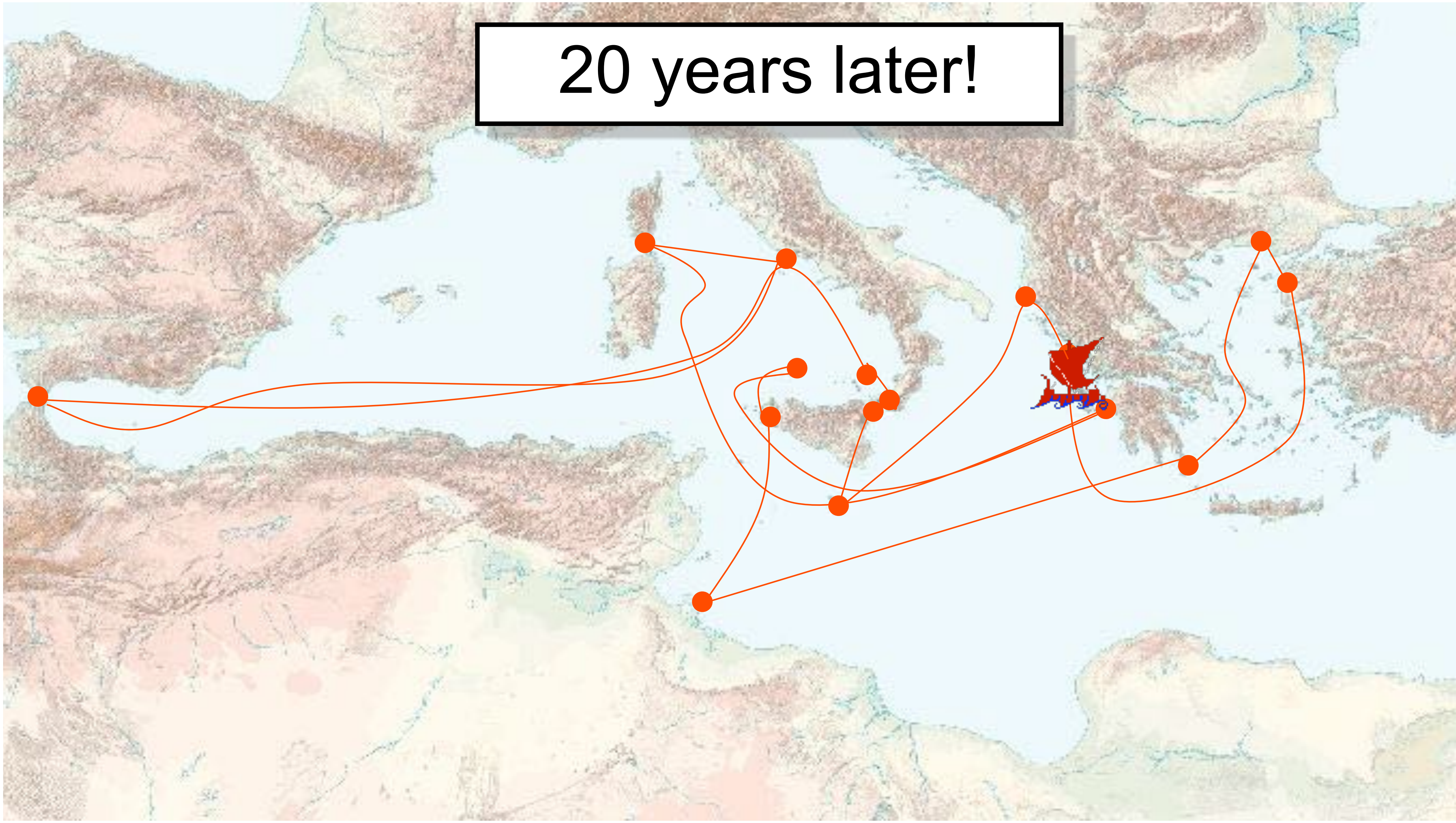

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Several Centuries before Euclid

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The Traveling Salesman Problem

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Given:

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The Traveling Salesman Problem

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A graph $G = (V, E)$
with edge lengths w_e

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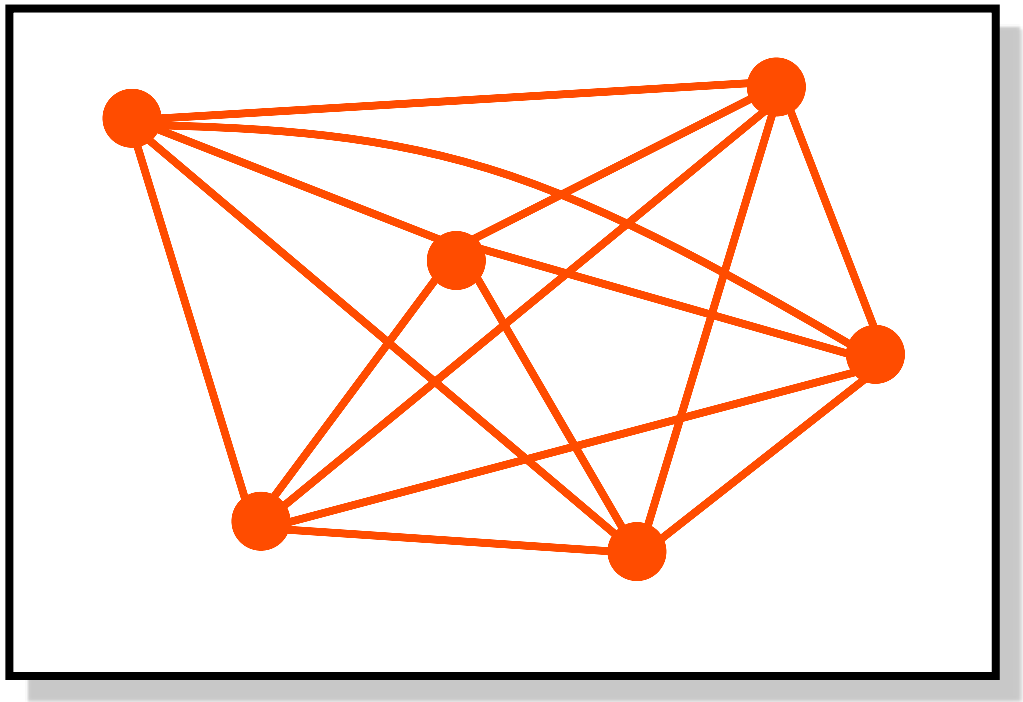

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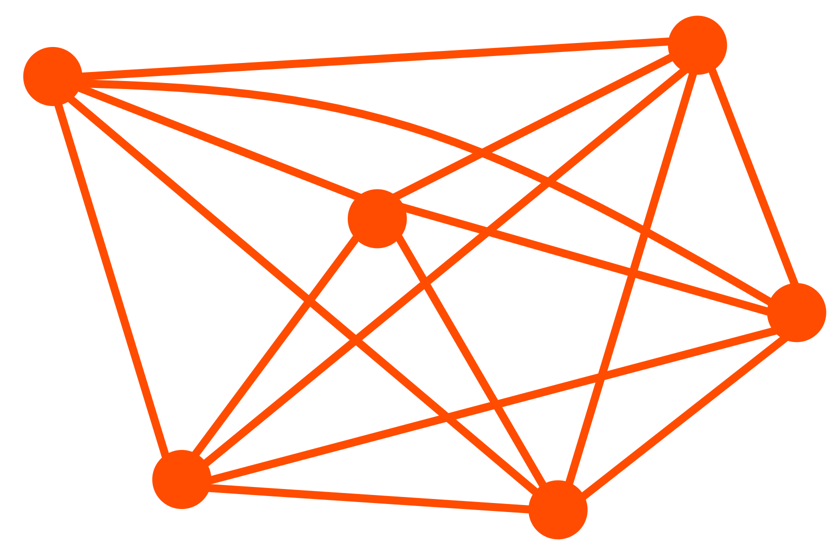

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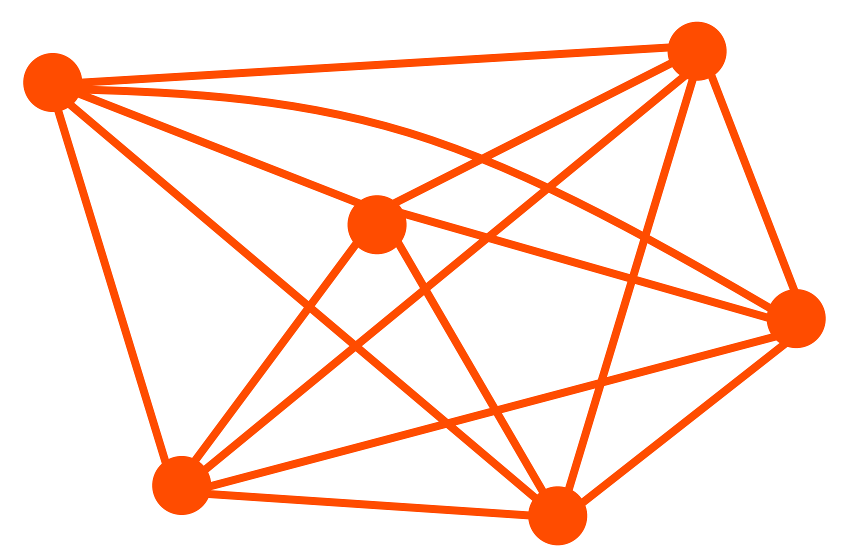

Wanted: A shortest roundtrip through all vertices.

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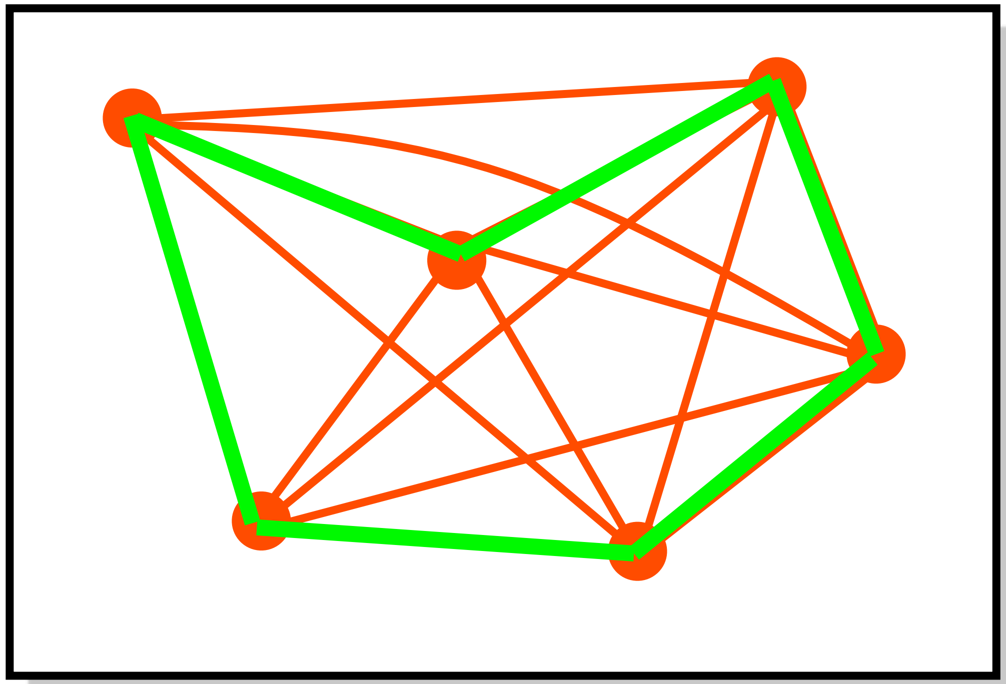

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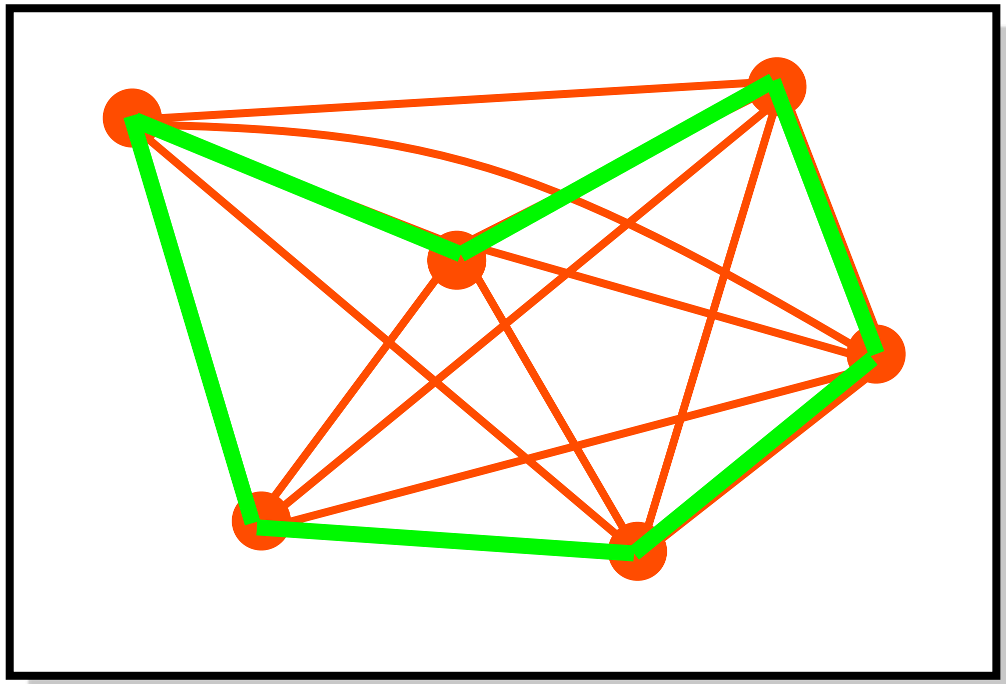

Wanted: A shortest roundtrip through all vertices.

Theorem: (Karp 1972) Finding a shortest tour is NP-hard.

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Back to Odysseus

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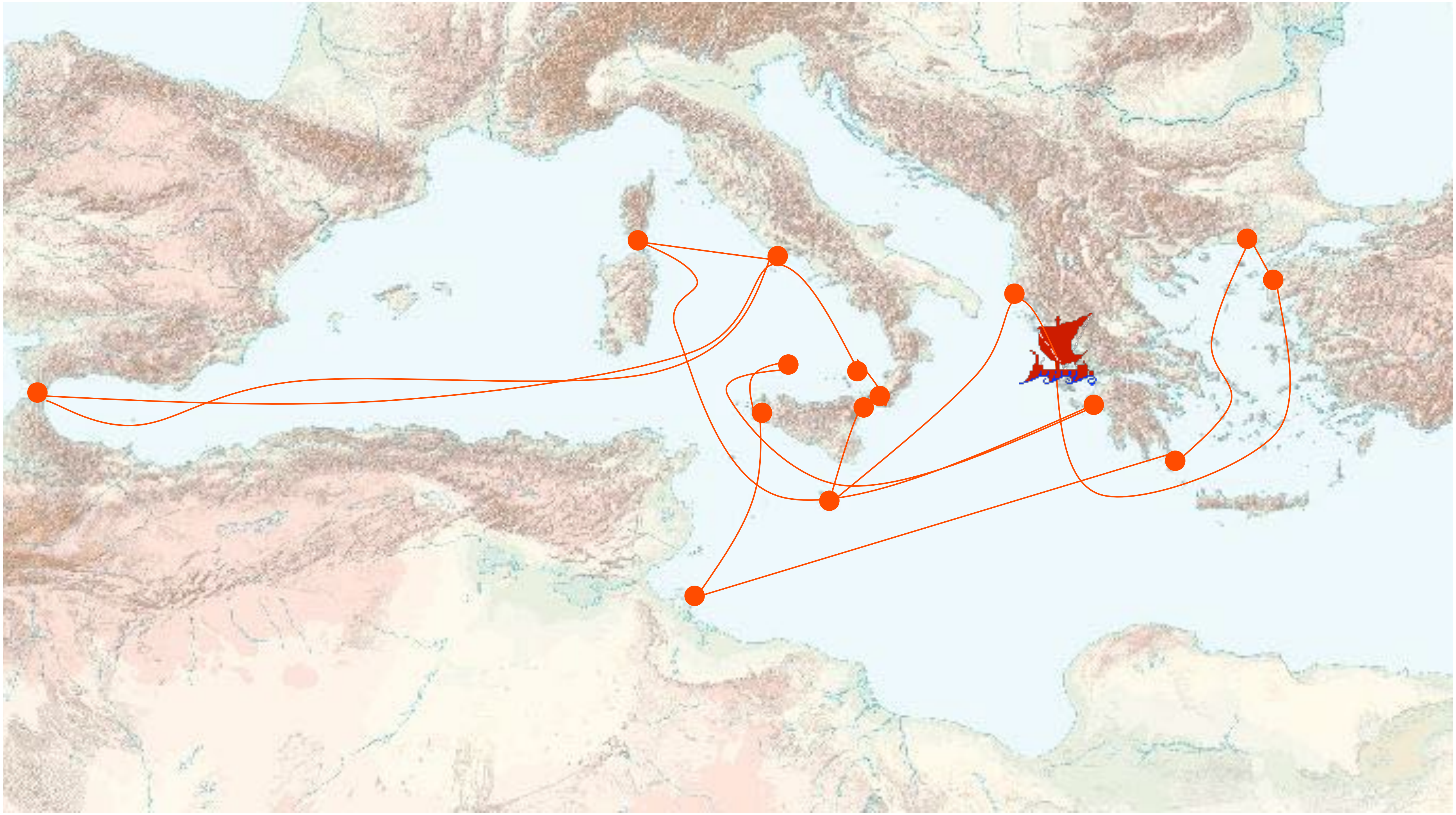

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Back to Odysseus

„Couldn‘t you make it any *slower*?“

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The Maximum TSP

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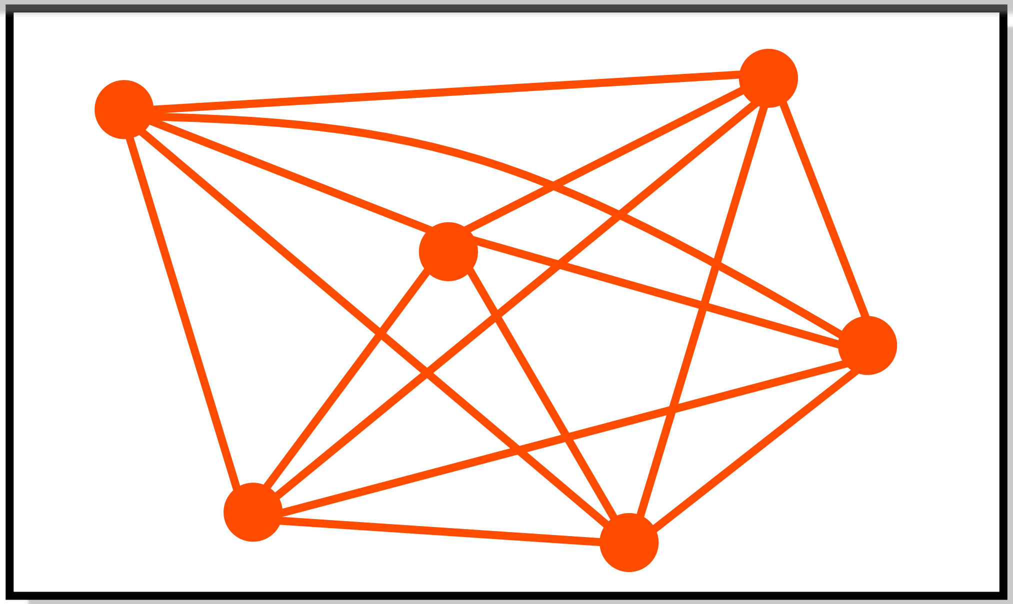

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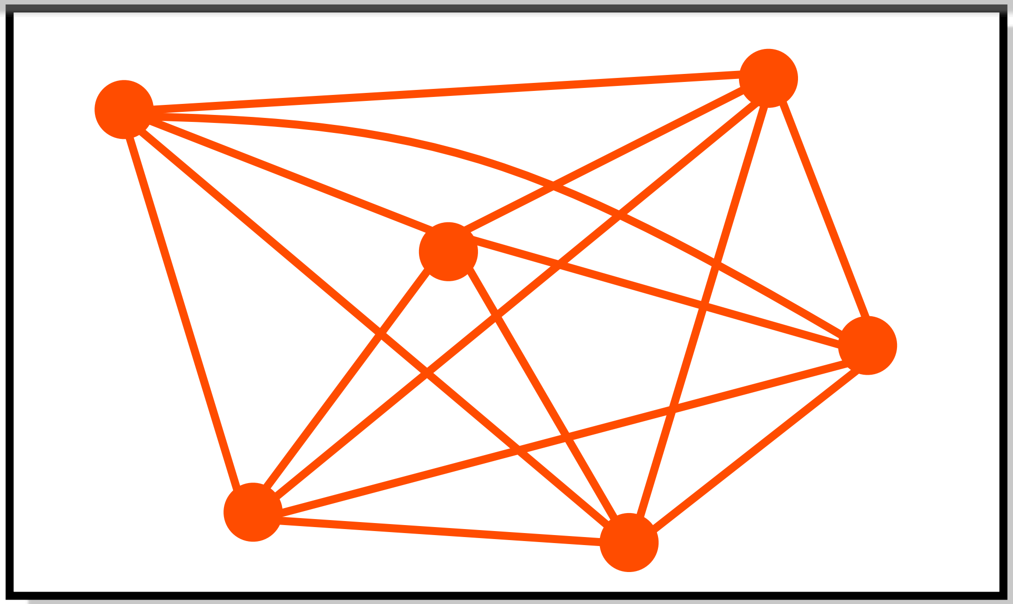

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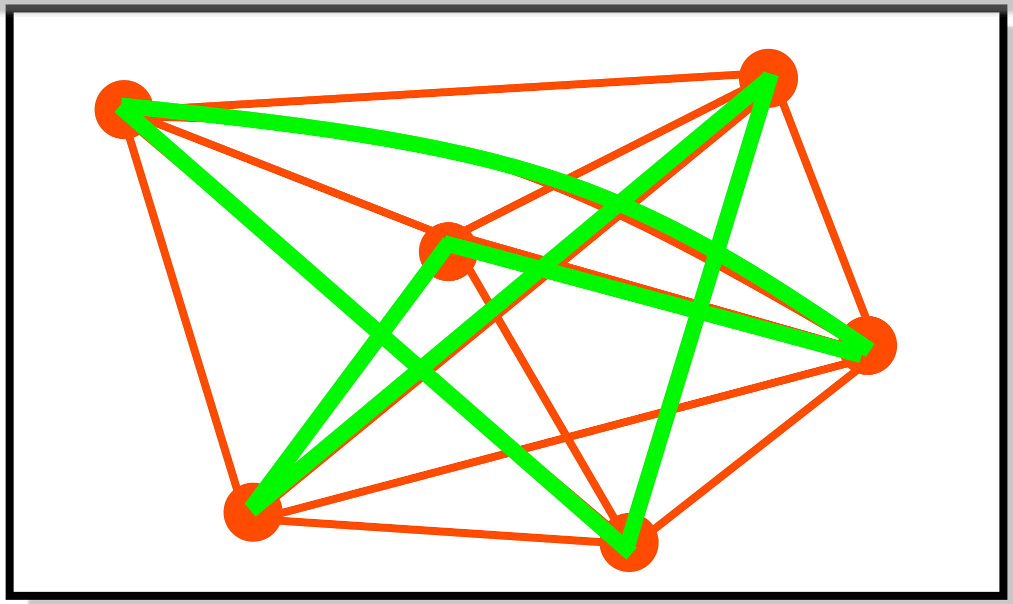

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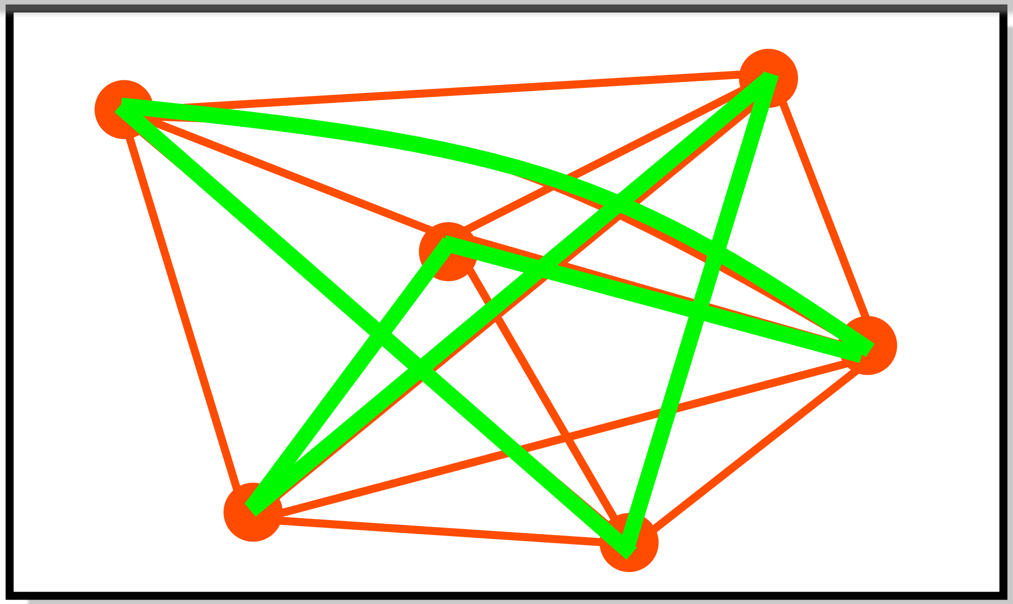

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Simple excercise:

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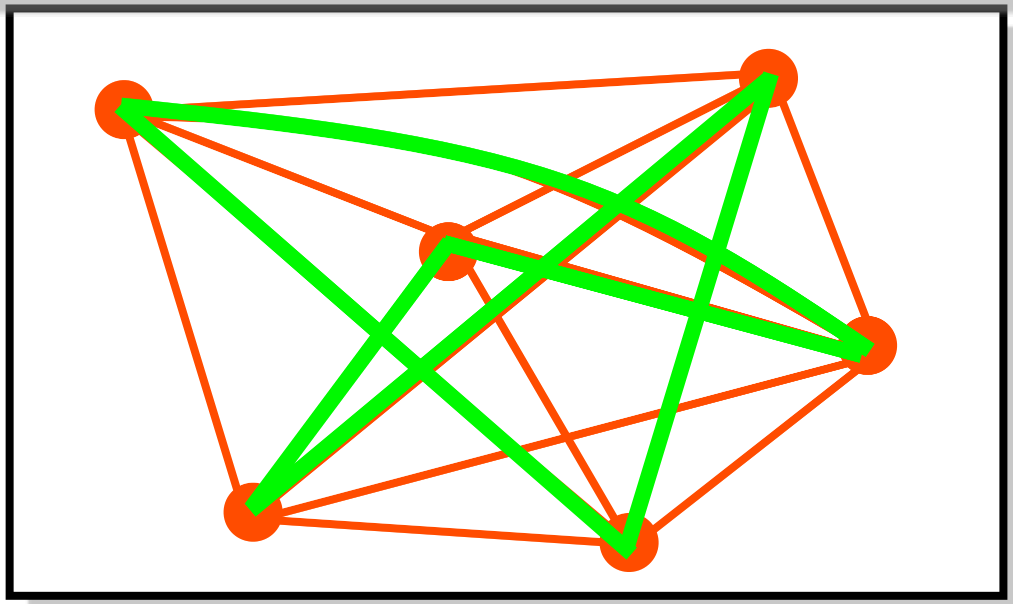

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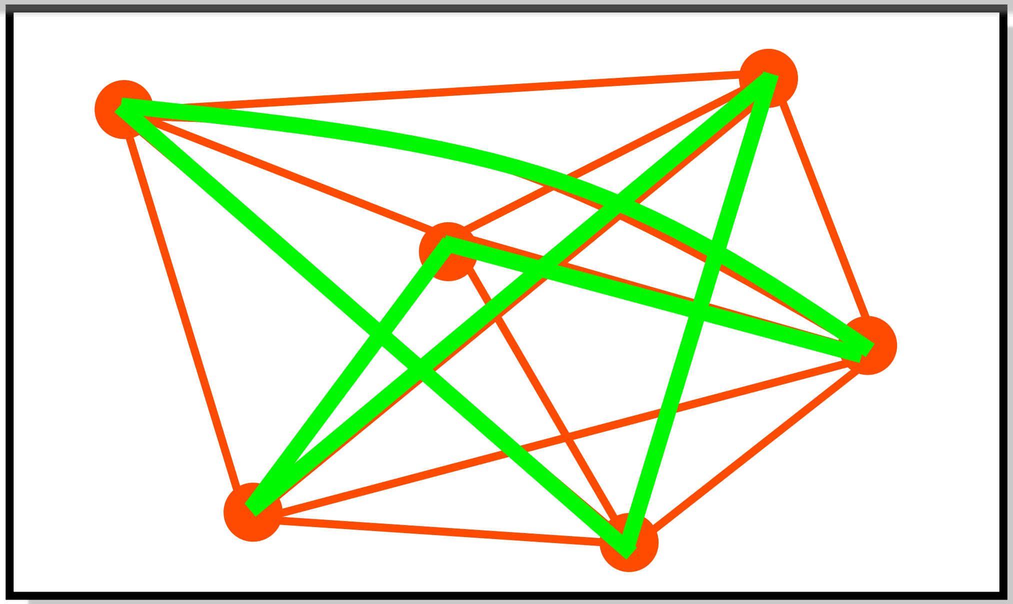

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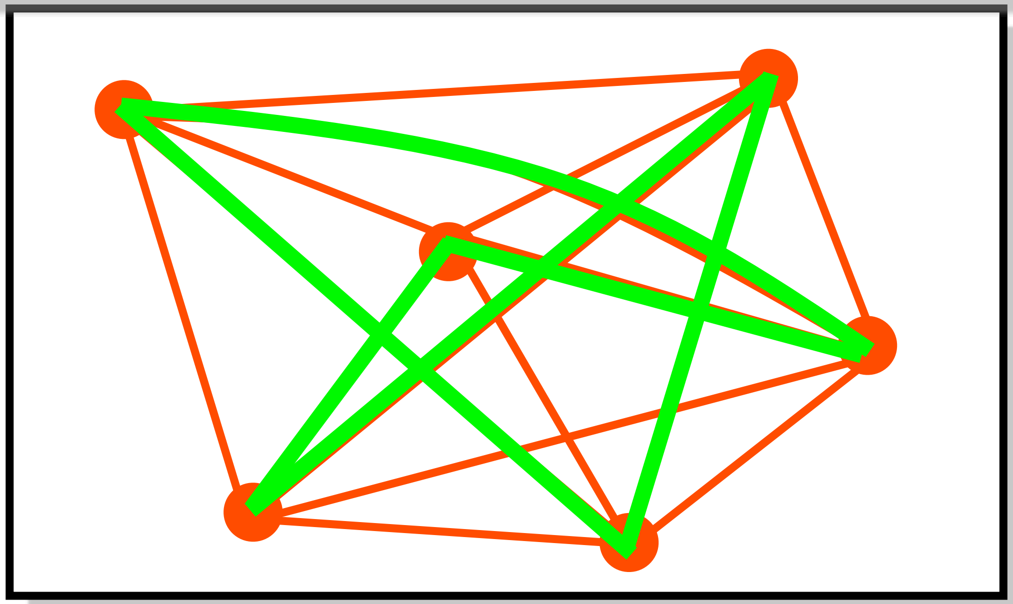

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Simple excercise:

Finding a longest tour is NP-hard.

Theorem: (Serdyukov 1984)
For metric distances, the Maximum TSP can be approximated in polynomial time within a factor of 3/4.

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The Maximum TSP

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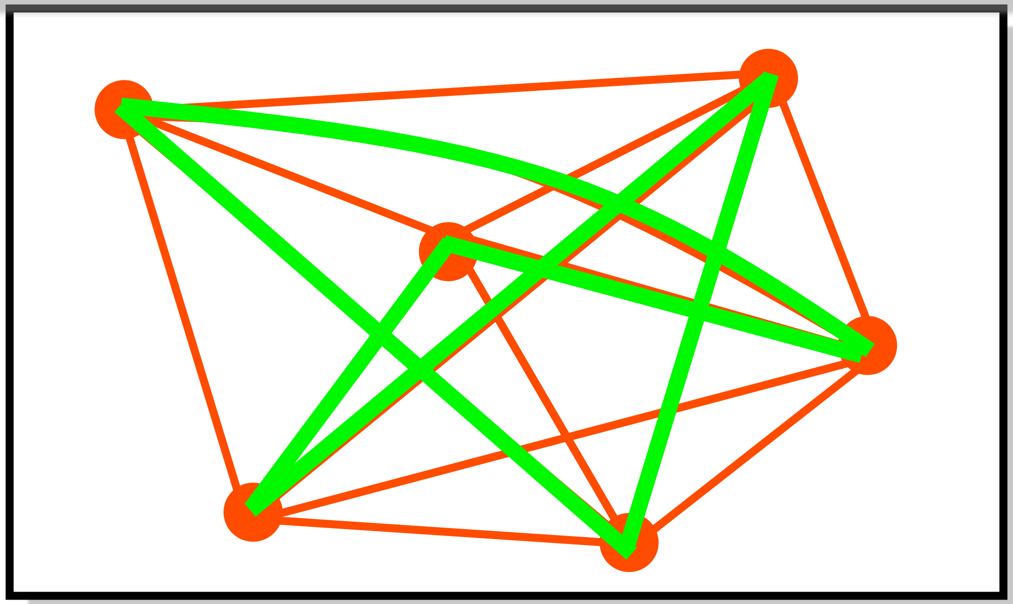

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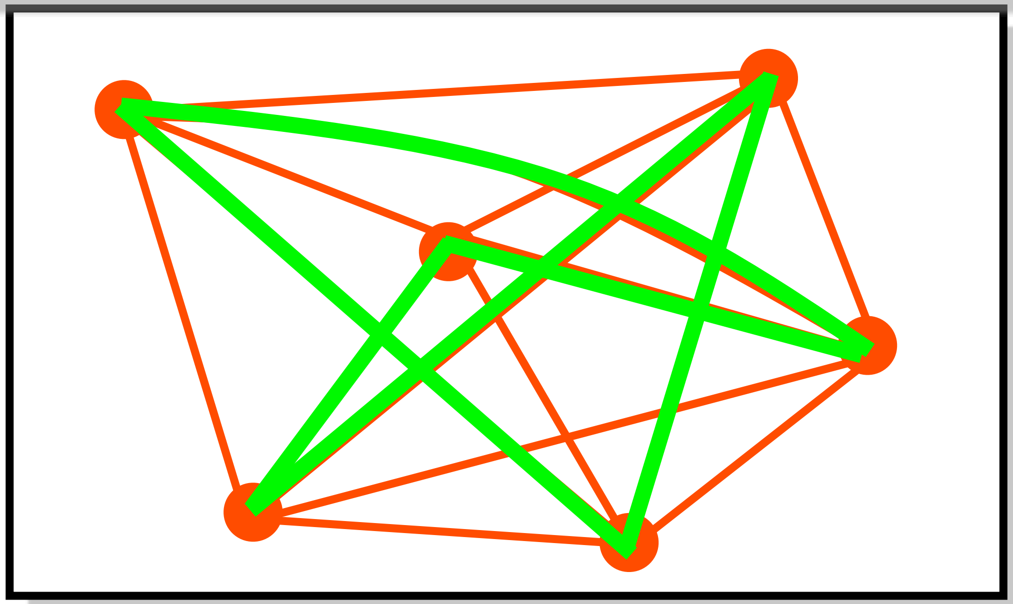

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Theorem: (Barvinok 1996)

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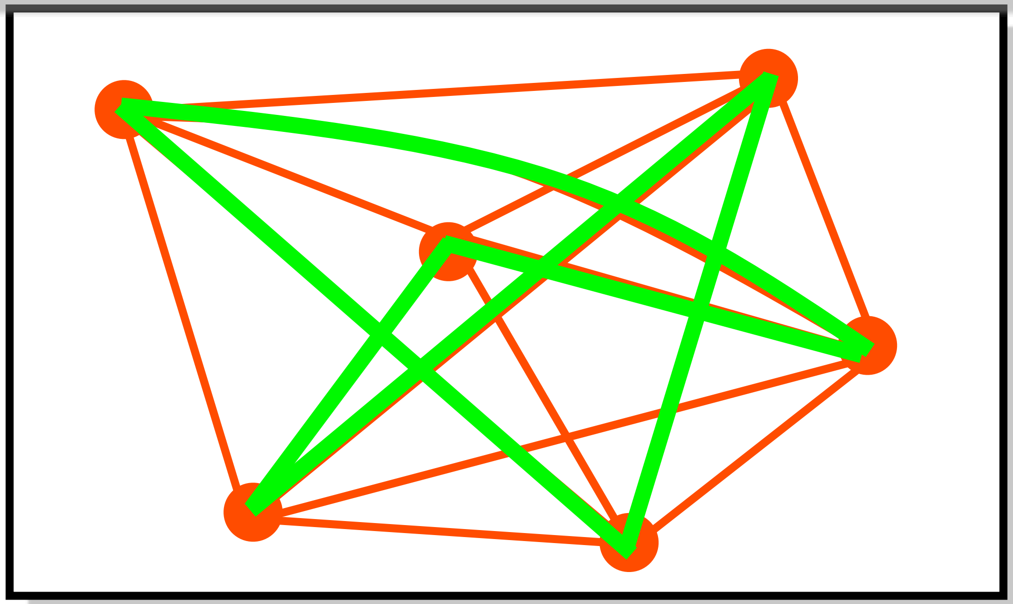

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Finding a longest tour is NP-hard.

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For metric distances, the Maximum TSP can be approximated in polynomial time within a factor of 3/4.

Theorem: (Barvinok 1996)

The Maximum TSP for geometric instances can be solved in polynomial time within a factor of $1/(1+\varepsilon)$.

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Touring the New World

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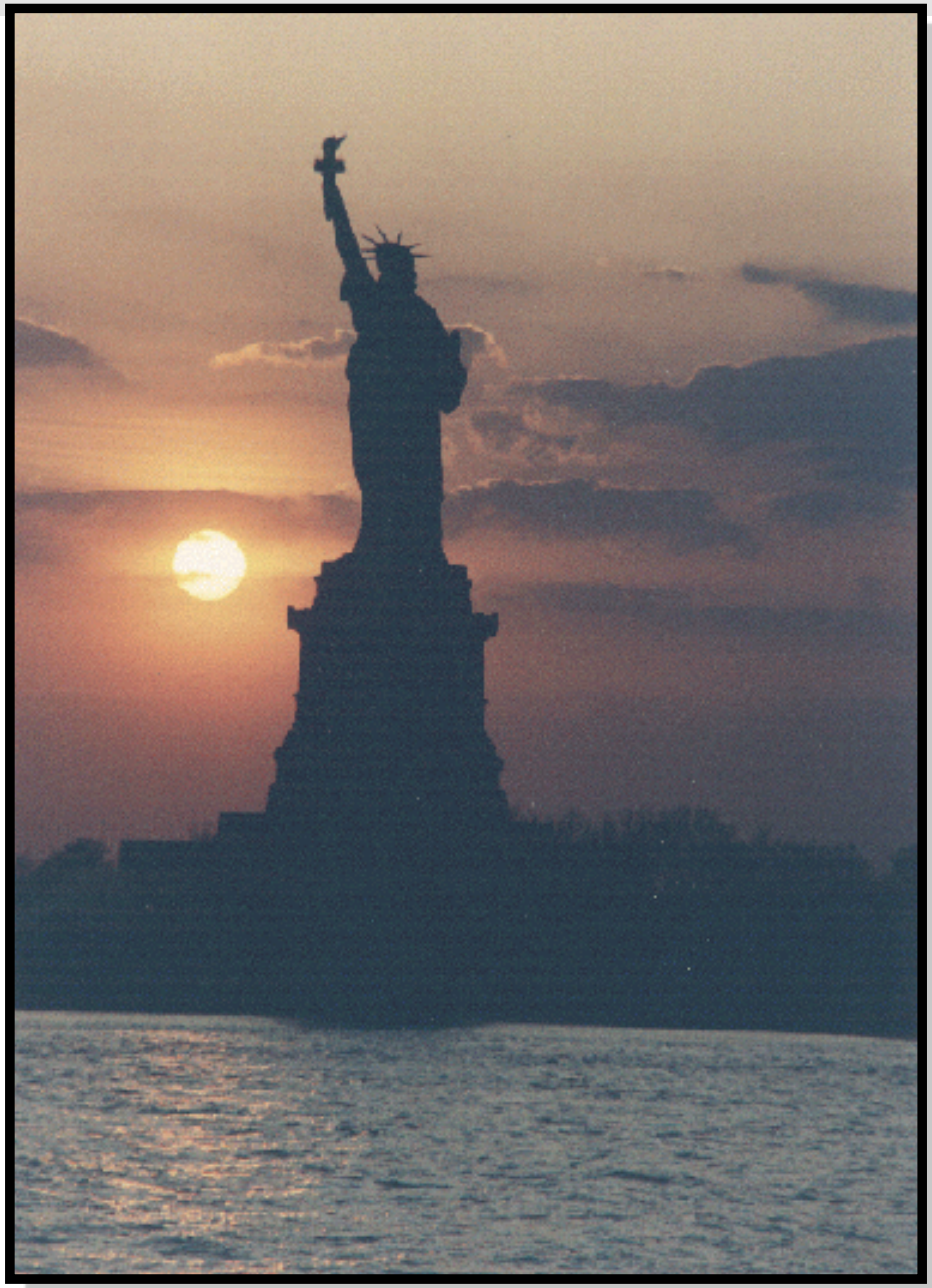
Touring the New World



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Touring the New World

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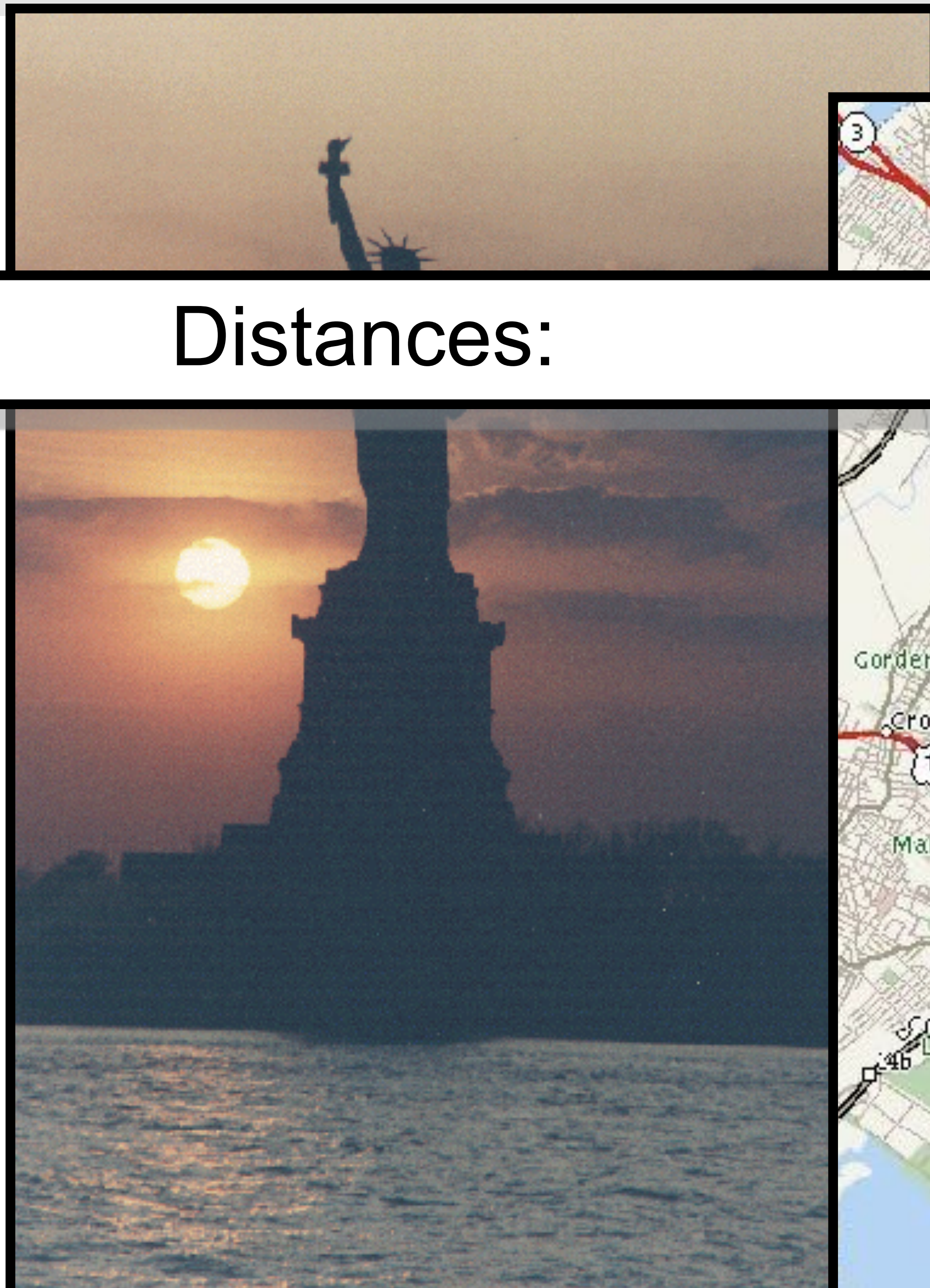
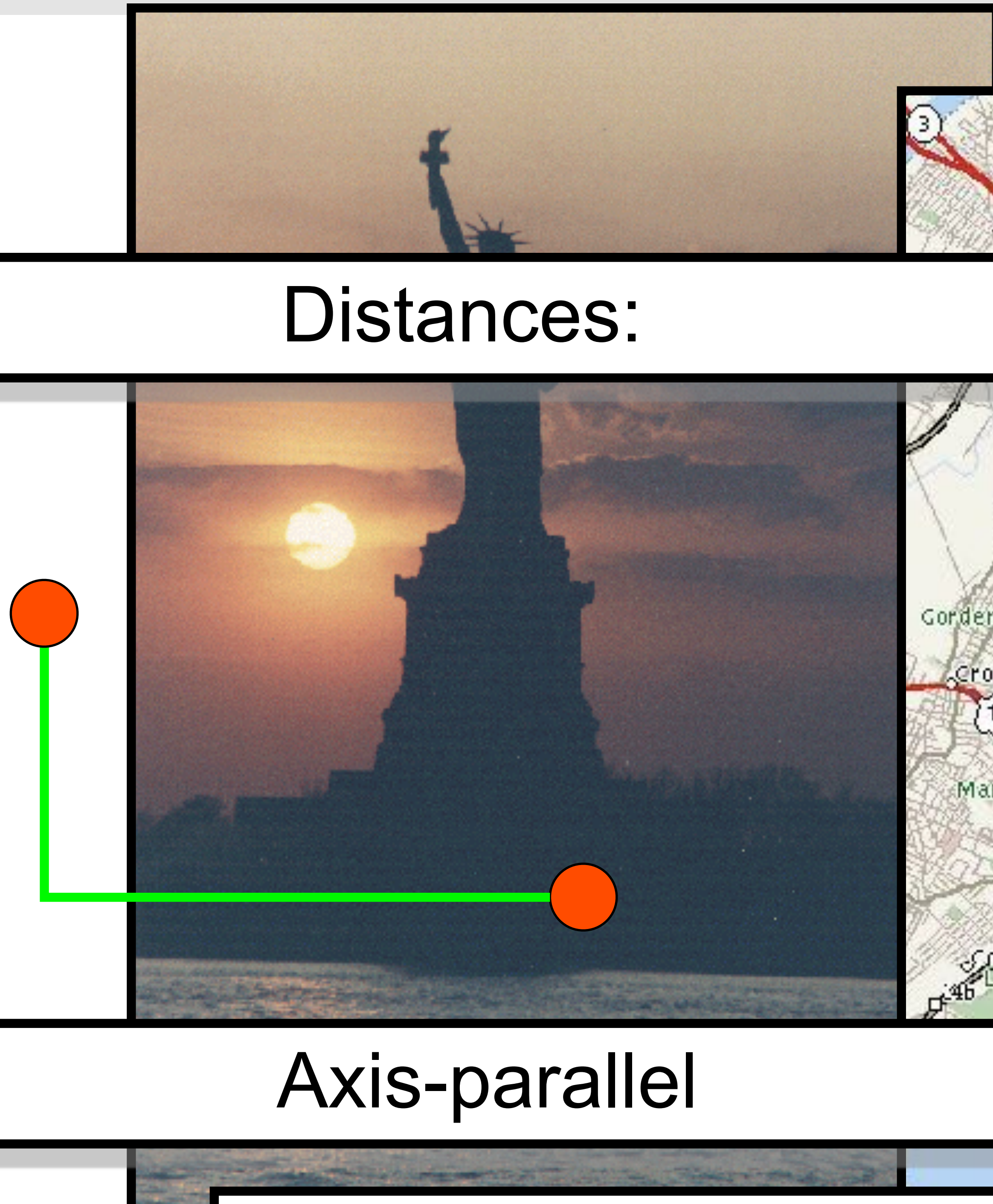


Distances:

Touring the New World



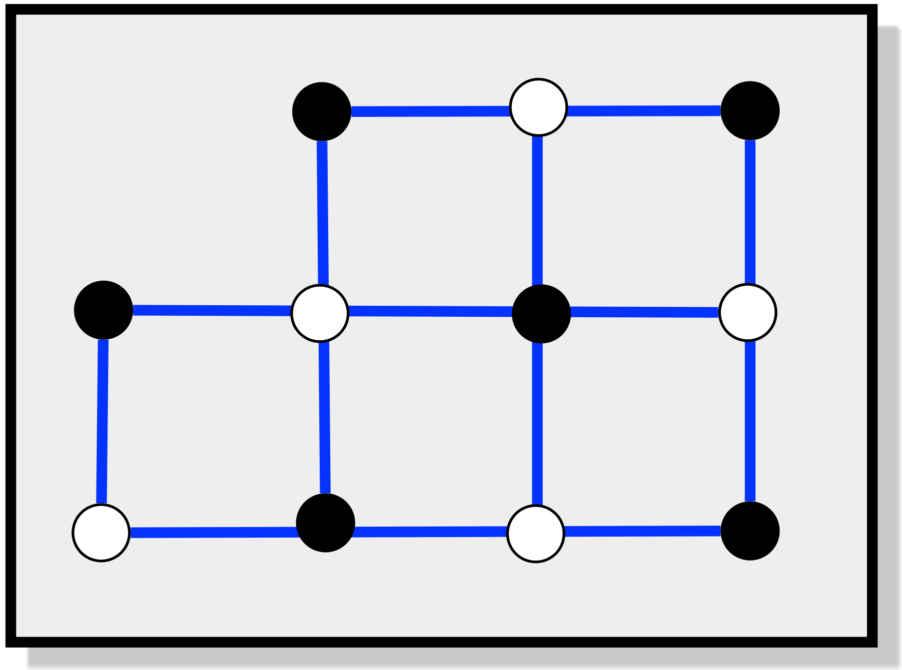
Distances:

Axis-parallel

„Manhattan metric“

Shortest Tours in Manhattan

Shortest Tours in Manhattan



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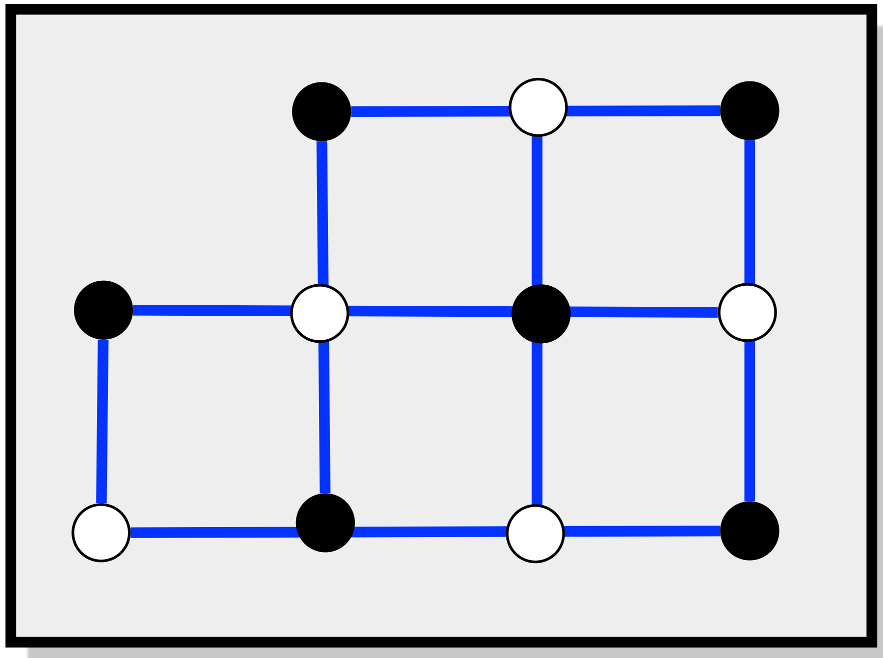
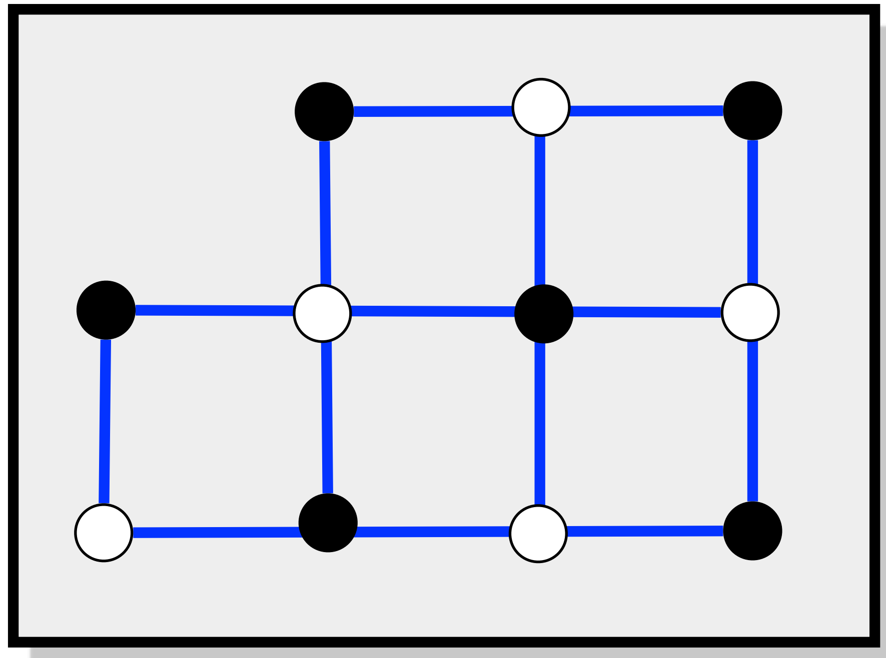


A Grid Graph

Shortest Tours in Manhattan



A Grid Graph

Theorem: (Itai, Papadimitriou, Szwarcfiter 1982)

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HAMILTON PATHS IN GRID GRAPHS*

ALON ITAI†, CHRISTOS H. PAPADIMITRIOU‡ AND JAYME LUIZ SZWARCFITER§

Abstract. A grid graph is a node-induced finite subgraph of the infinite grid. It is rectangular if its set of nodes is the product of two intervals. Given a rectangular grid graph and two of its nodes, we give necessary and sufficient conditions for the graph to have a Hamilton path between these two nodes. In contrast, the Hamilton path (and circuit) problem for general grid graphs is shown to be NP-complete. This provides a new, relatively simple, proof of the result that the Euclidean traveling salesman problem is NP-complete.

Key words. Hamilton circuit, Hamilton path, grid graphs, rectangular grid graphs, NP-complete problem, Euclidean traveling salesman problem

1. Introduction. Let G^∞ be the infinite graph whose vertex set consists of all points of the plane with integer coordinates and in which two vertices are connected if and only if the (Euclidean) distance between them is equal to 1. A *grid graph* is a finite, node-induced subgraph of G^∞. Thus, a grid graph is completely specified by its vertex set. Let v_x and v_y be the *coordinates* of the vertex v. We say that vertex v is *even* if $v_x+v_y\equiv 0 \pmod 2$; otherwise, v is *odd*. It is immediate that all grid graphs are bipartite, with the edges connecting even and odd vertices.

Let $R(m,n)$ be the grid graph whose vertex set is $V(R(m,n))=\{v: 1\leq v_x\leq m$ and $1\leq v_y\leq n\}$. A *rectangular graph* is a grid graph which, for some m and n, is isomorphic to $R(m,n)$. Thus m and n, the *dimensions*, specify a rectangular graph up to isomorphism.

Let s and t be distinct vertices of a graph G. We say that the *Hamilton path problem* (G,s,t) has a solution if there exists a Hamilton path from s to t in G. In this paper we examine the Hamilton path problem for grid graphs; rectangular grid graphs were examined first in [LM]. In § 2 we show that the Hamilton path and Hamilton circuit problems for general grid graphs are NP-complete. Consider now a bipartite graph $B=(V^0\cup V^1, E)$. If $|V^0|=|V^1|+1$, then all Hamilton paths of B must start and end at vertices of V^0. If $(R(m,n),s,t)$, with $m\times n$ odd, has a solution, then the number of even vertices is greater by one than that of the odd vertices. Hence, a necessary condition for the solvability of $(R(m,n),s,t)$ is that both s and t be even. In § 3 it is shown that this condition is also *sufficient* for nontrivial (i.e., $m,n>1$) odd rectangular graphs. If $m\times n$ is even, then a solution is possible only if s and t have different parity. However, this condition is not sufficient. There are three families of configurations for which even though s and t have different parity $(R(m,n),s,t)$ has no solution. In § 3 we give the precise necessary and sufficient conditions for a Hamilton path problem to have a solution. Partial results in this direction were first proved in [LM].

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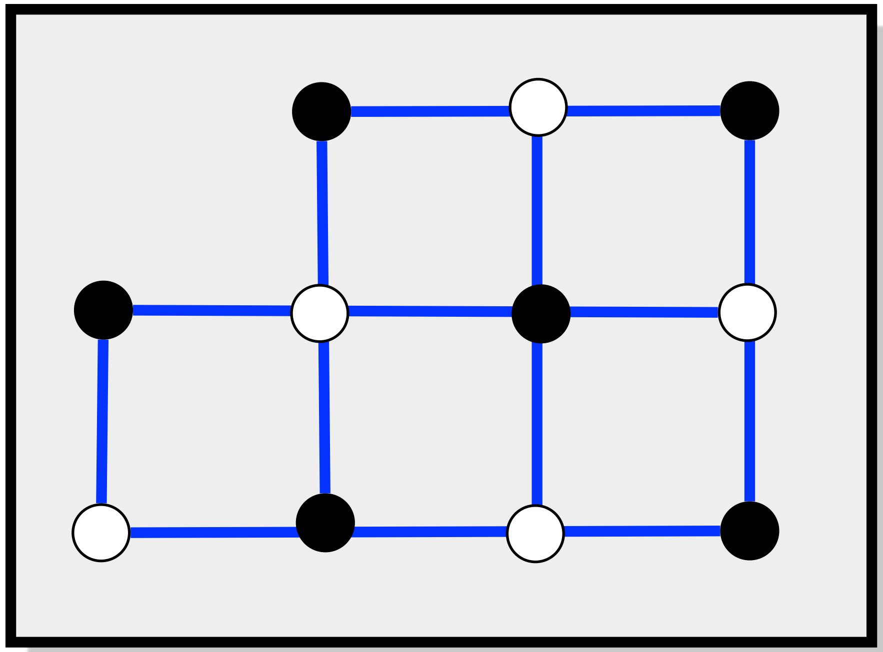
‡ Laboratory for Computer Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, and National Technical University of Athens, Greece. The work of this author was supported by the National Science Foundation under grant MCS 76-01193.
§ Universidade Federal do Rio de Janeiro, Brasil. Present address: Computer Science Division, University of California, Berkeley, California 94720. The work of this author was supported by the Conselho Nacional de Desenvolvimento Cientifico e Technologico (CNPq), Brasil, processo 574/78.

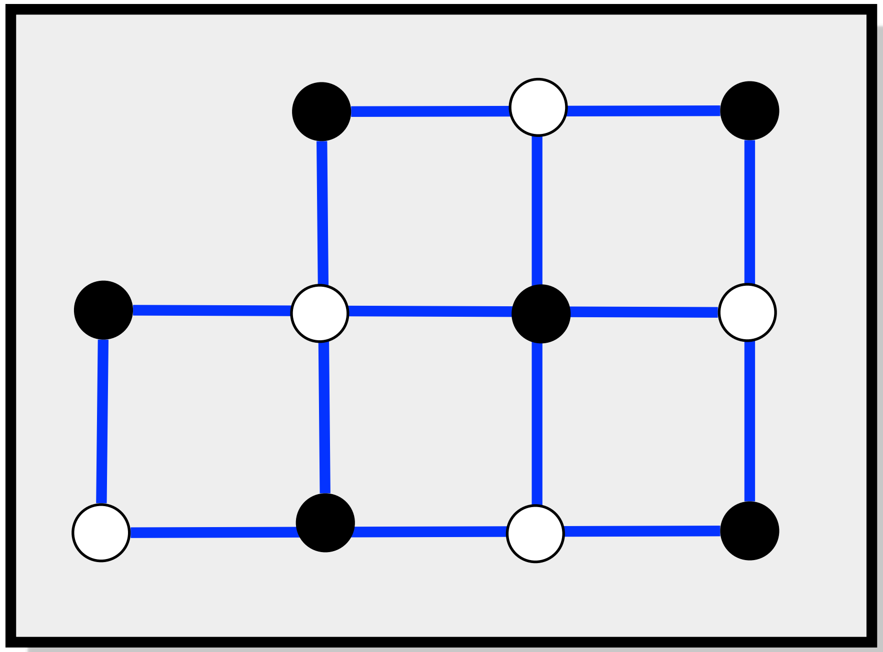
Shortest Tours in Manhattan



A Grid Graph

Theorem: (Itai, Papadimitriou, Szwarcfiter 1982)

Shortest Tours in Manhattan



A Grid Graph

Theorem: (Itai, Papadimitriou, Szwarcfiter 1982)

Finding a *shortest* tour in a grid graph is NP-hard.

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Longest Tours in Manhattan

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Simplicity and Hardness of the Maximum Traveling Salesman Problem under Geometric Distances*

Sándor P. Fekete[†]

Abstract

Recently, Barvinok, Johnson, Woeginger, and Woodroofe have shown that the *Maximum TSP*, i. e., the problem of finding a traveling salesman tour of maximum length, can be solved in polynomial time, provided that distances are computed according to a polyhedral norm in $\mathbb{R}^d$, for some fixed d. The most natural case of this class of problems arises for rectilinear distances in the plane $\mathbb{R}^2$, where the unit ball is a square. With the help of some additional improvements by Tamir, the method by Barvinok et al. yields an $O(n^2 \log n)$ algorithm for this case by making elegant use of geometry, graph theory, and optimization, including some rather powerful tools.

In this paper, we present a simple algorithm with $O(n)$ running time for computing the length of a longest tour for a set of points in the plane with rectilinear distances. The algorithm does not use any indirect addressing, so its running time remains valid even in comparison based models in which sorting requires $\Omega(n \log n)$ time, which implies the same lower bound on verifying a Hamiltonian cycle. In addition, our approach gives a simple characterization of *all* optimal solutions. These results give a good idea what makes the (polyhedral) max TSP so much easier than its minimization counterpart.

Resolving the complexity status of the max TSP for Euclidean distances in spaces of fixed dimension has been stated by Barvinok et al. as a main open problem. In this paper, the results on simplicity are complemented by a proof that the Maximum TSP under Euclidean distances in $\mathbb{R}^d$ for any fixed $d \geq 3$ is NP-hard, shedding new light on the well-studied difficulties of Euclidean distances. In addition, our result implies NP-hardness of the Maximum TSP under polyhedral norms if the number k of facets of the unit ball is not fixed. As a corollary, we get NP-hardness of the Maximum Scatter TSP for geometric instances, where the objective is to find a tour that maximizes the shortest edge. This resolves a conjecture by Arkin, Chiang, Mitchell, Skiena, and Yang in the affirmative.

1 Introduction

The Traveling Salesman Problem (TSP) is one of the classical problems of combinatorial optimization: Given a set $\{v_1, v_2, \ldots, v_n\}$ of vertices together with the distance $d(v_i, v_j)$ between every pair of distinct vertices v_i, v_j, the goal is to find a permutation π of the vertices (a "tour") that minimizes (Minimum TSP) or maximizes (Maximum TSP) the total tour length

$$d\left(v_{\pi(n)}, v_{\pi(1)}\right) + \sum_{i=1}^{n-1} d\left(v_{\pi(i)}, v_{\pi(i+1)}\right).$$

Geometric instances of the TSP have always been of particular interest: vertices v_i correspond to points $p_i = (x_1, \ldots, x_d)$ in space $\mathbb{R}^d$, and distances $d(v_i, v_j)$ are given by some geometric norm $\|p_i - p_j\|$. The most common norms considered include the Euclidean norm L_2 and the Manhattan norm L_1, which are both special cases of the L_p norms. The L_1 norm is also an example of a *polyhedral norm*, where the set of points at distance 1 from the origin is given by a centrally symmetric polyhedron with k facets.

Two key questions regarding the complexity of the Minimum TSP on geometric instances have been answered. Itai, Papadimitriou, and Swarcfiter [11] showed that the Minimum TSP is NP-hard for any fixed dimension $d \geq 2$ and any L_p or polyhedral norm. On the other hand, Arora [2, 3] and Mitchell [13] showed that all these geometric instances allow a polynomial-time approximation scheme (PTAS), i. e., a sequence of algorithms A_s that compute a solution within a factor of $1 + \frac{1}{s}$ of the optimum, in time that is polynomial for any fixed s. This highlights the special role of geometry, since it is well known that for general Minimum TSP instances, no PTAS can exist. (See Trevisan [17], and Papadimitriou and Yannakakis [14] for such results in more restricted situations.)

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Longest Tours in Manhattan

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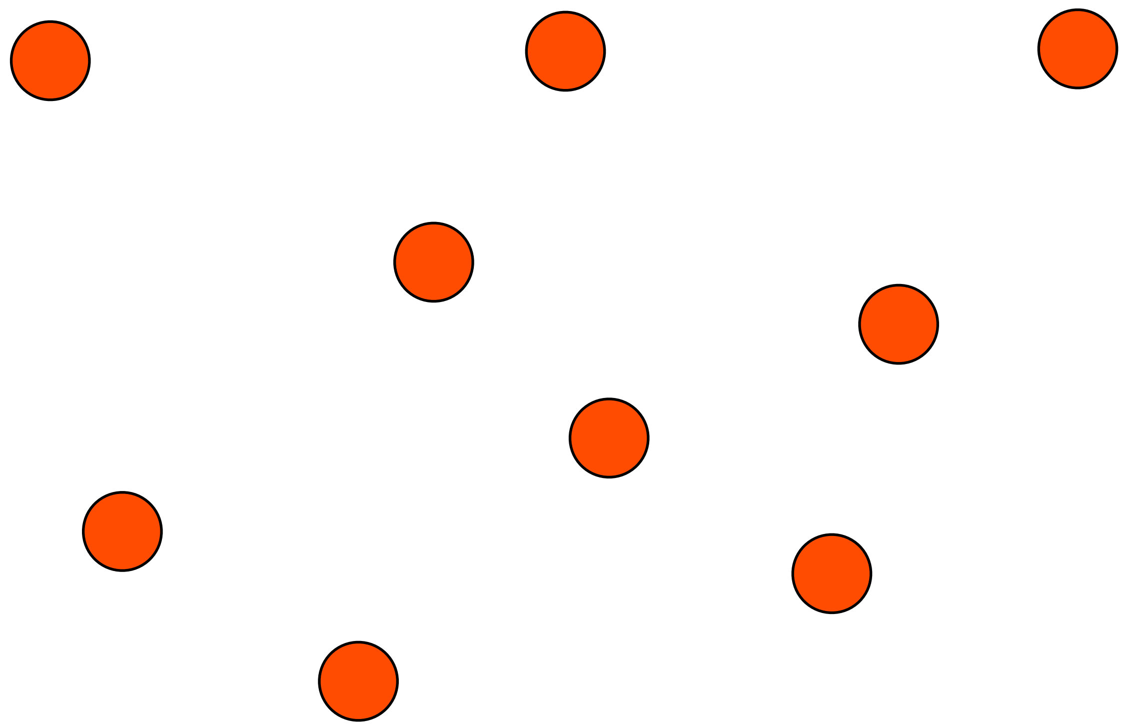
Longest Tours in Manhattan



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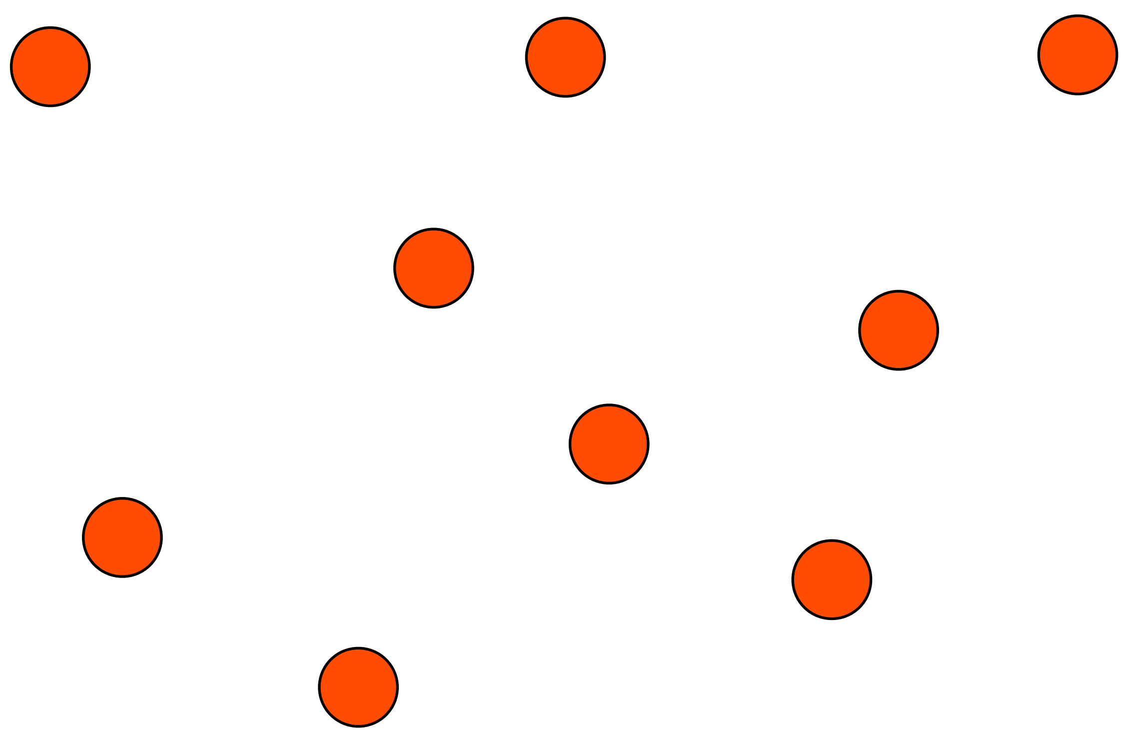

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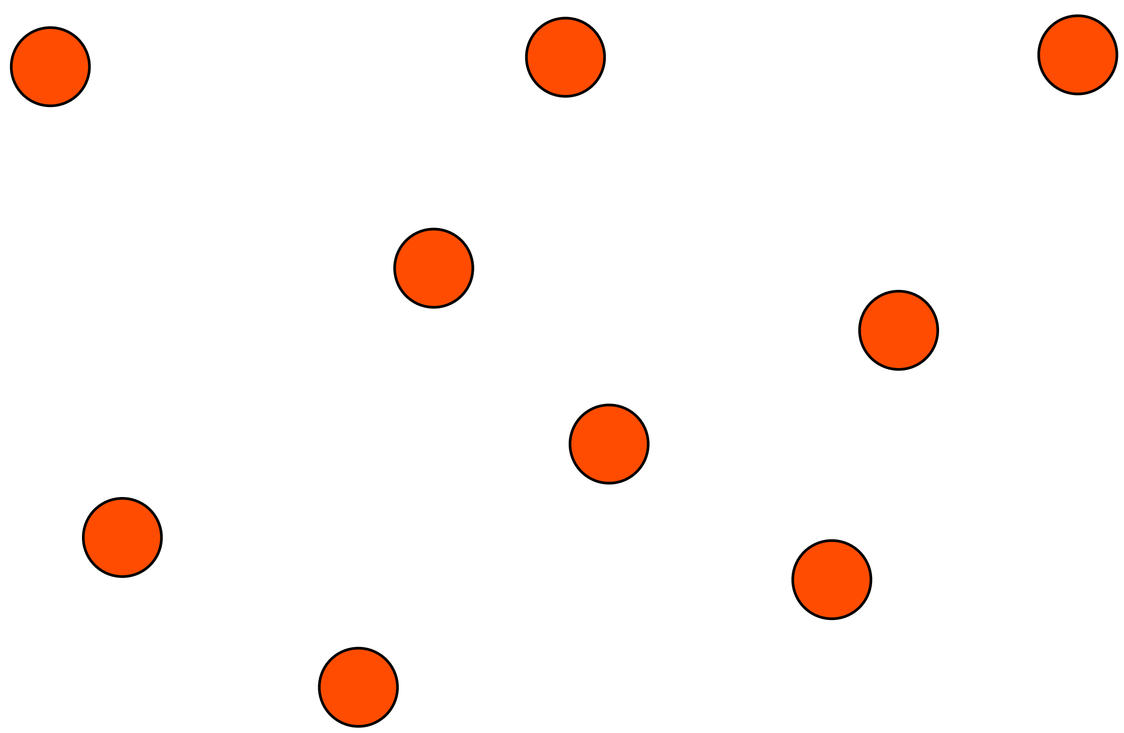

- Find a point that minimizes the sum of Manhattan distances:

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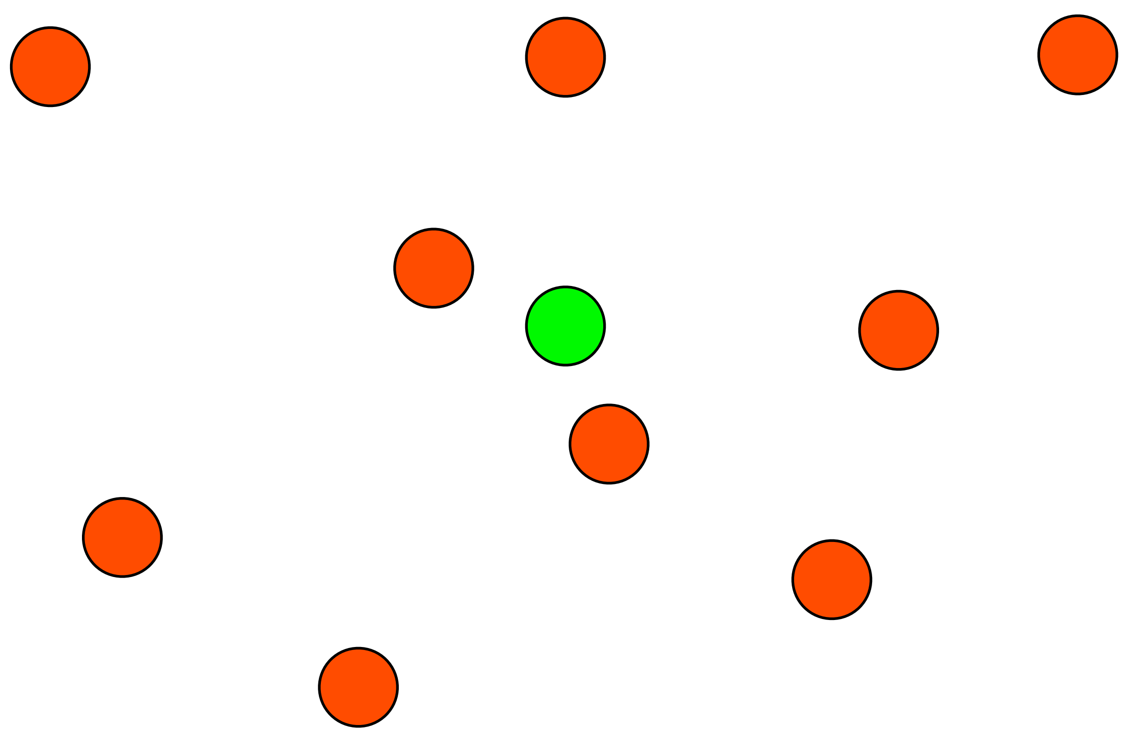
Longest Tours in Manhattan



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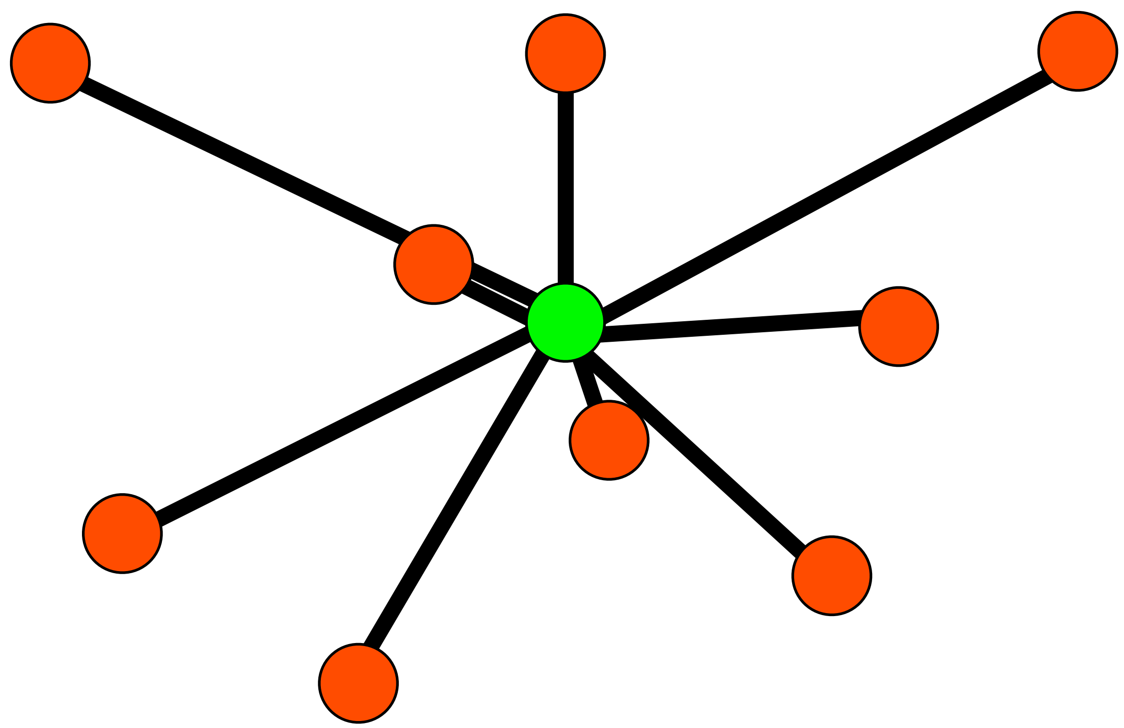

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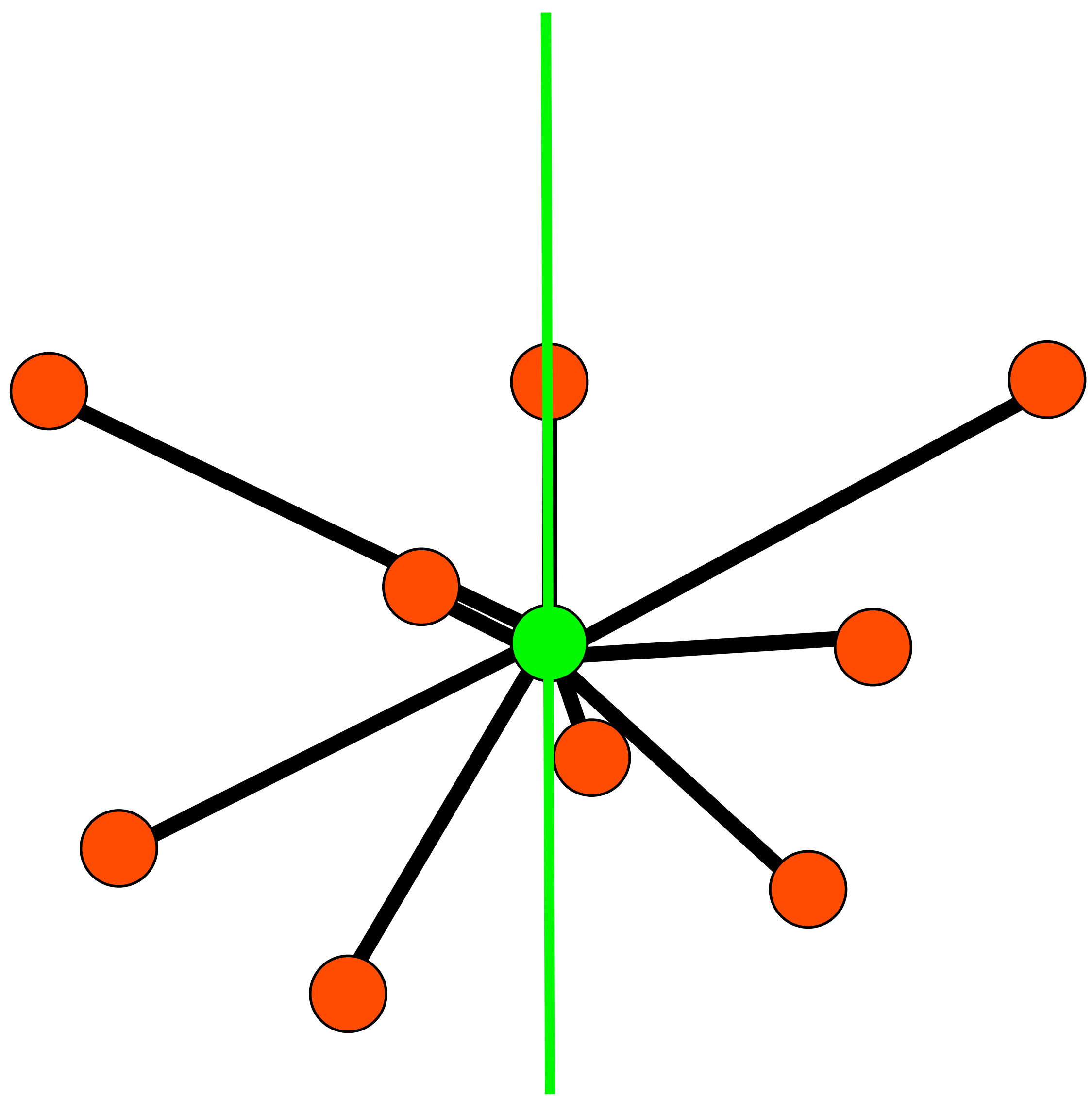

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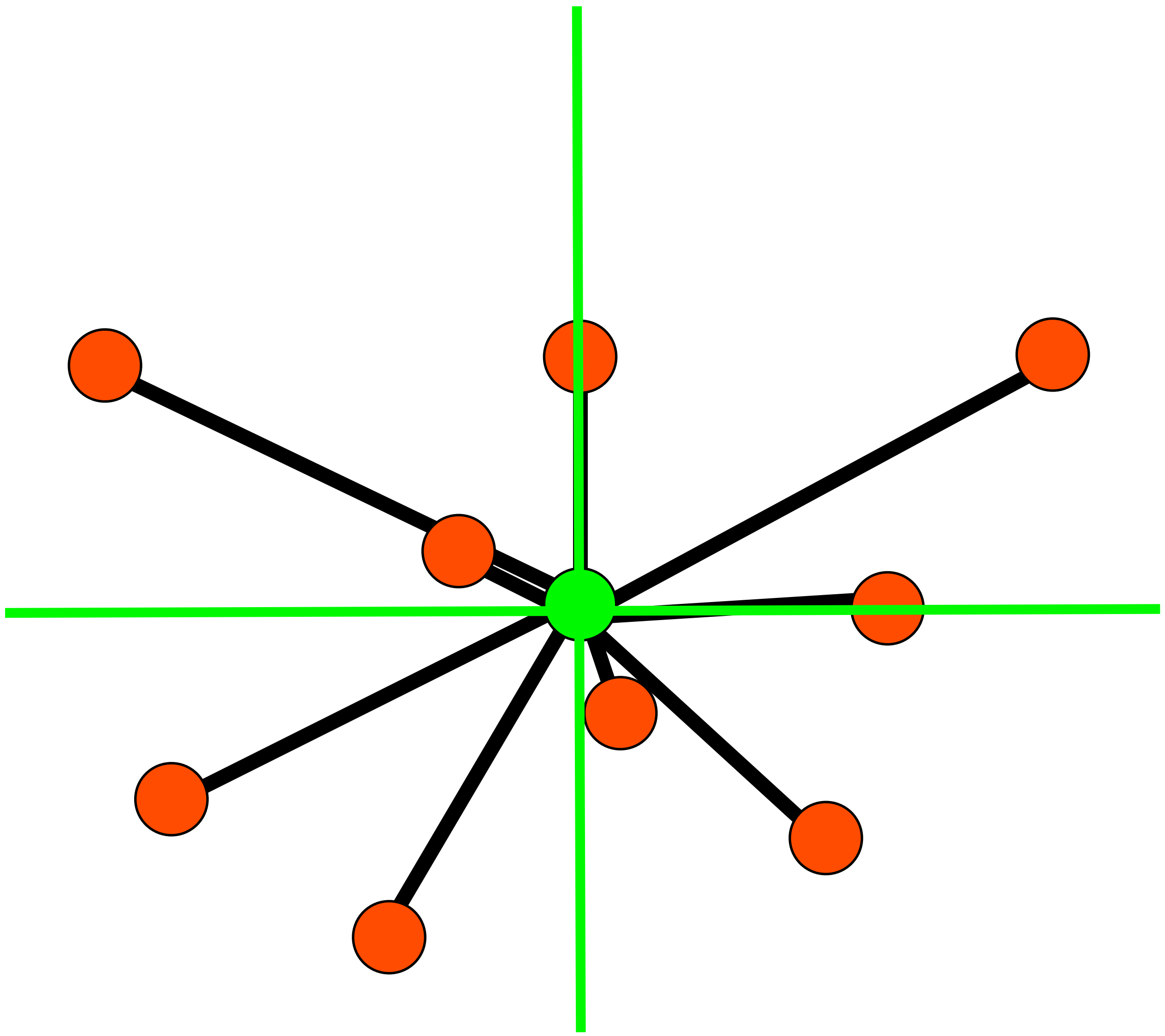

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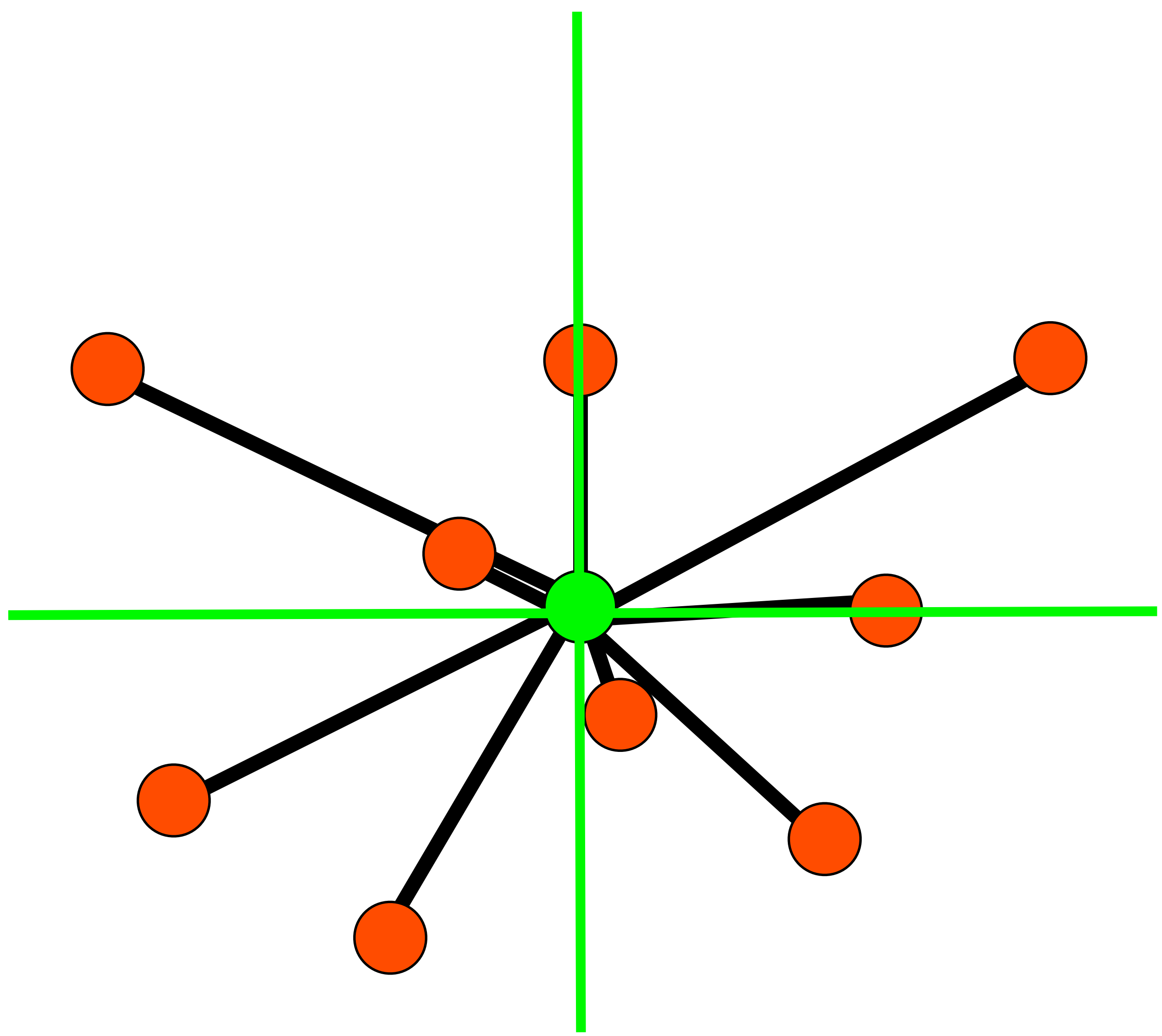

- Find a point that minimizes the sum of Manhattan distances:

  median in x and y!

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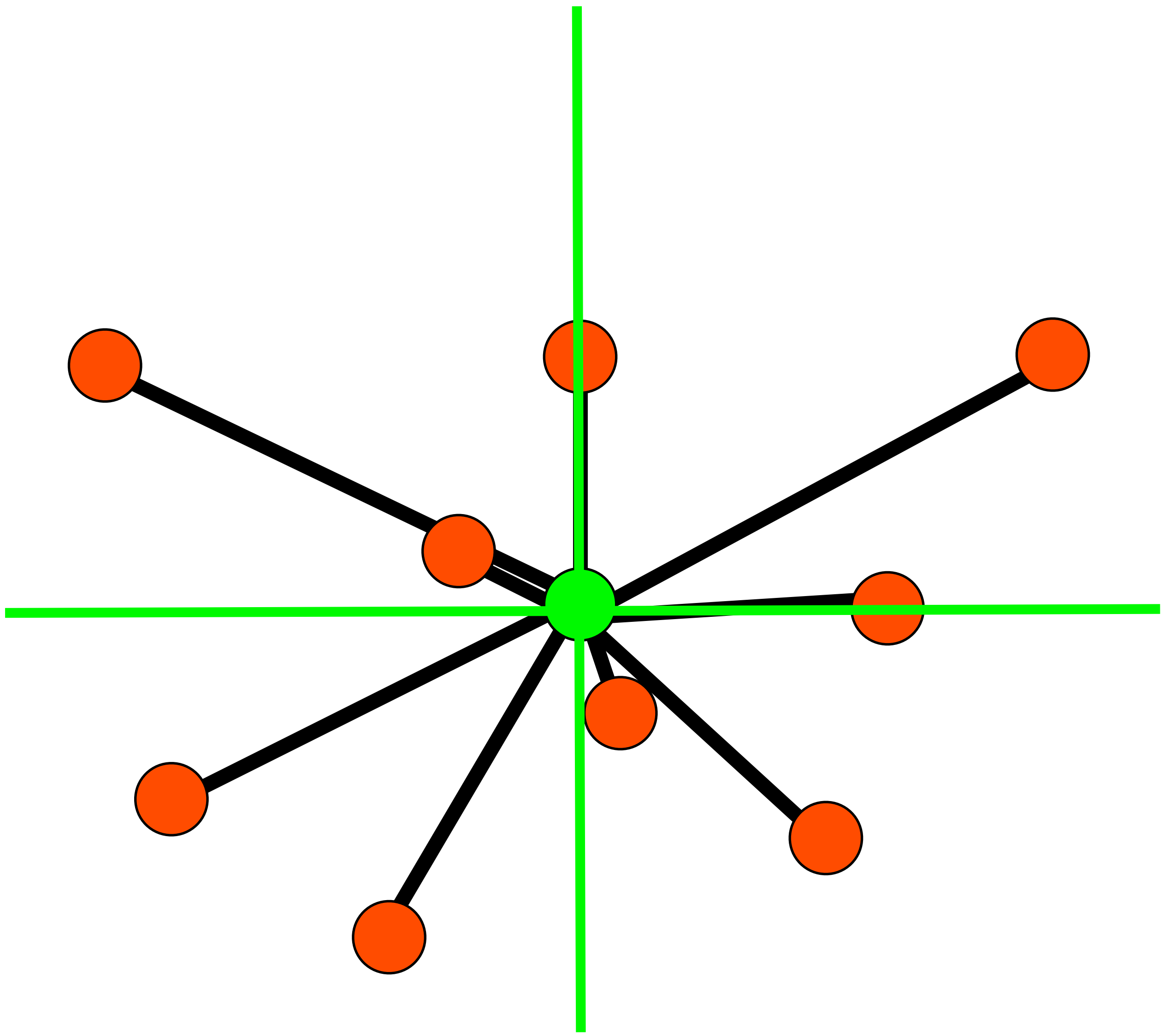
Longest Tours in Manhattan



- Find a point that minimizes the sum of Manhattan distances:

  median in x and y!

- Find long connections.

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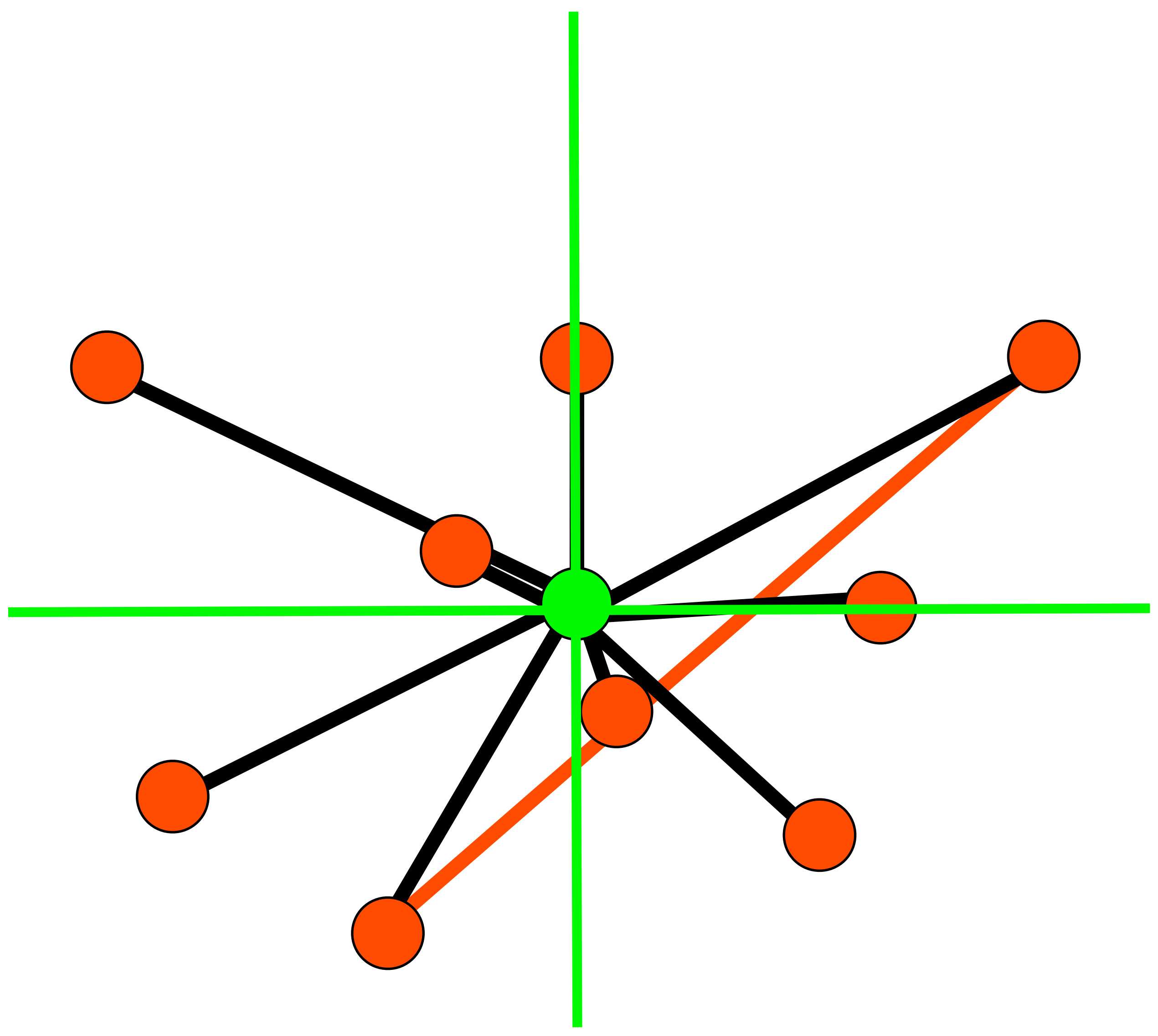

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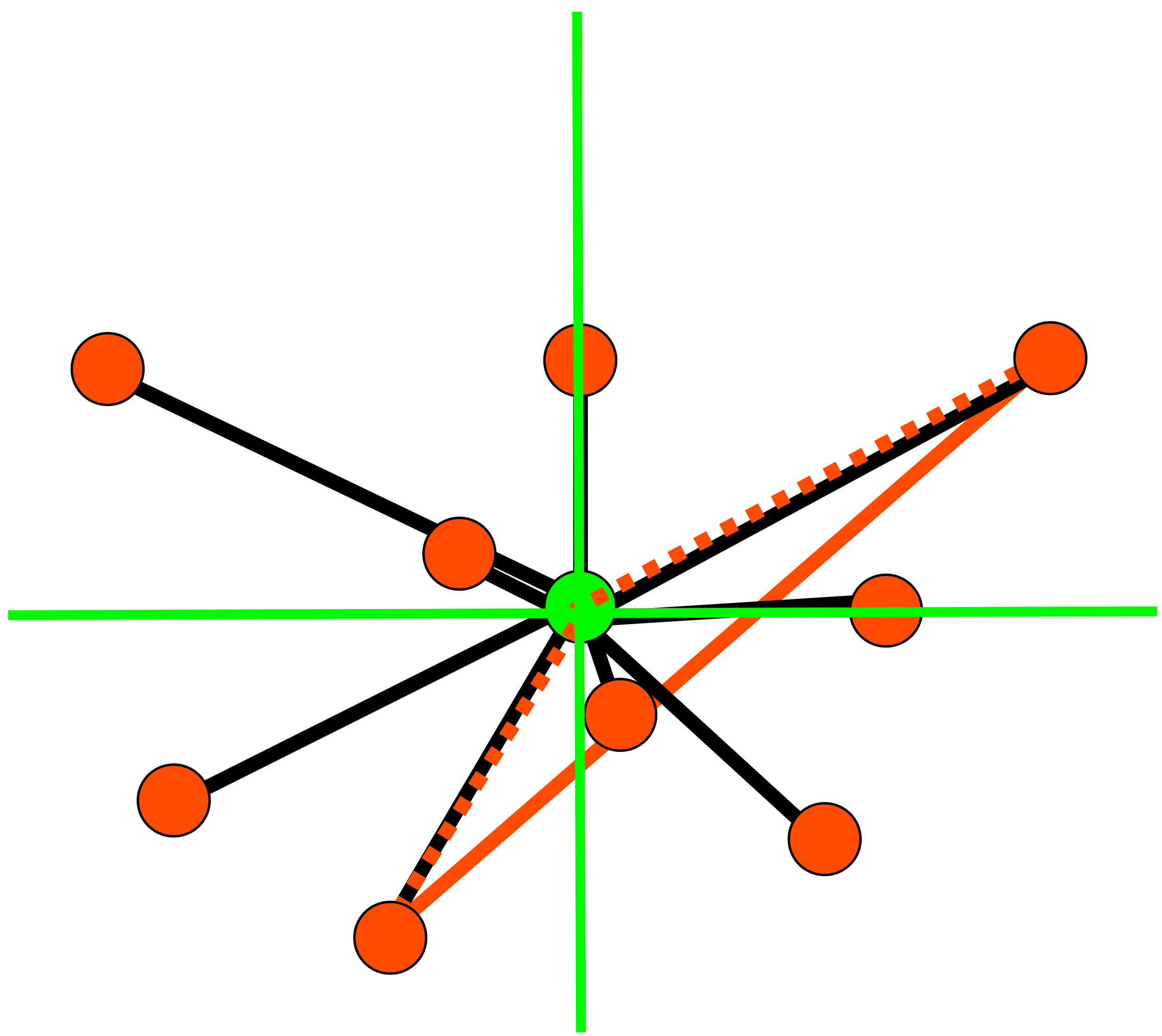

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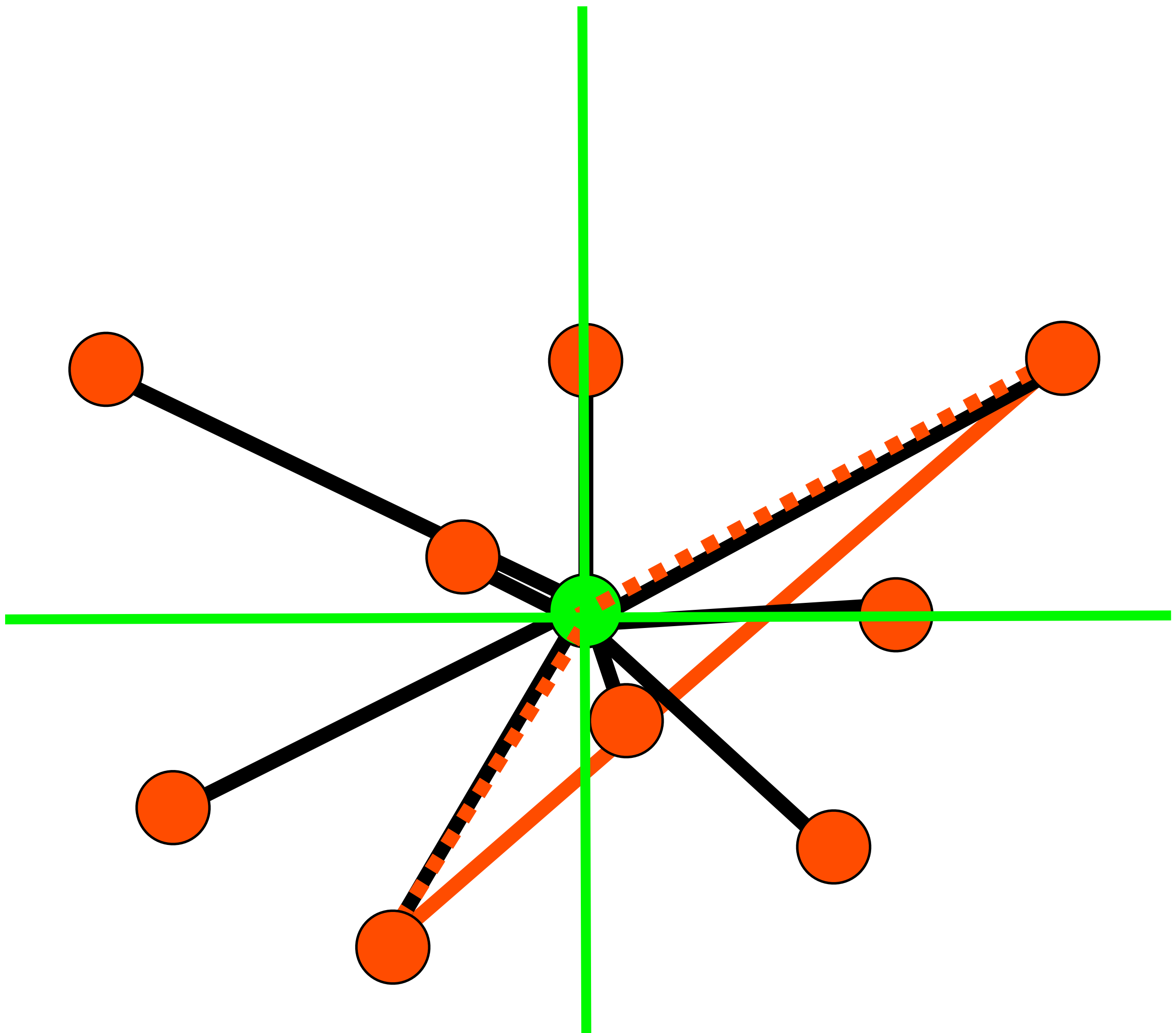

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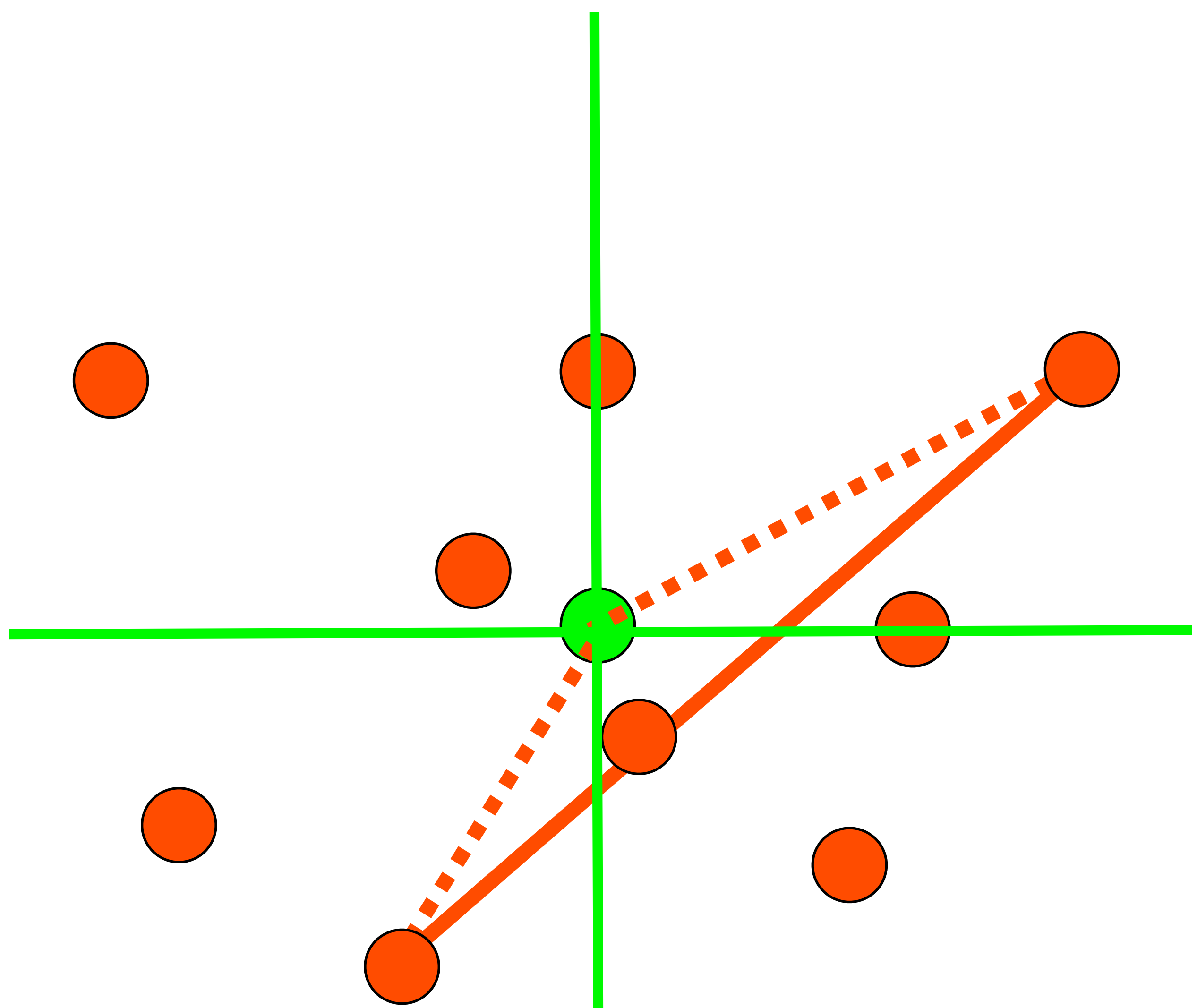

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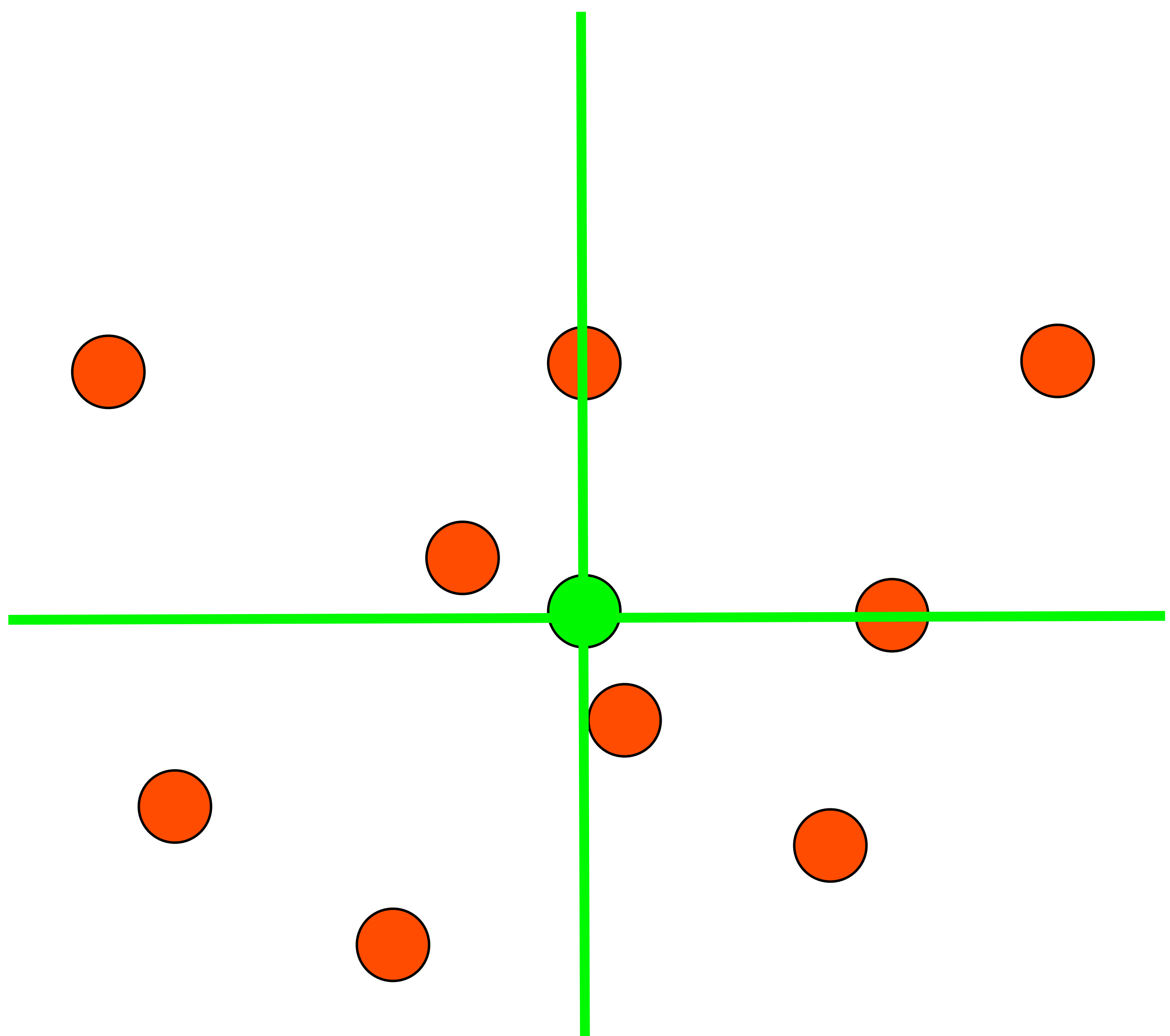

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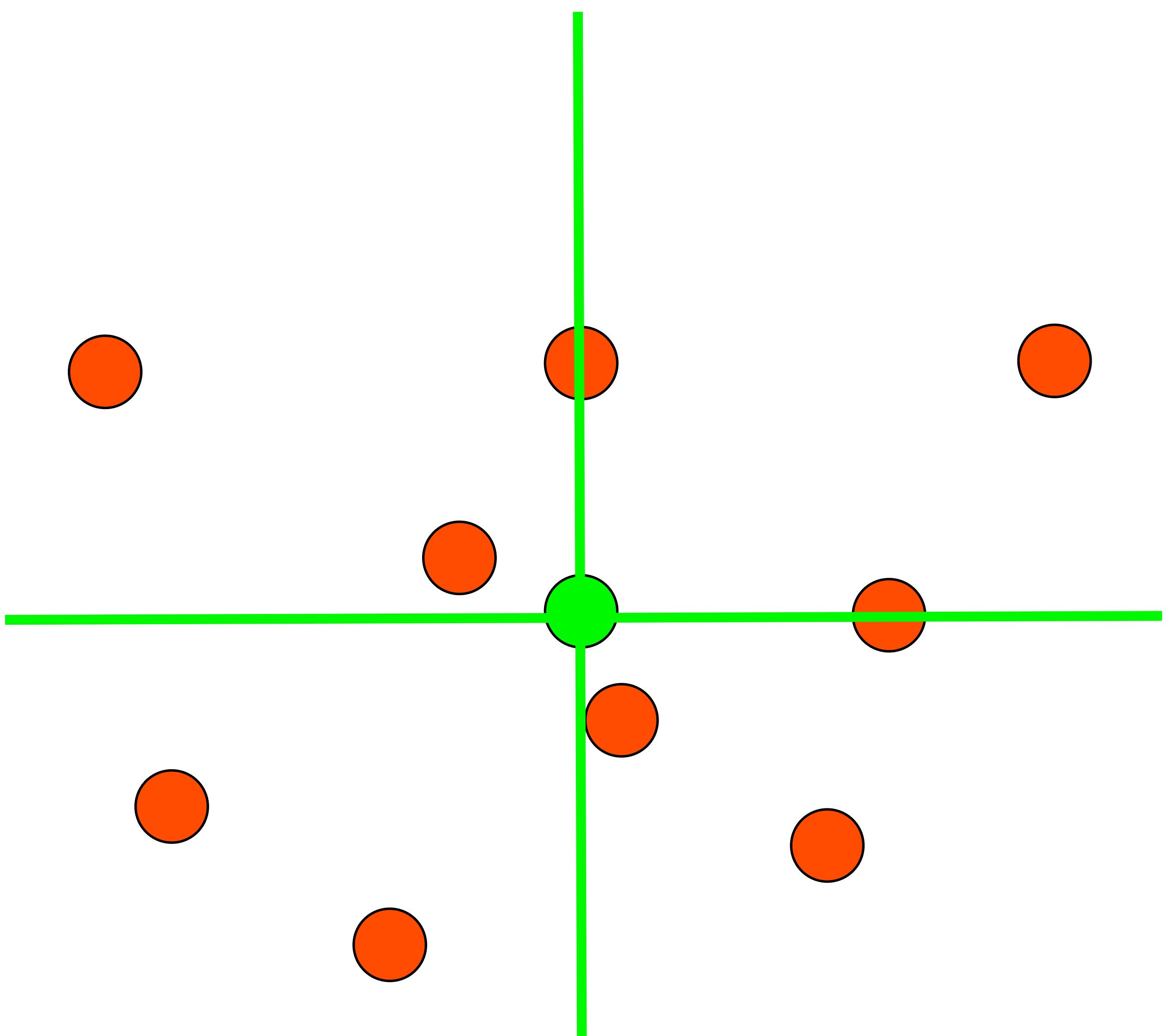

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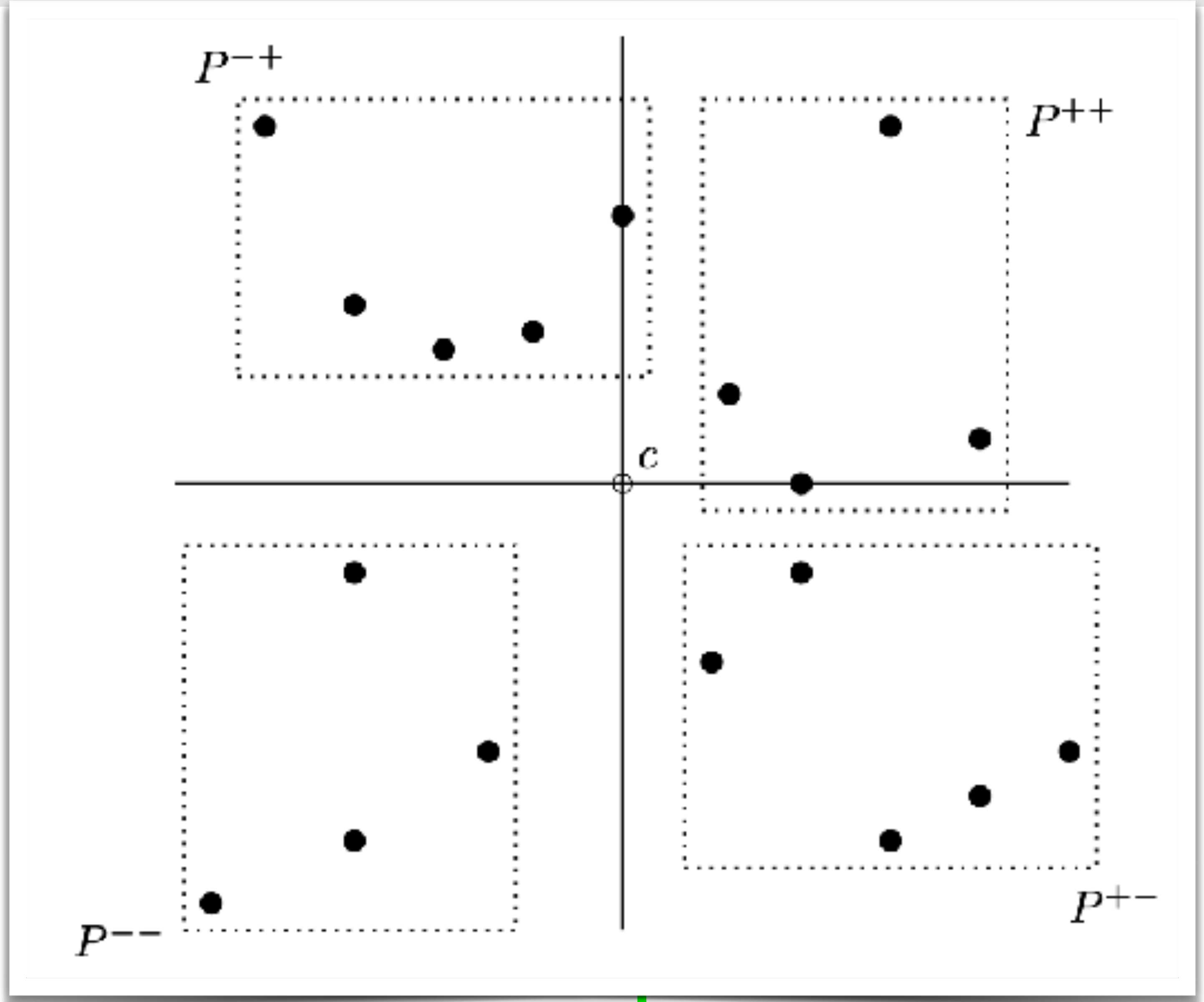

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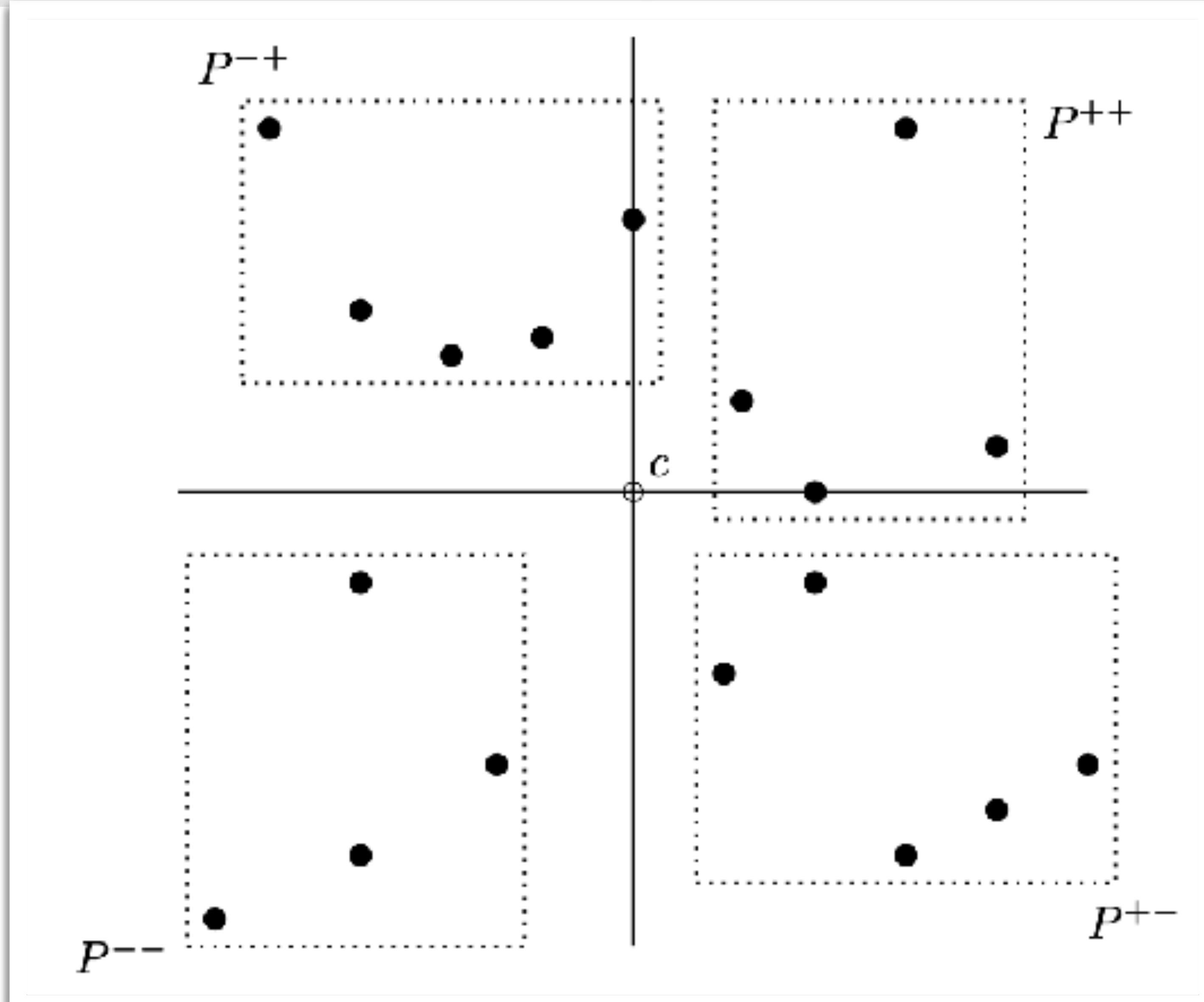

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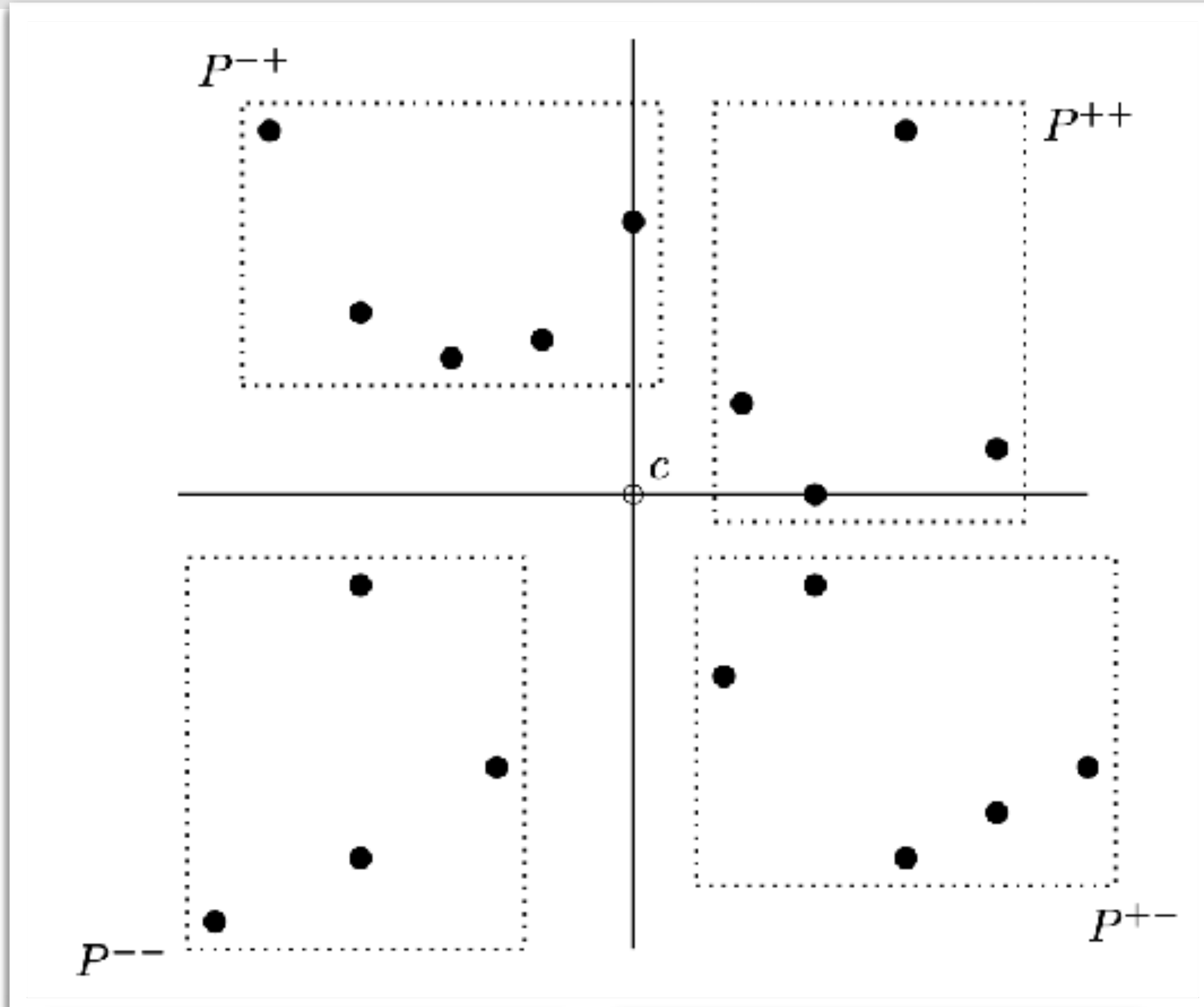

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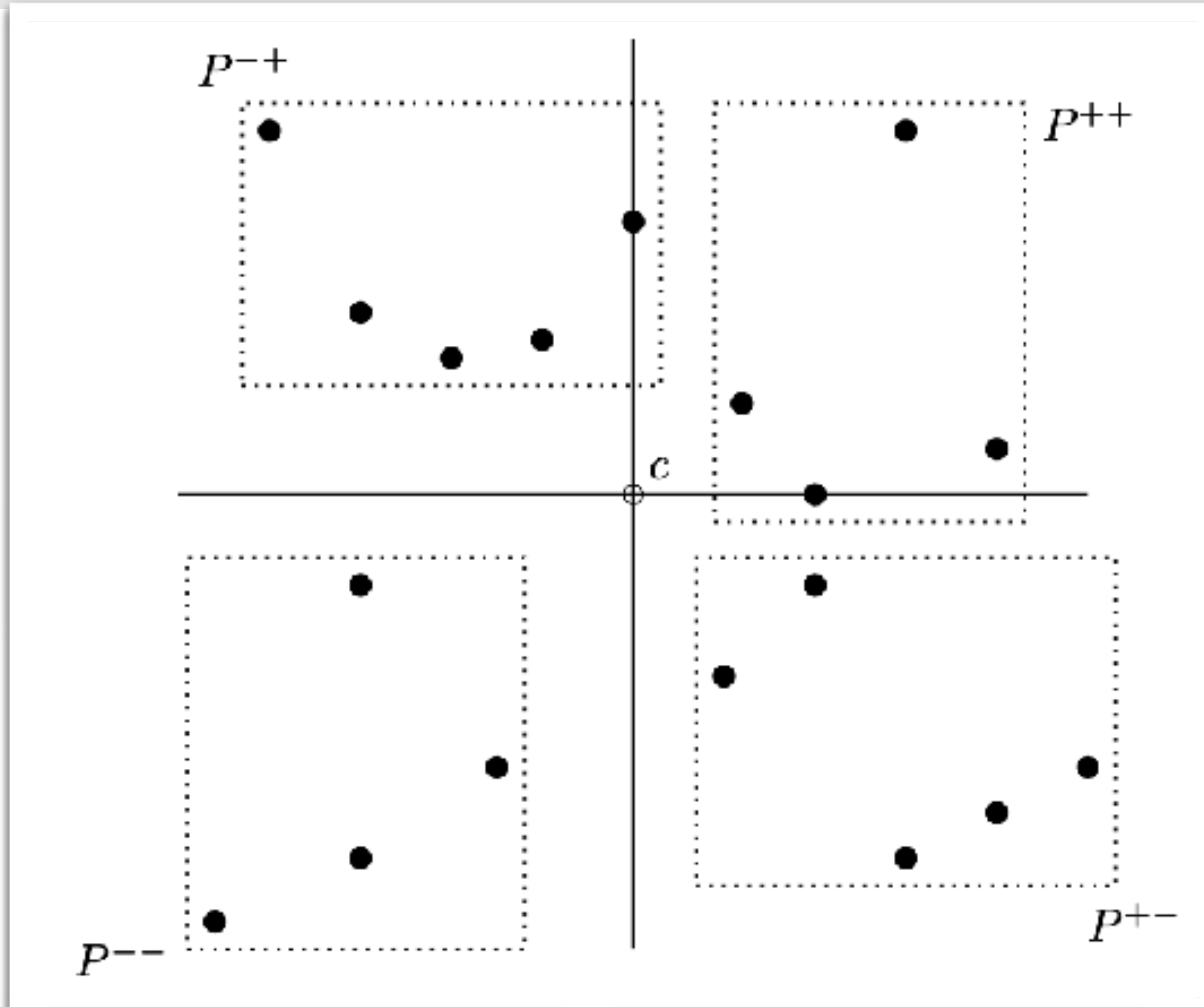

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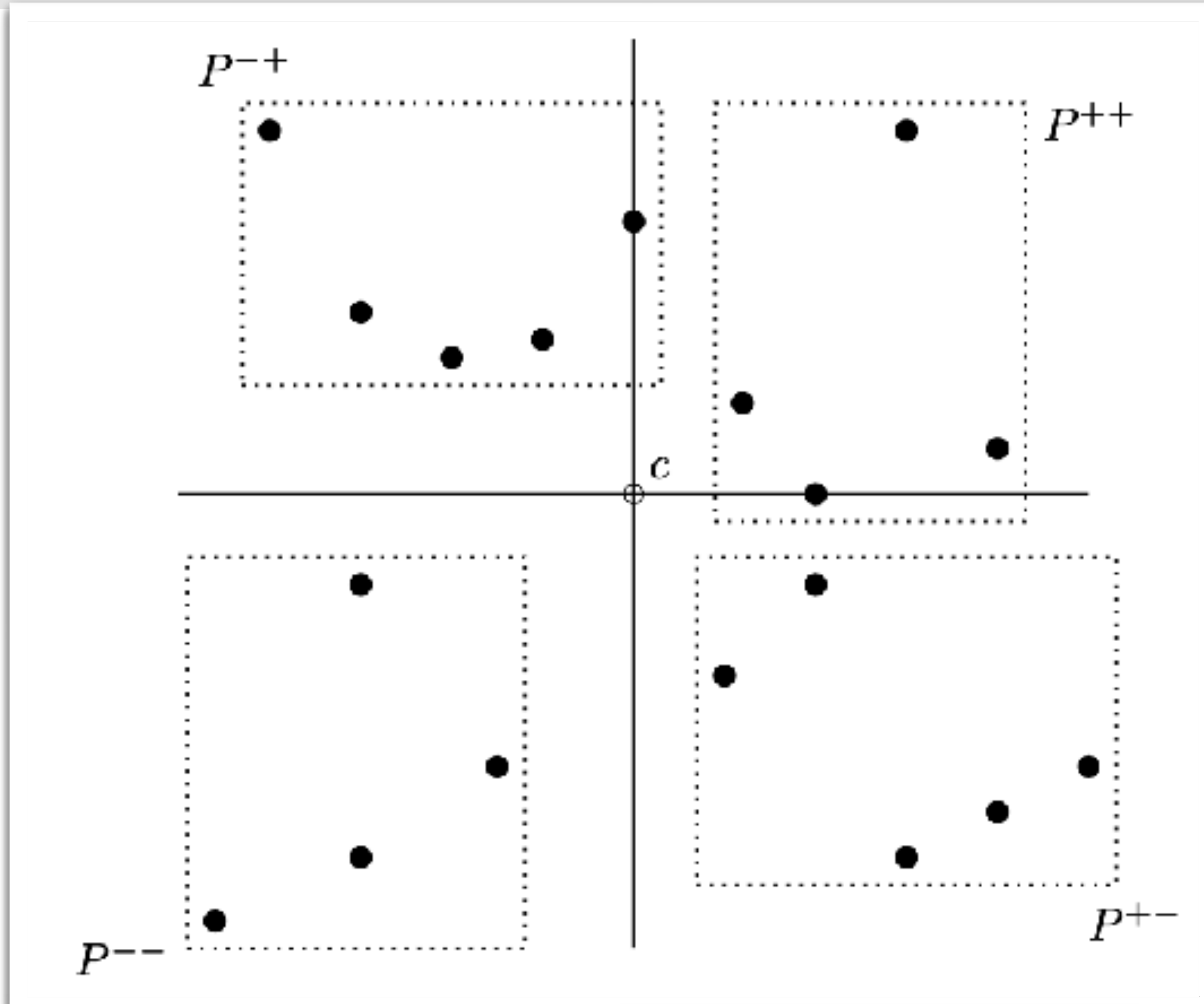

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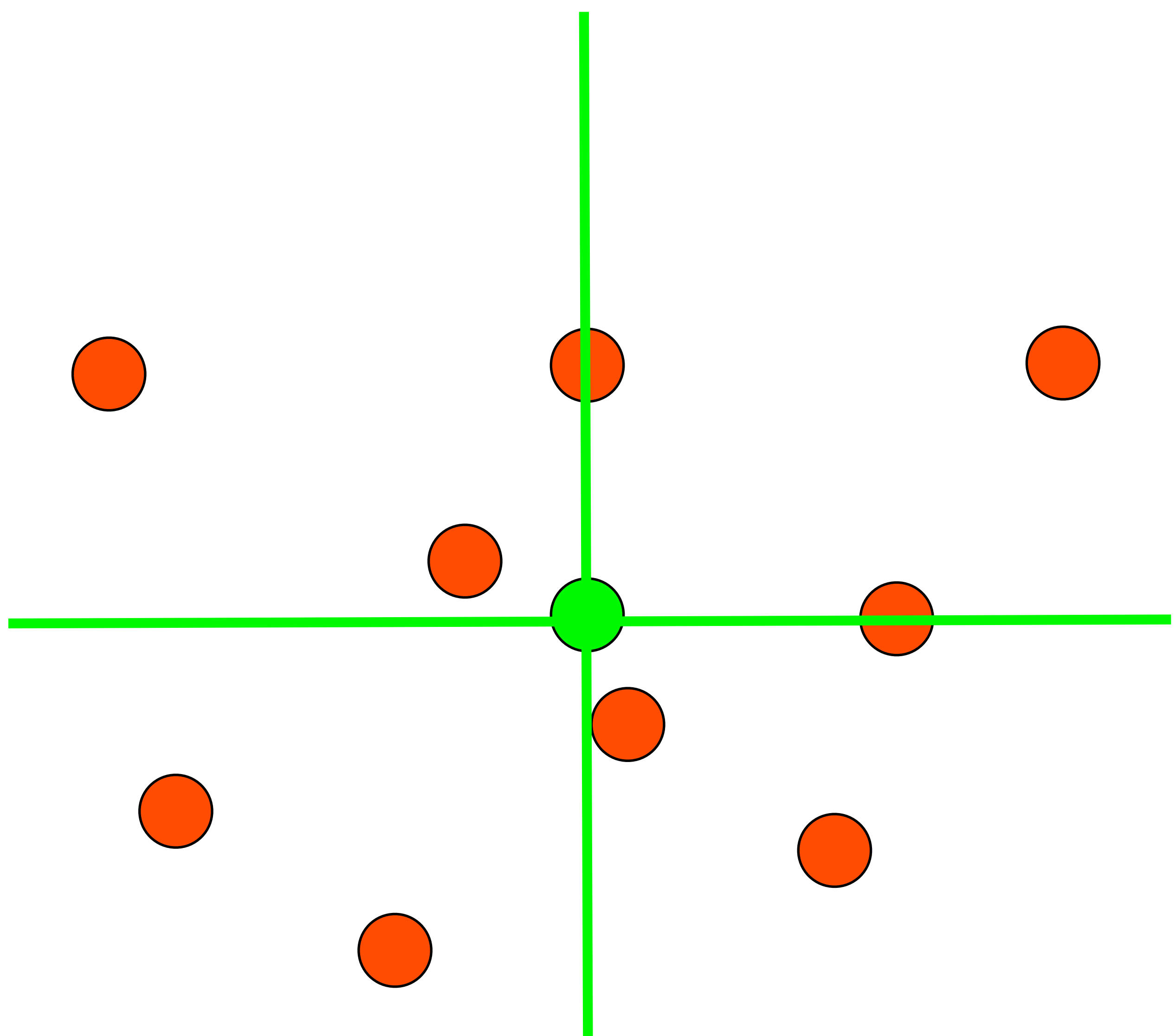

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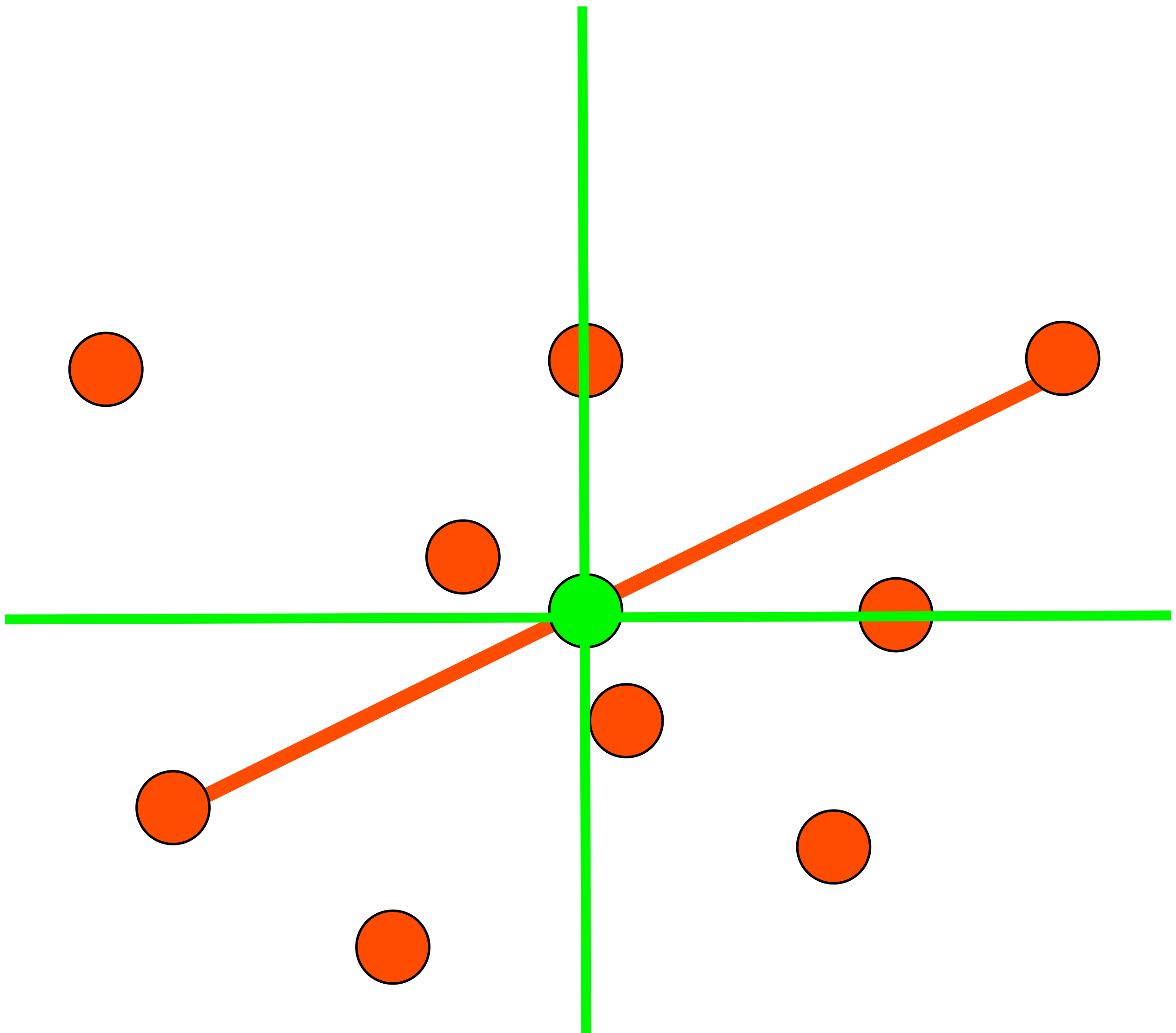

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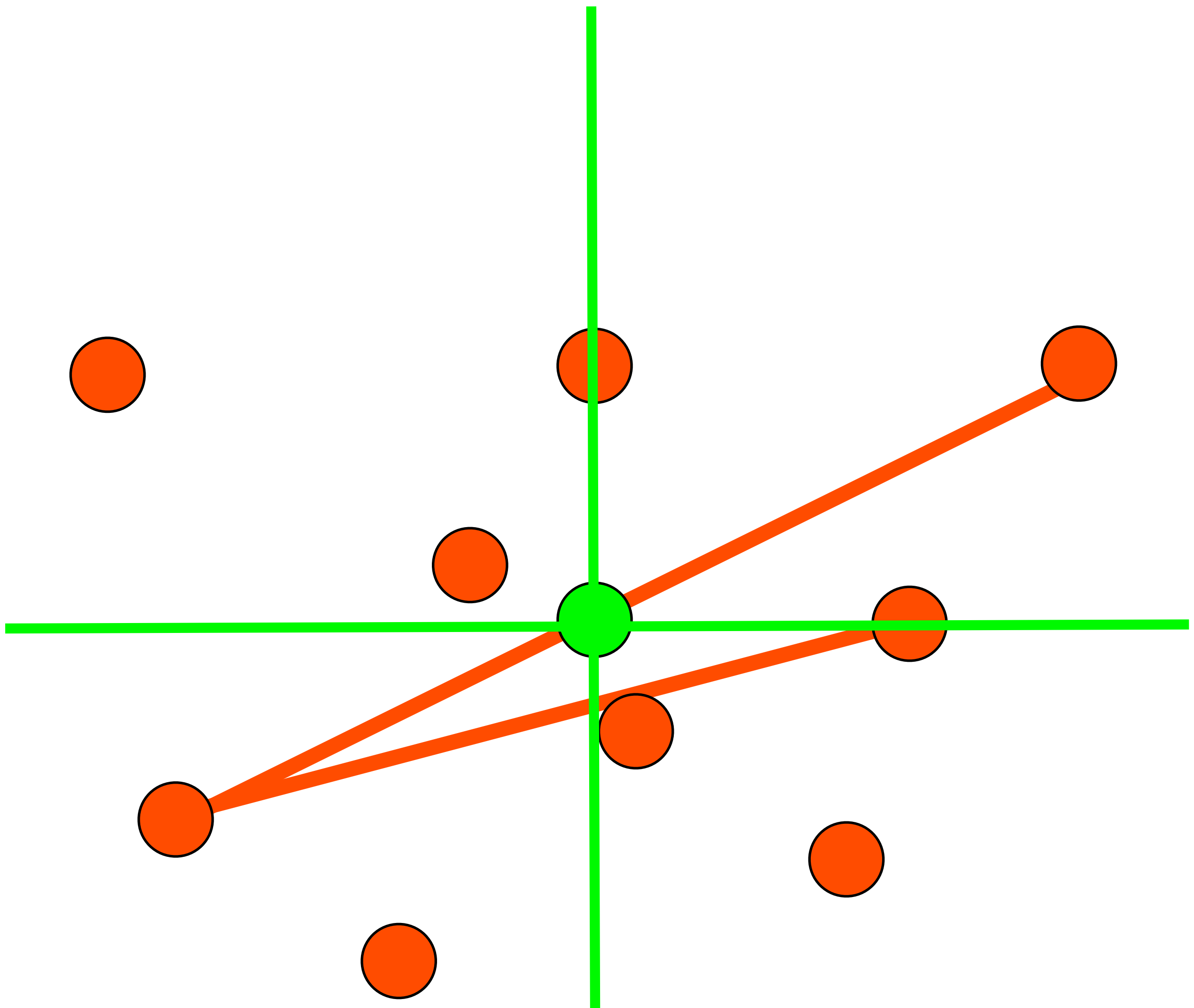

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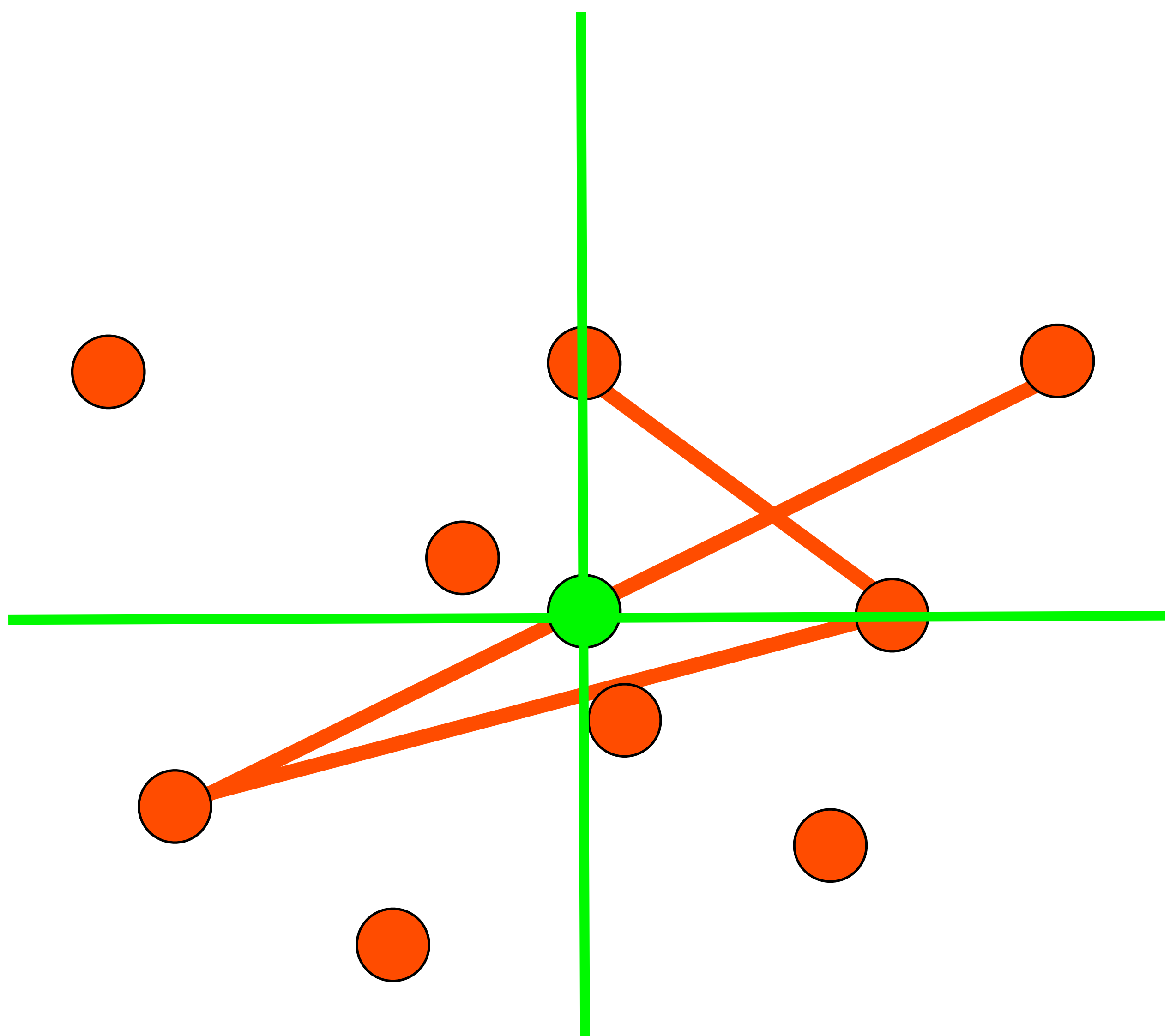

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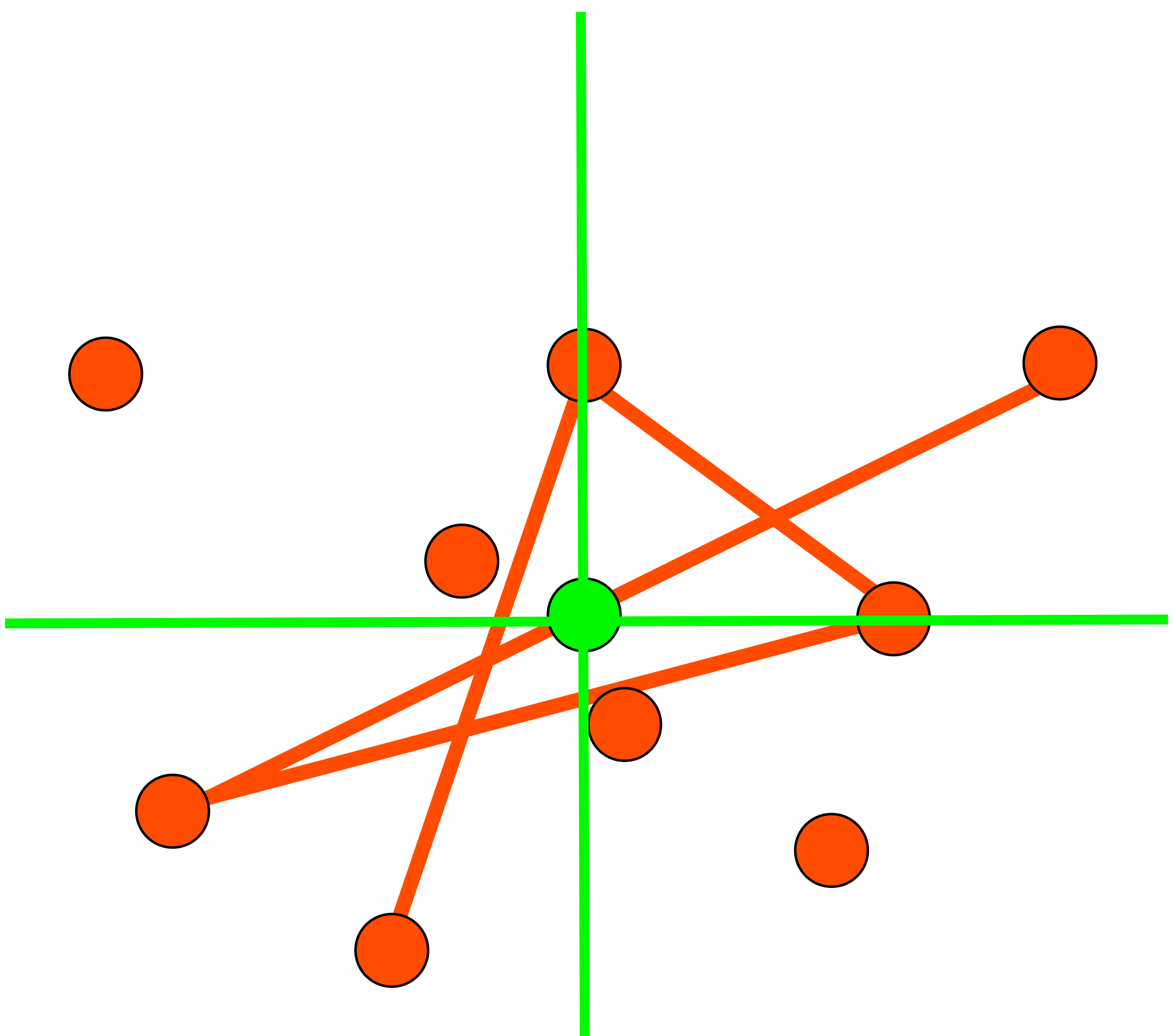

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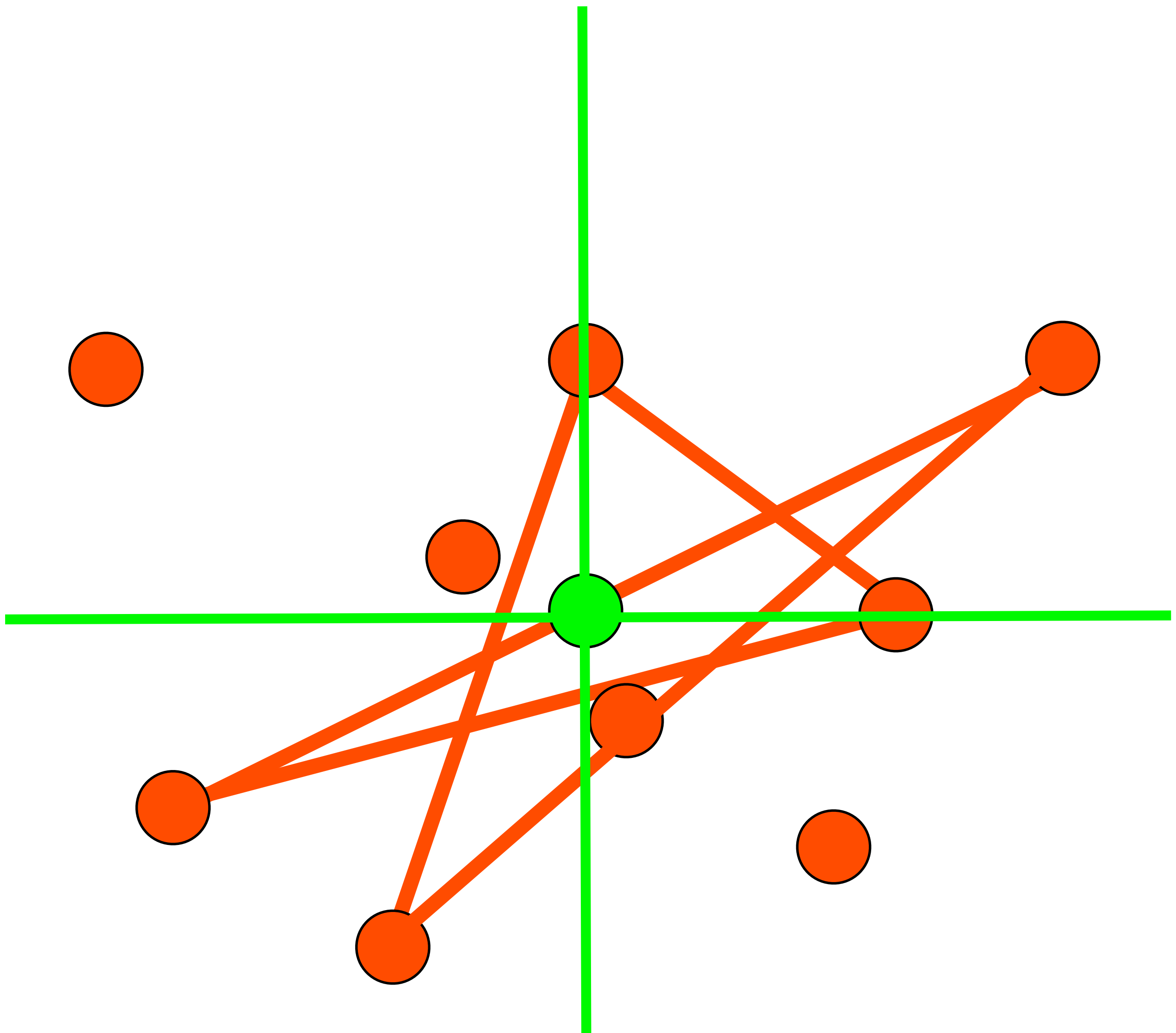

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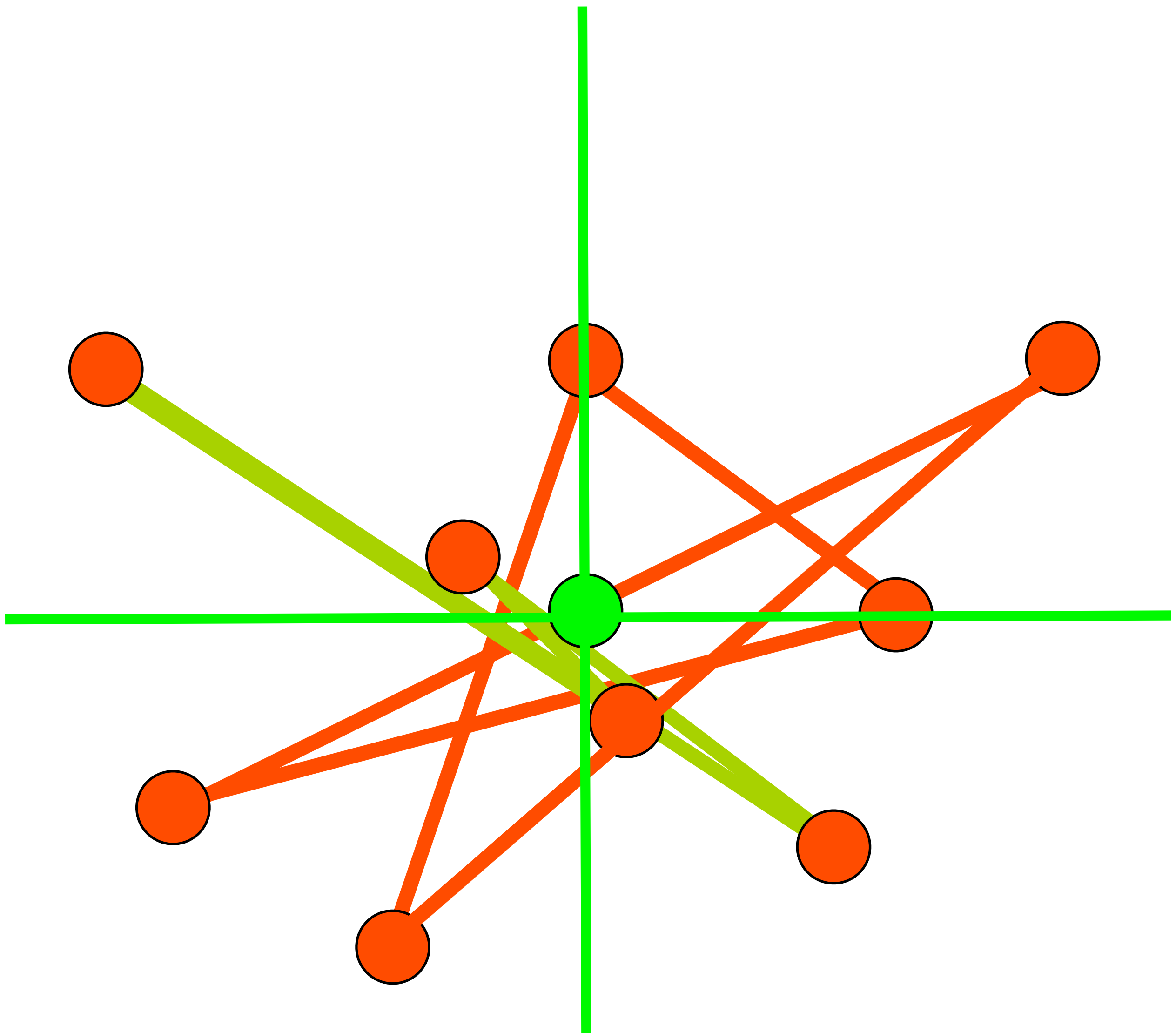

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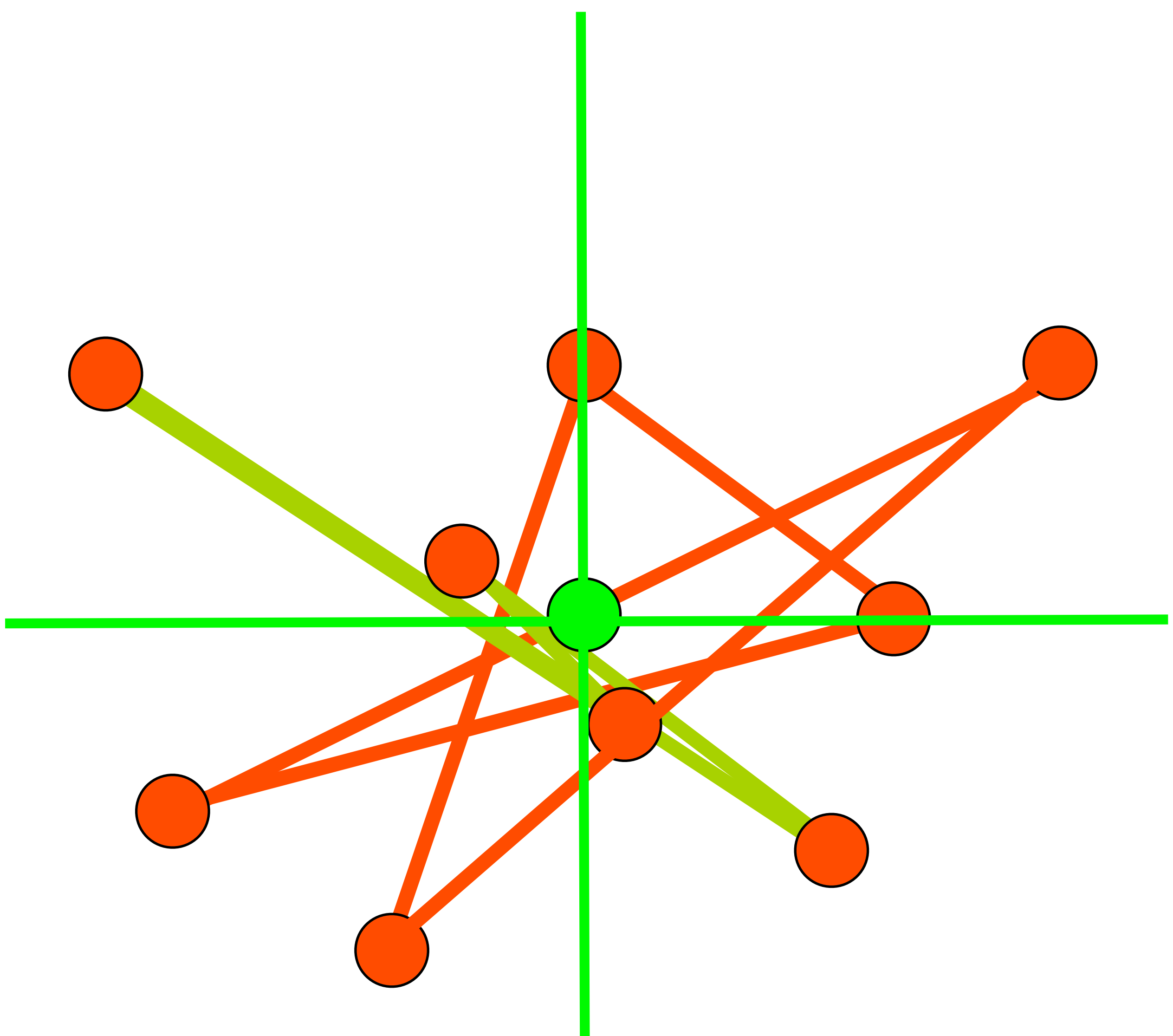

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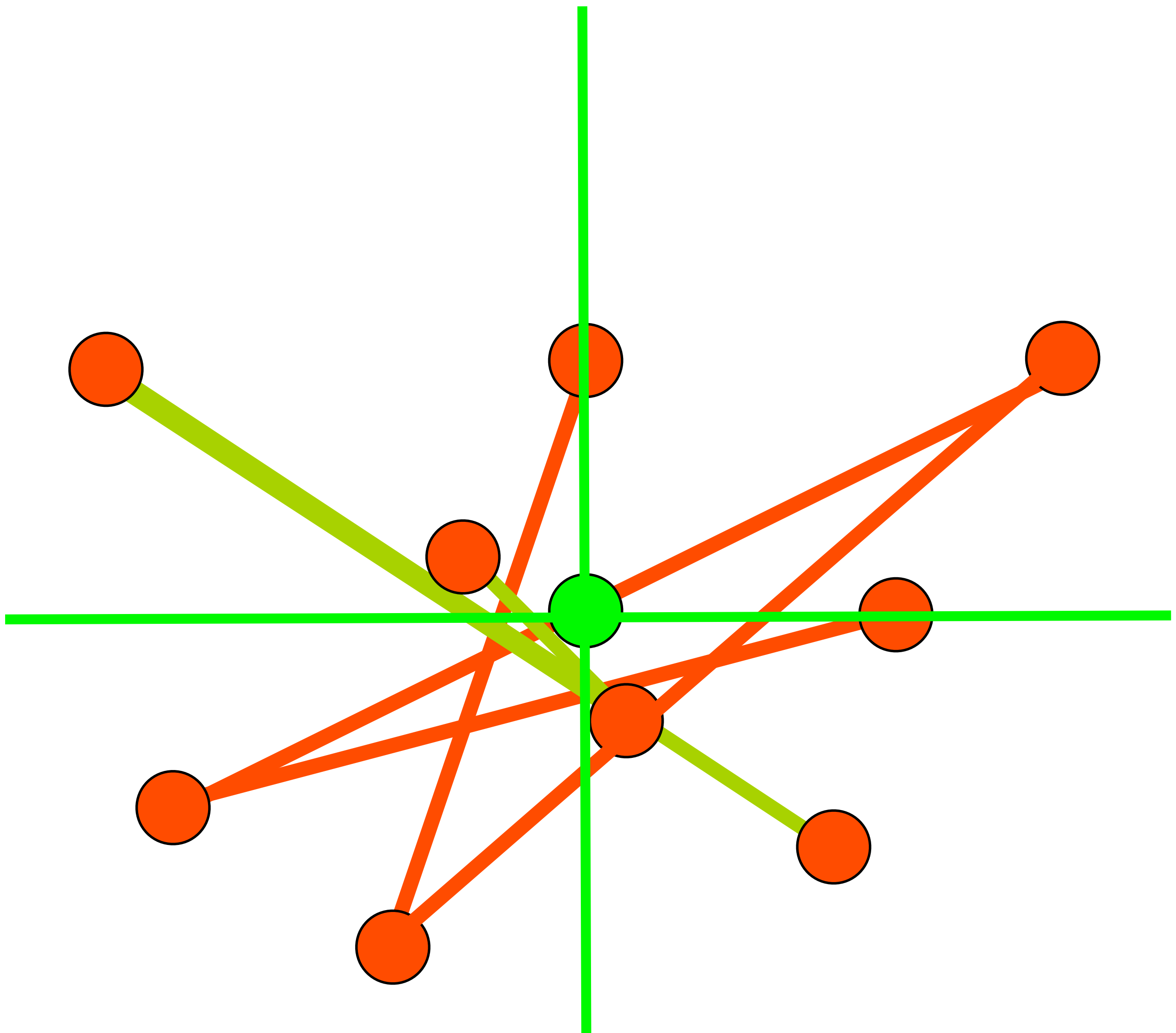

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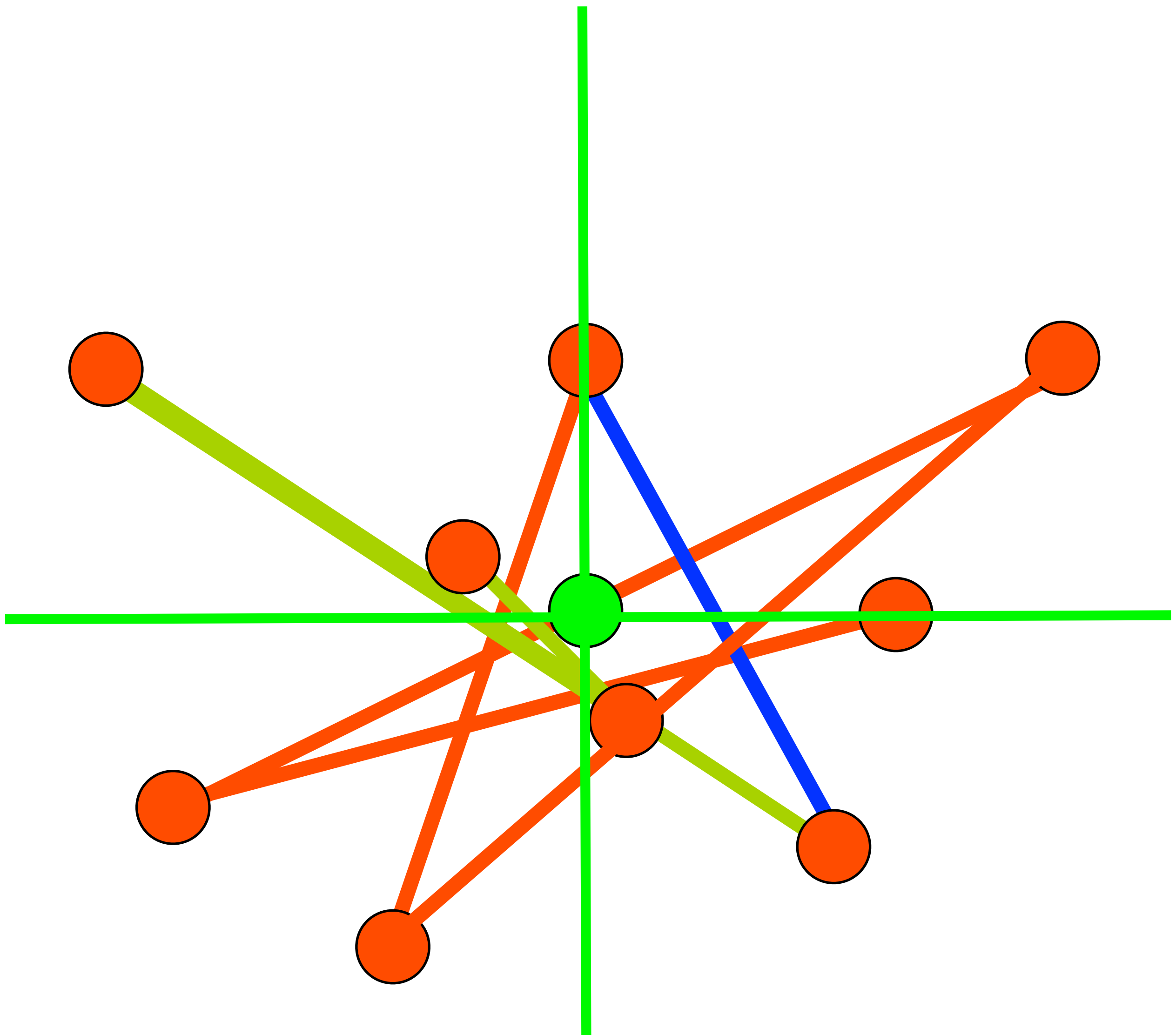

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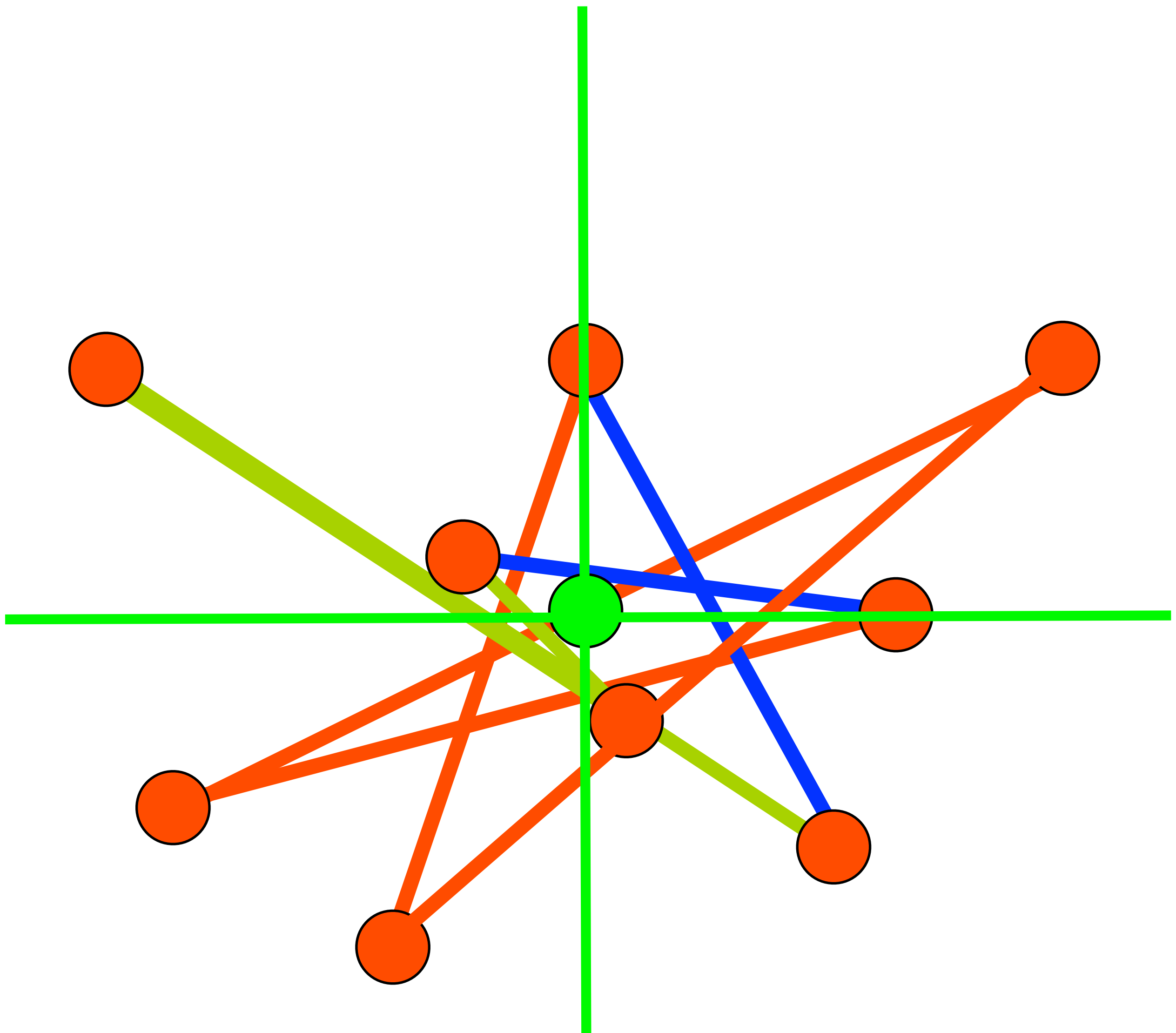

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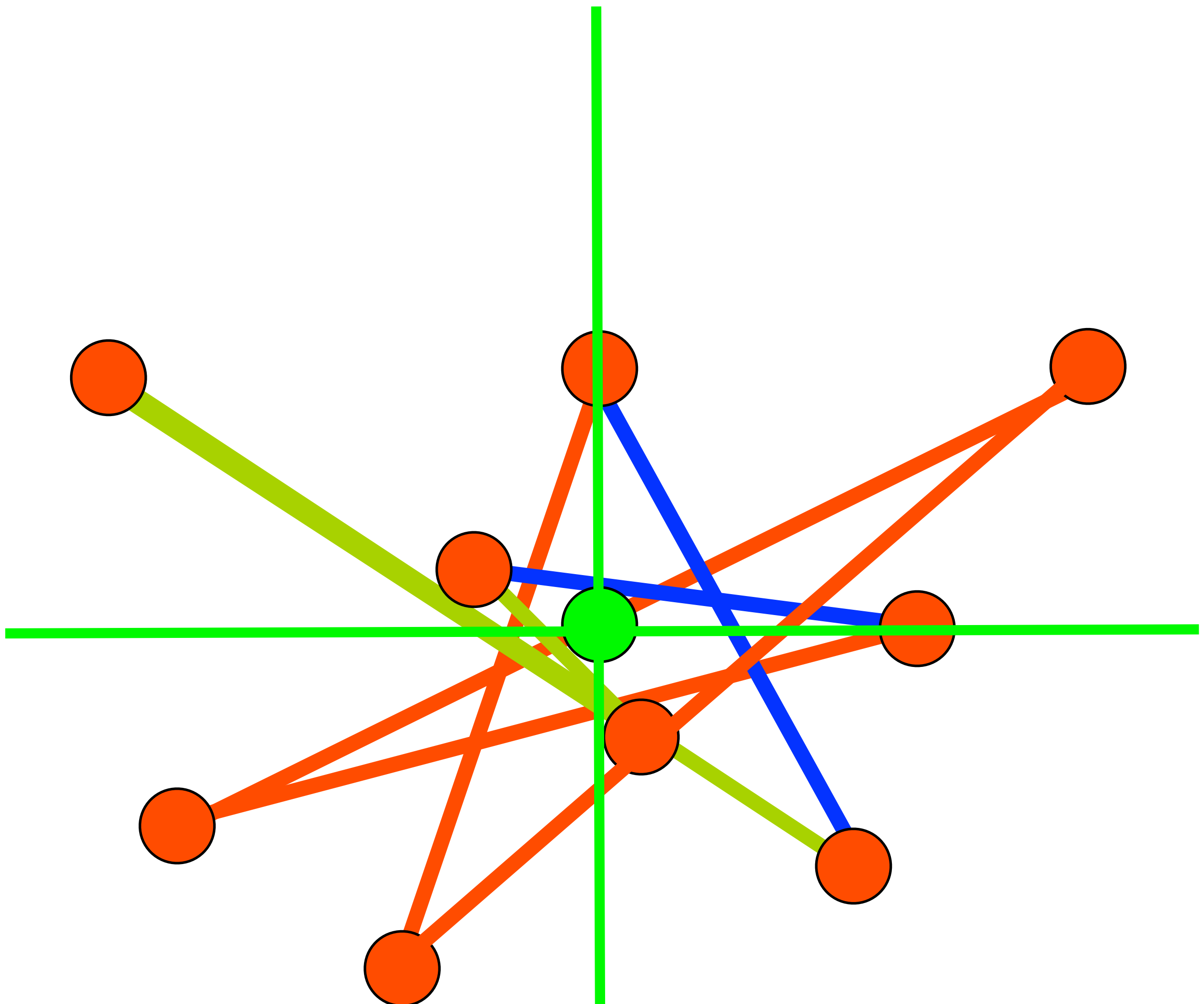

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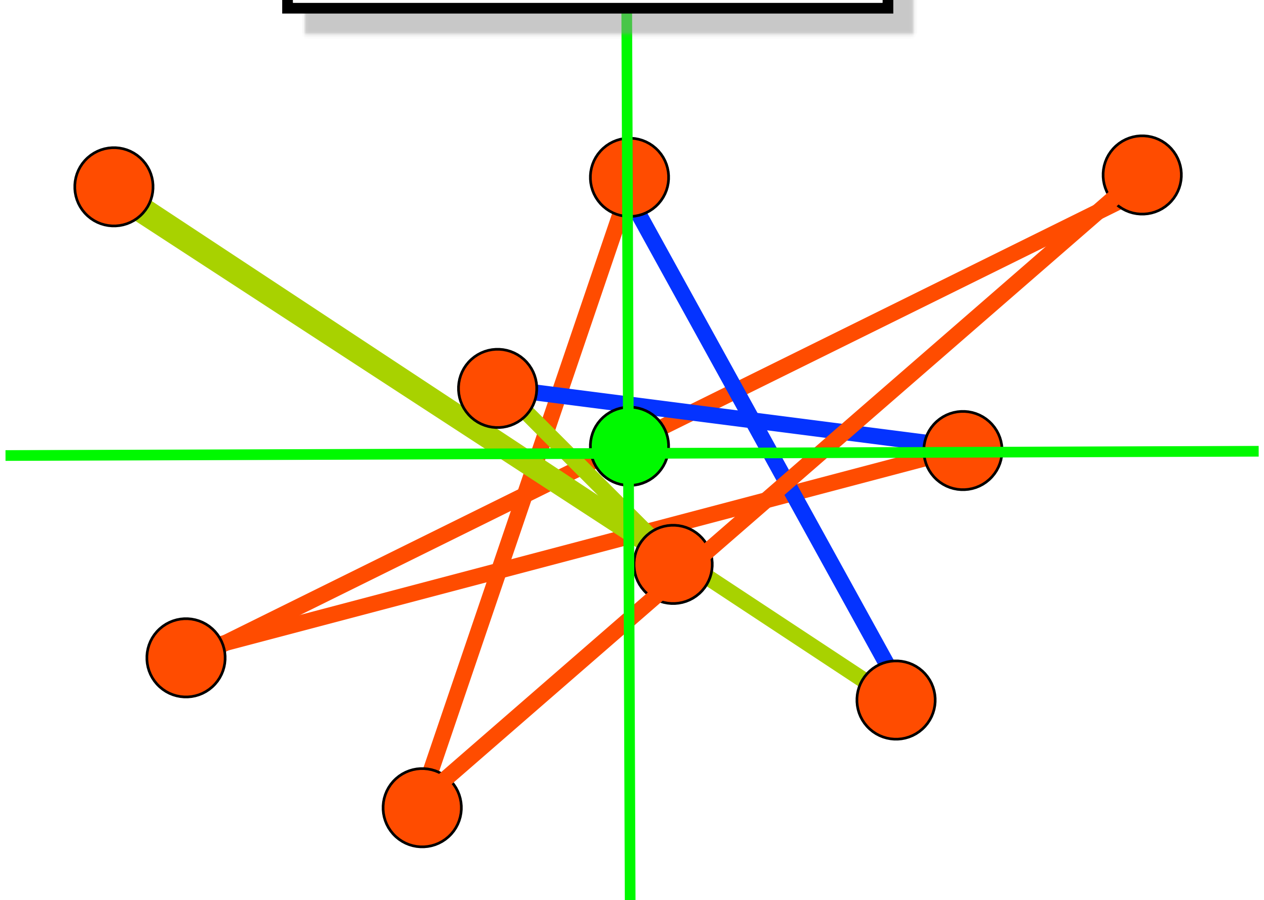
Optimal!!



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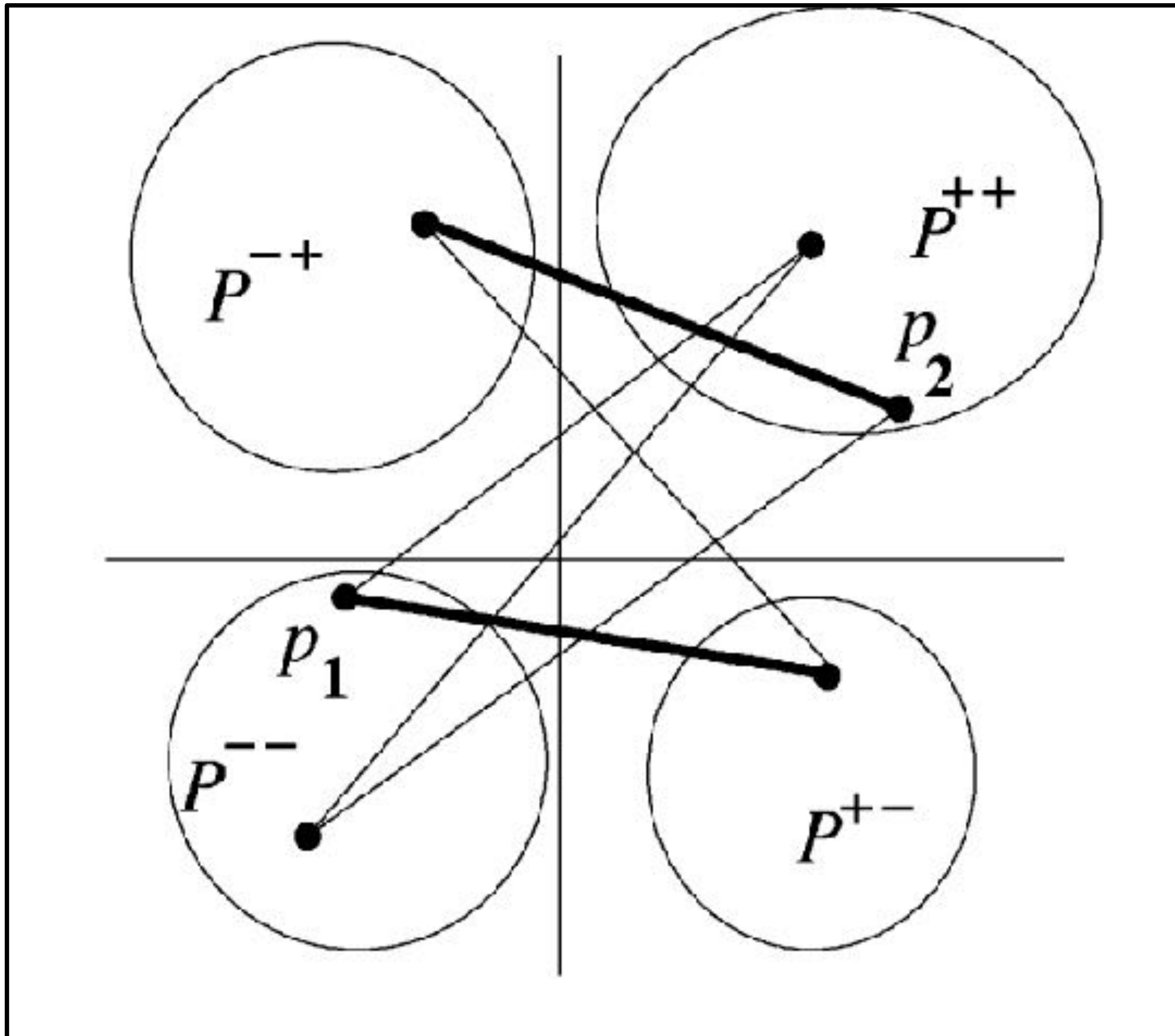

FIG. 5. Getting an optimal tour for even n.

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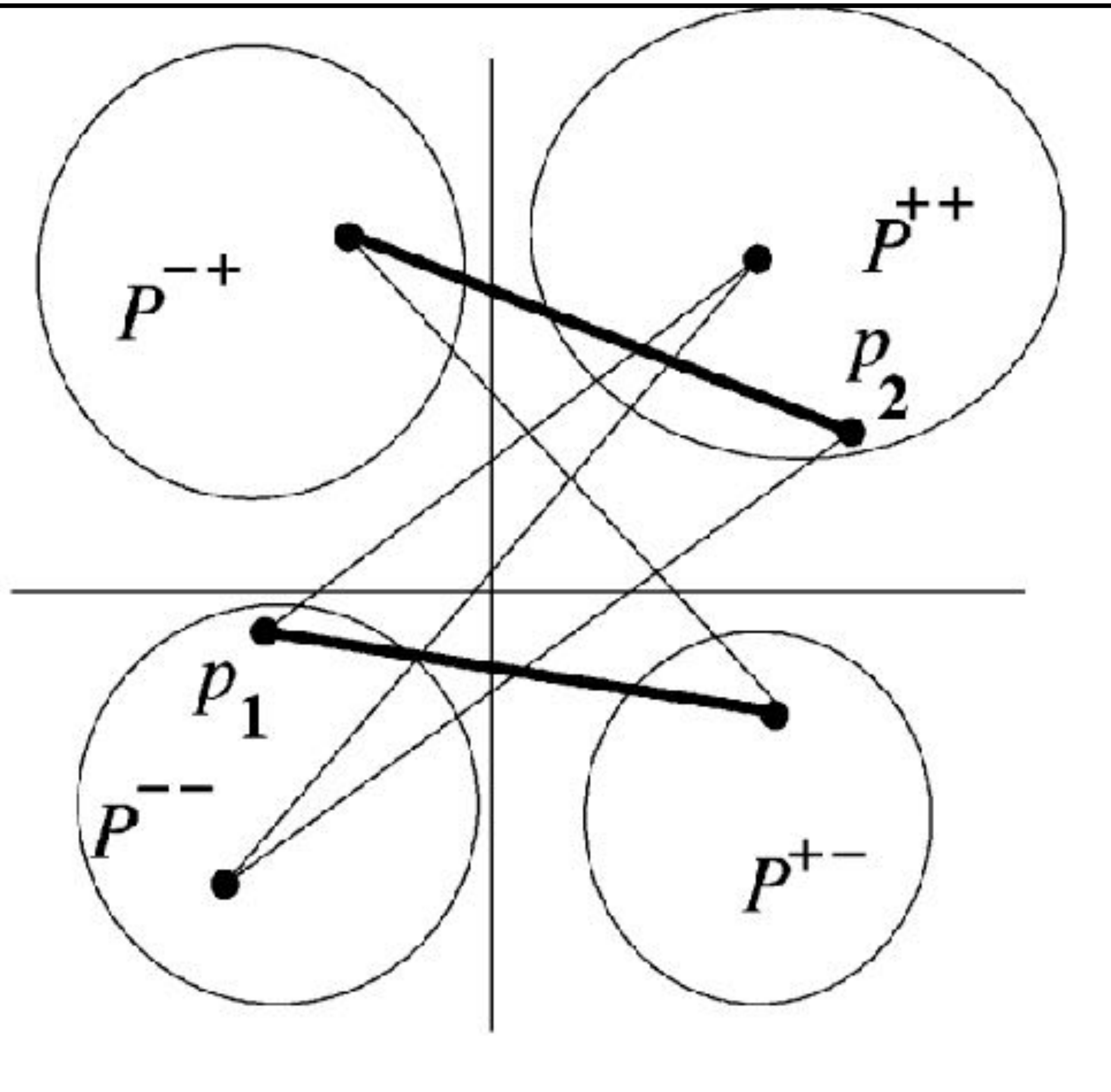
Longest Tours in Manhattan



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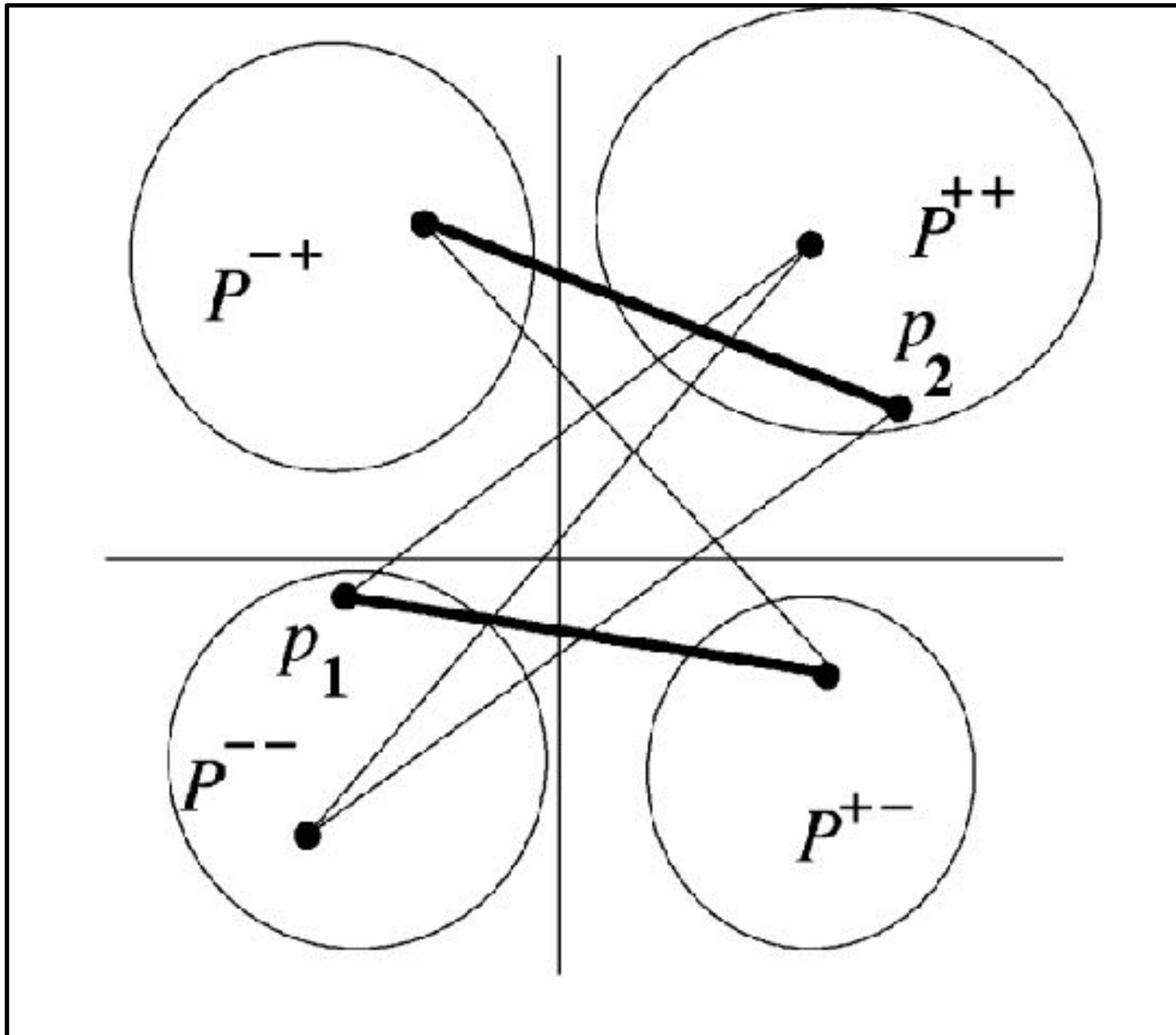
Theorem:

Longest Tours in Manhattan



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A longest tour for points in the plane with Manhattan distances can be computed in *linear* time.

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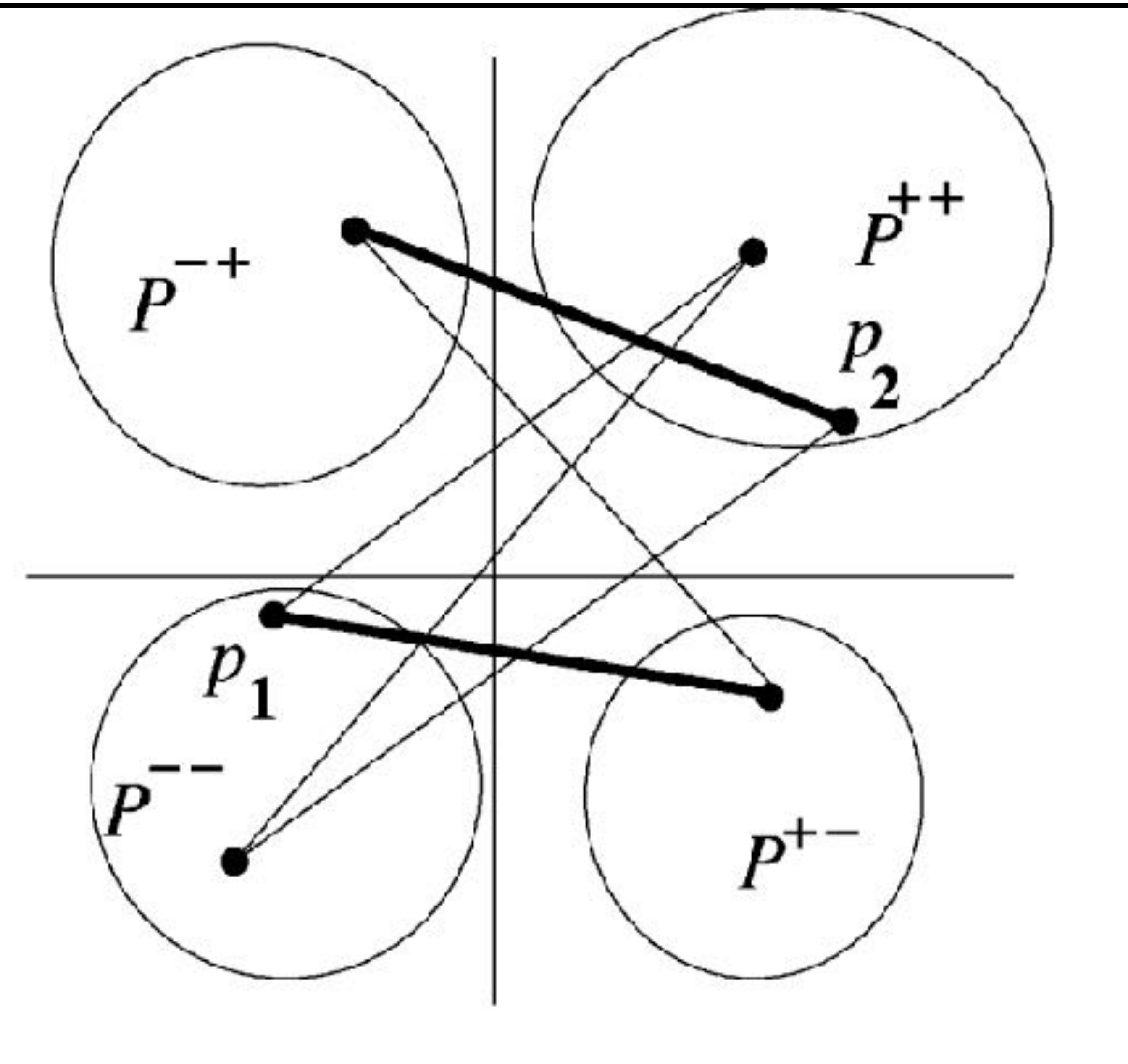
Longest Tours in Manhattan



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Euclidean distances??

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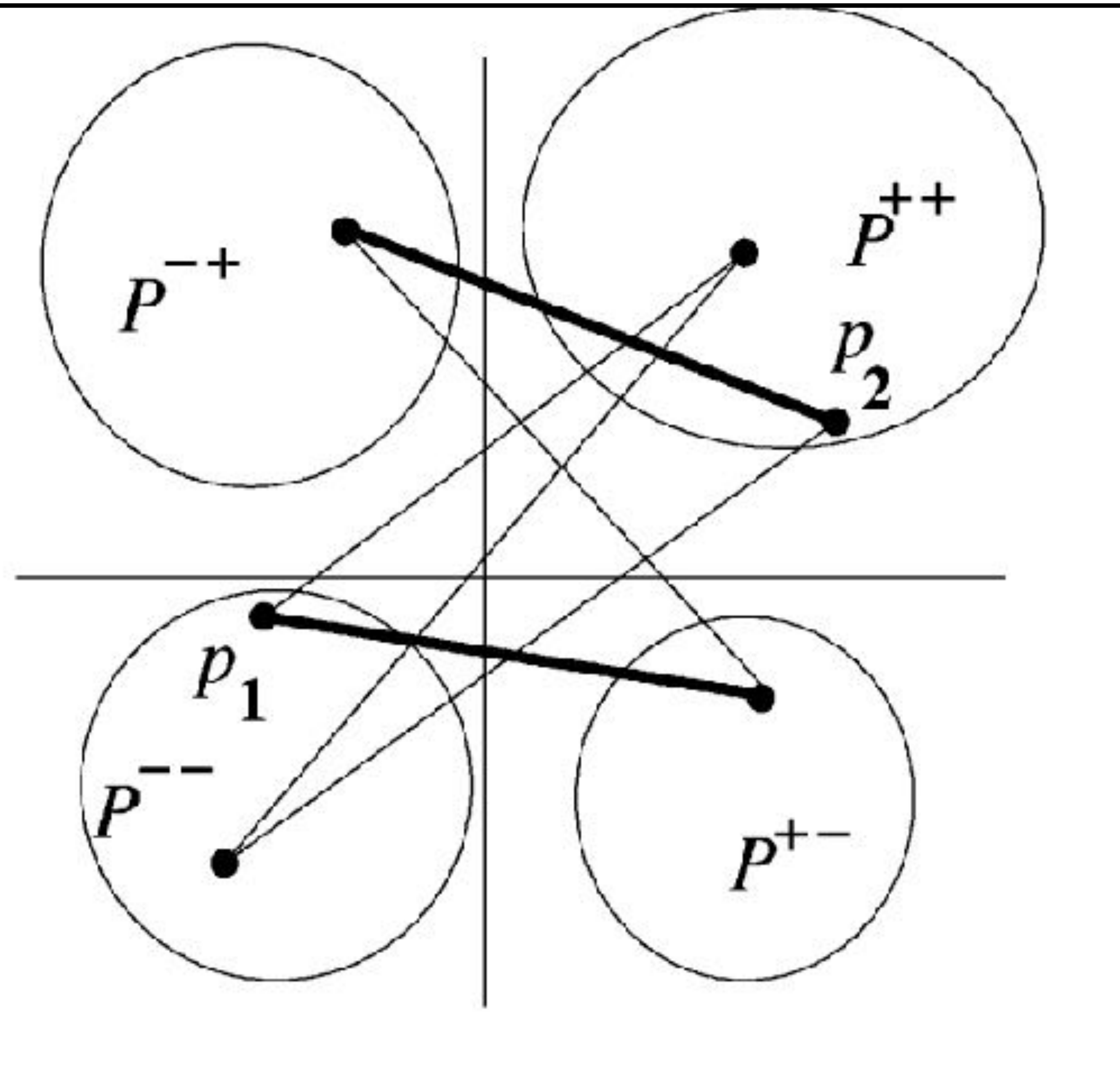

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Think Globally

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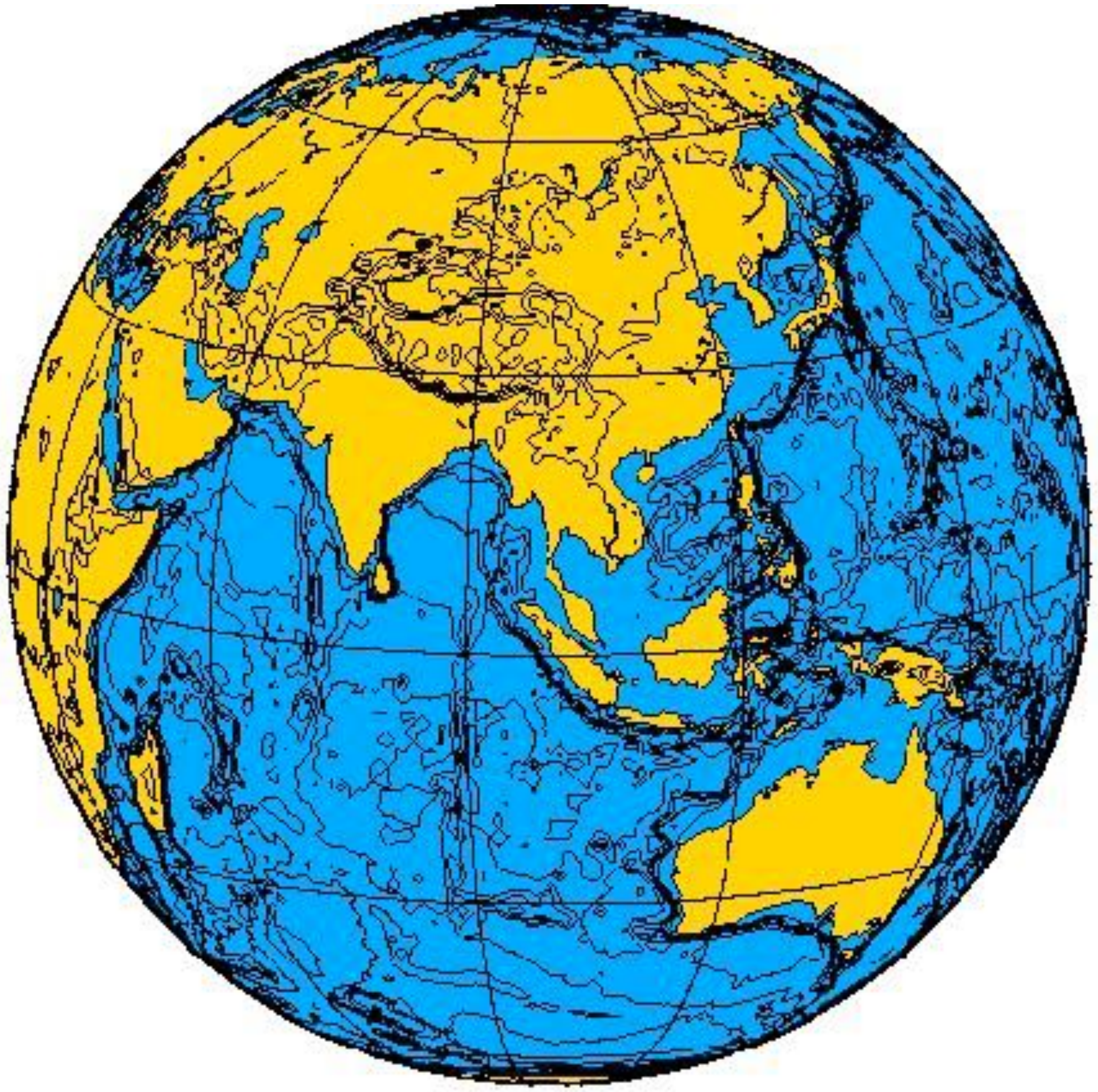

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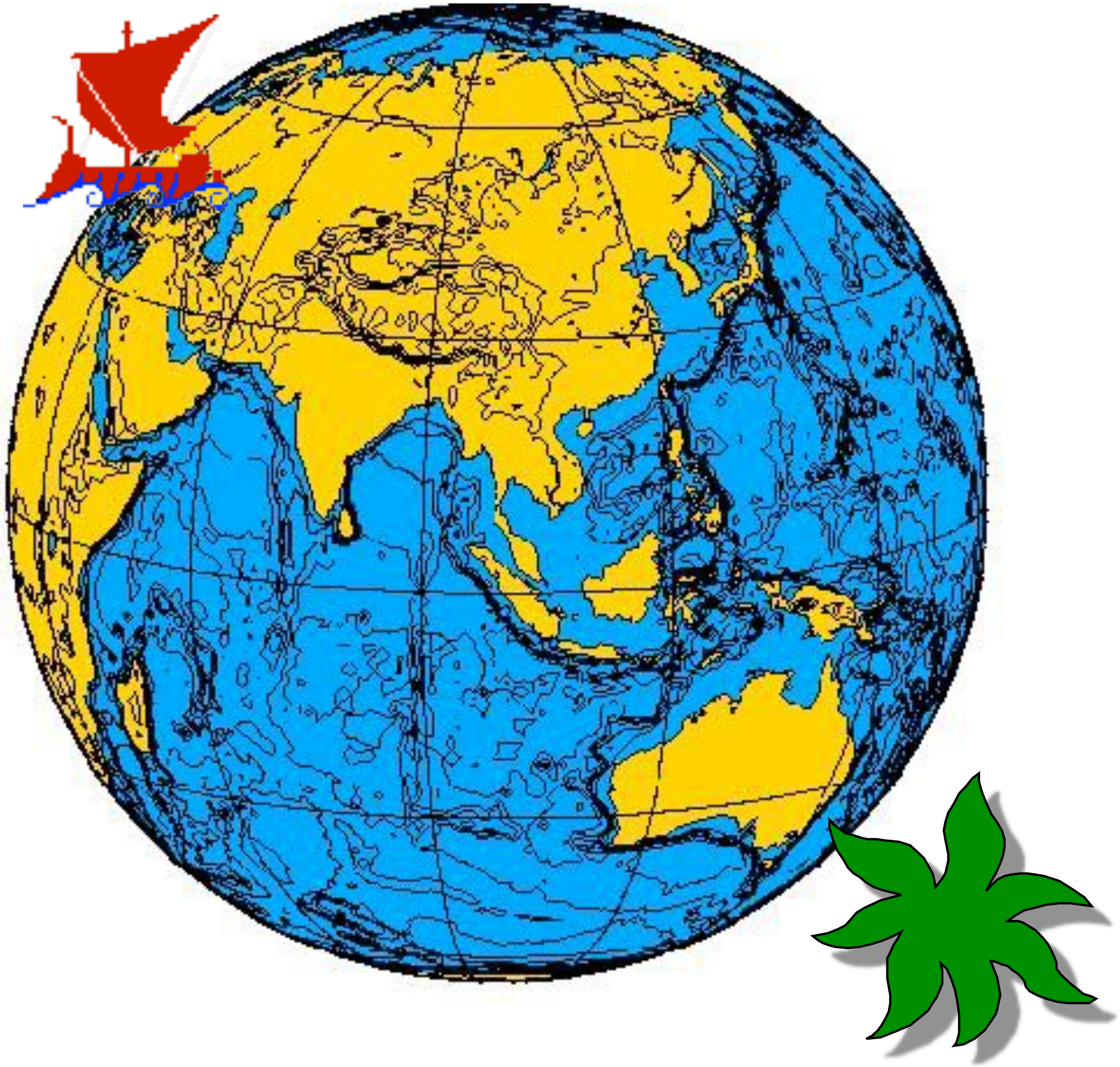

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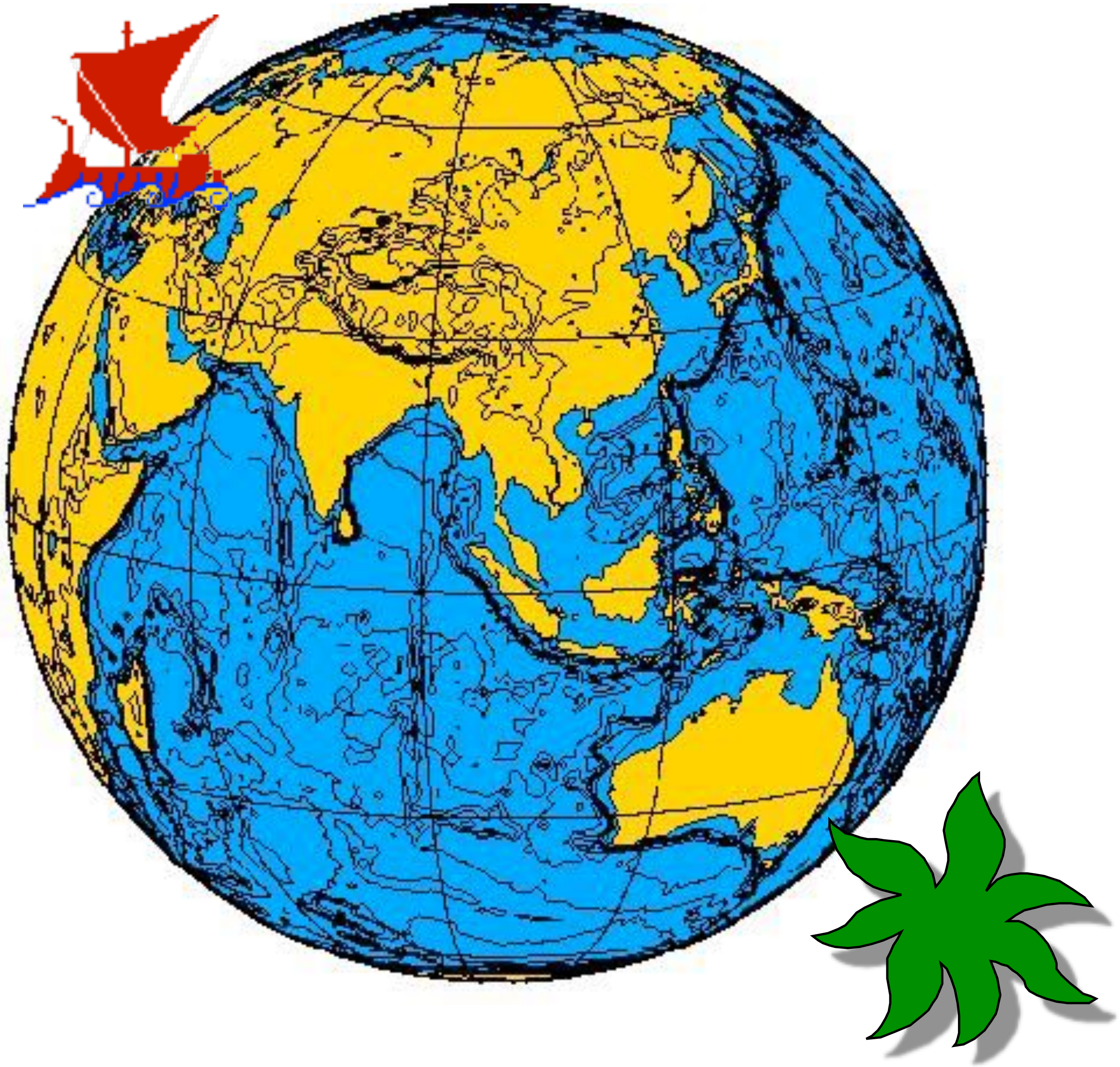
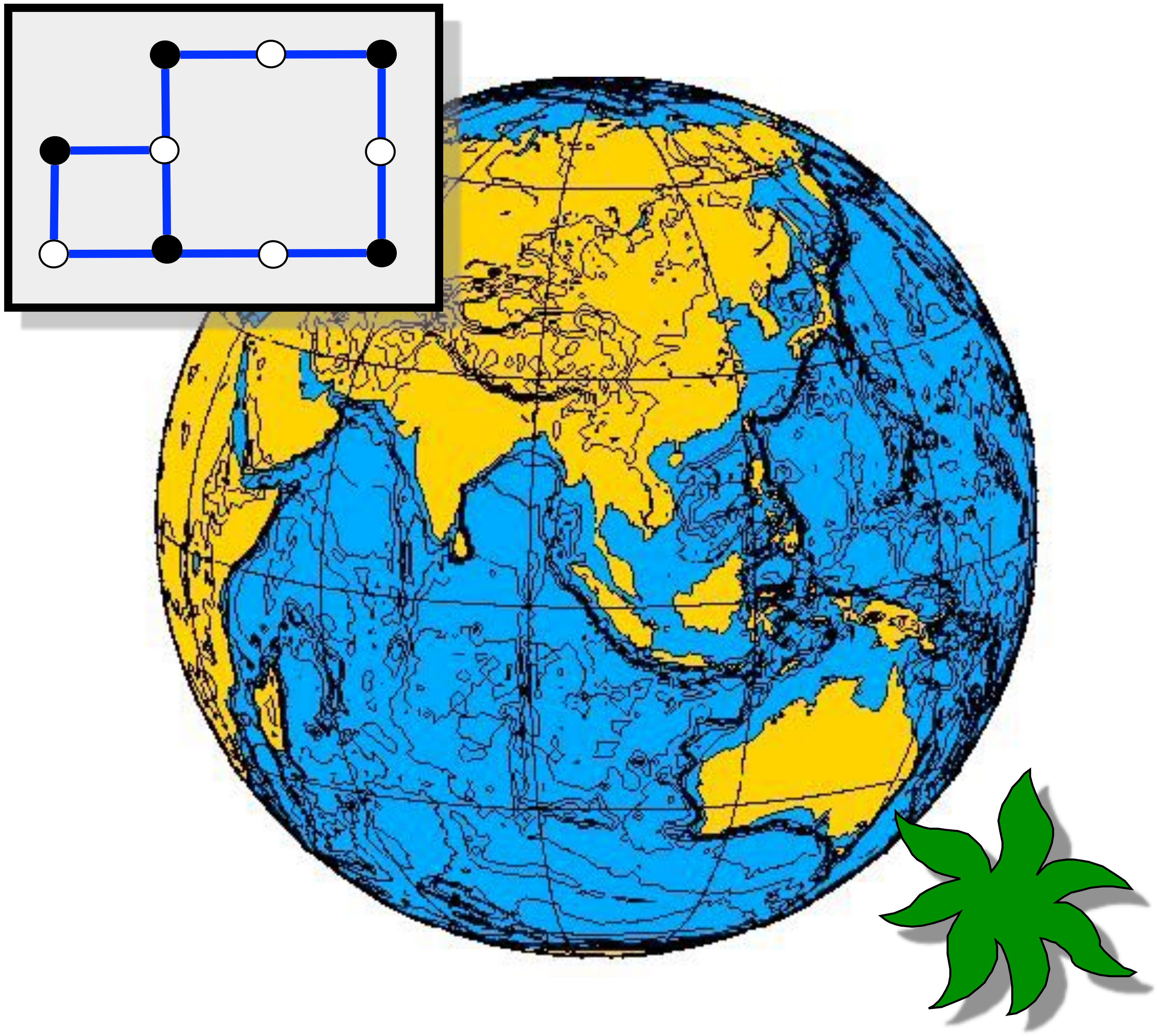


- Consider a grid graph embedded on a sphere.

Think Globally



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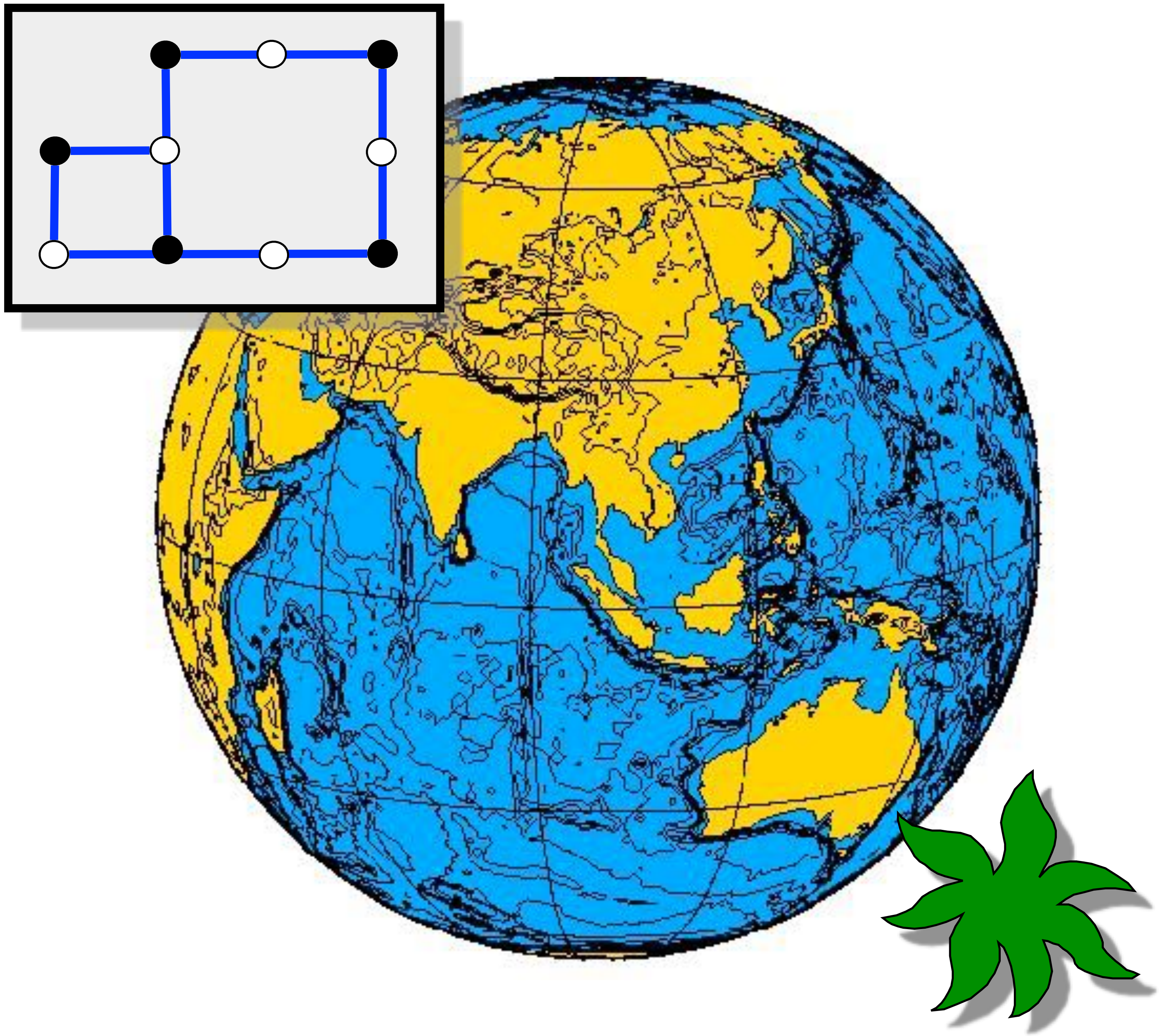
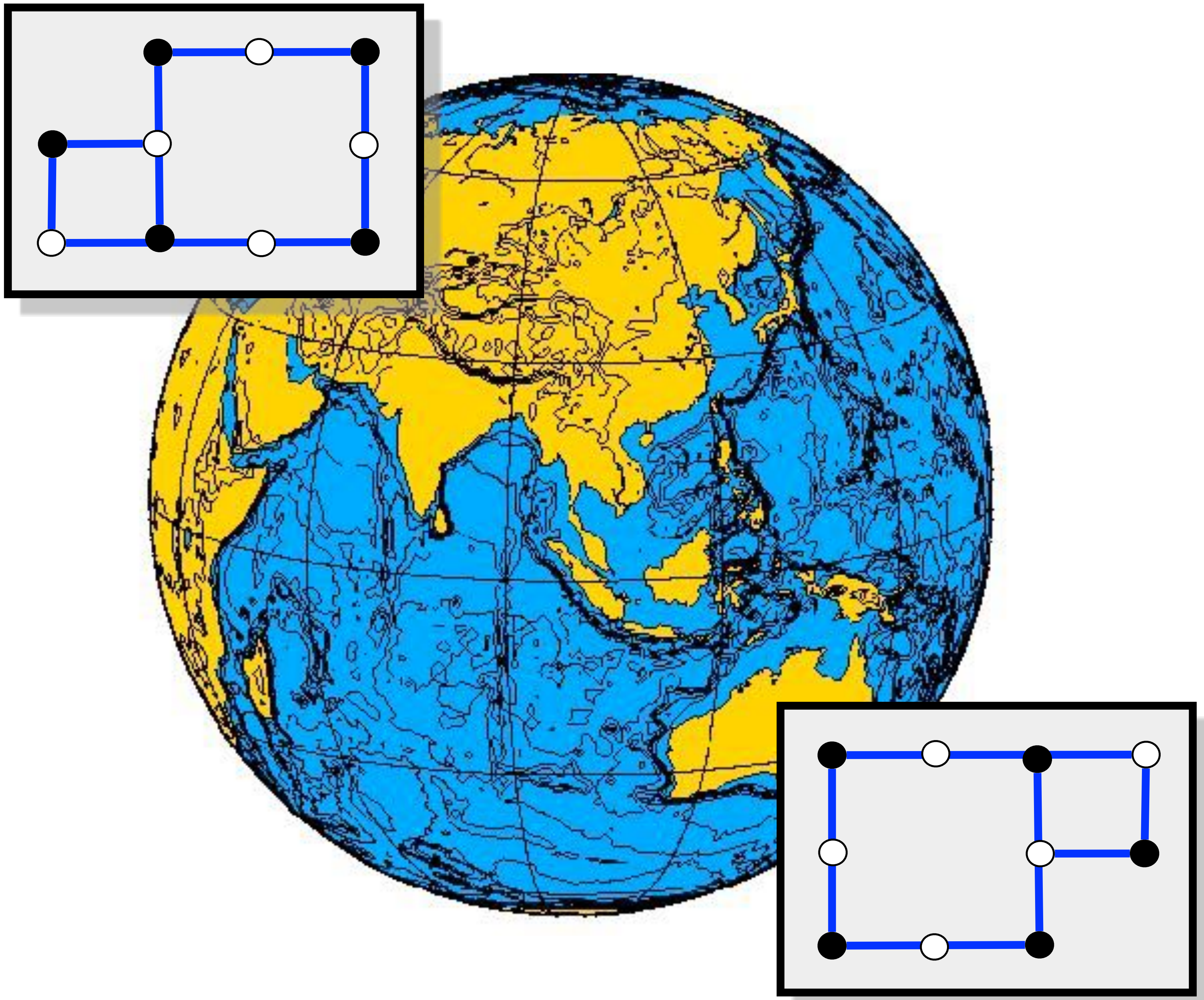


- Consider a grid graph embedded on a sphere.
- Add second copy at the antipodes.

Think Globally



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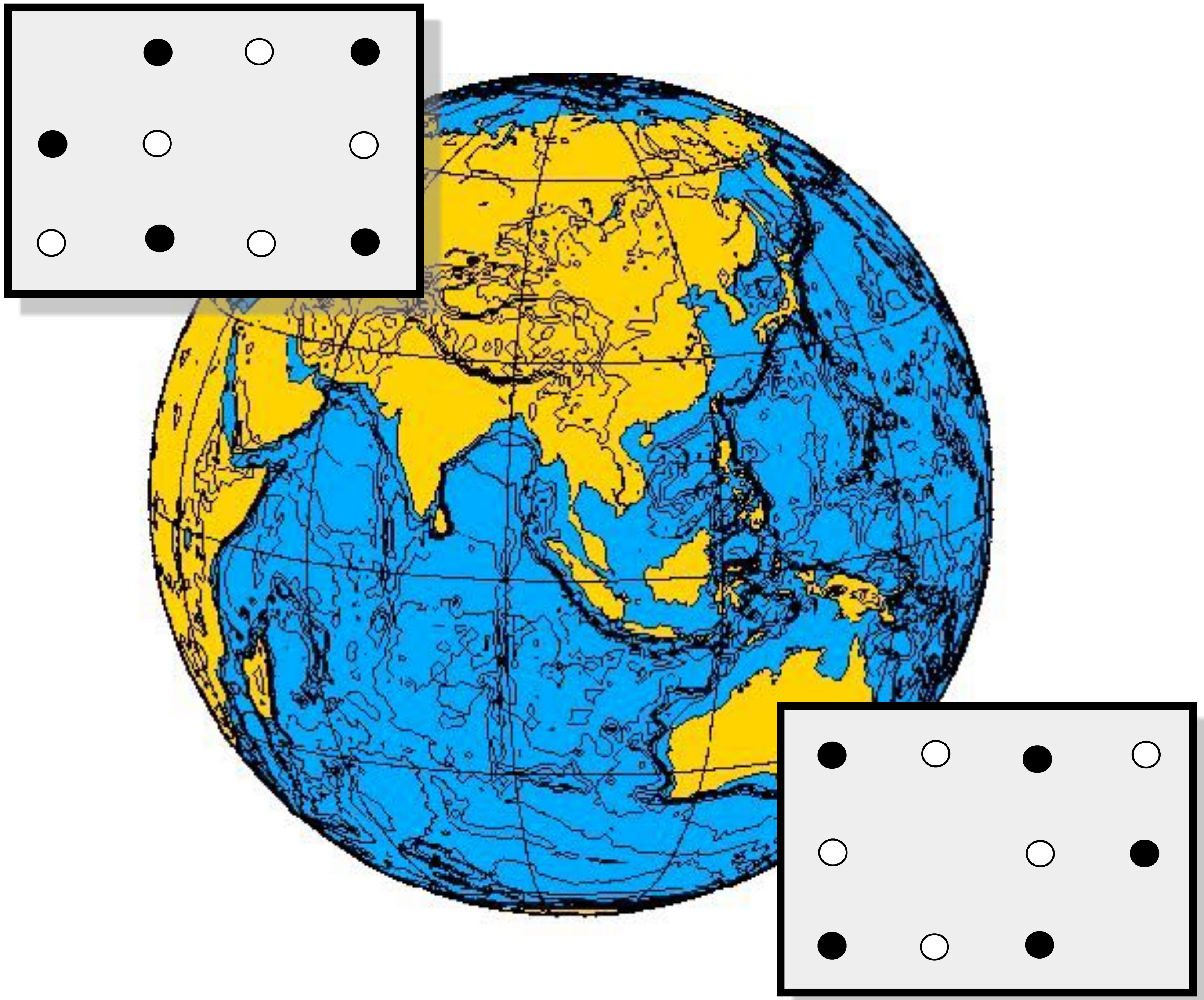

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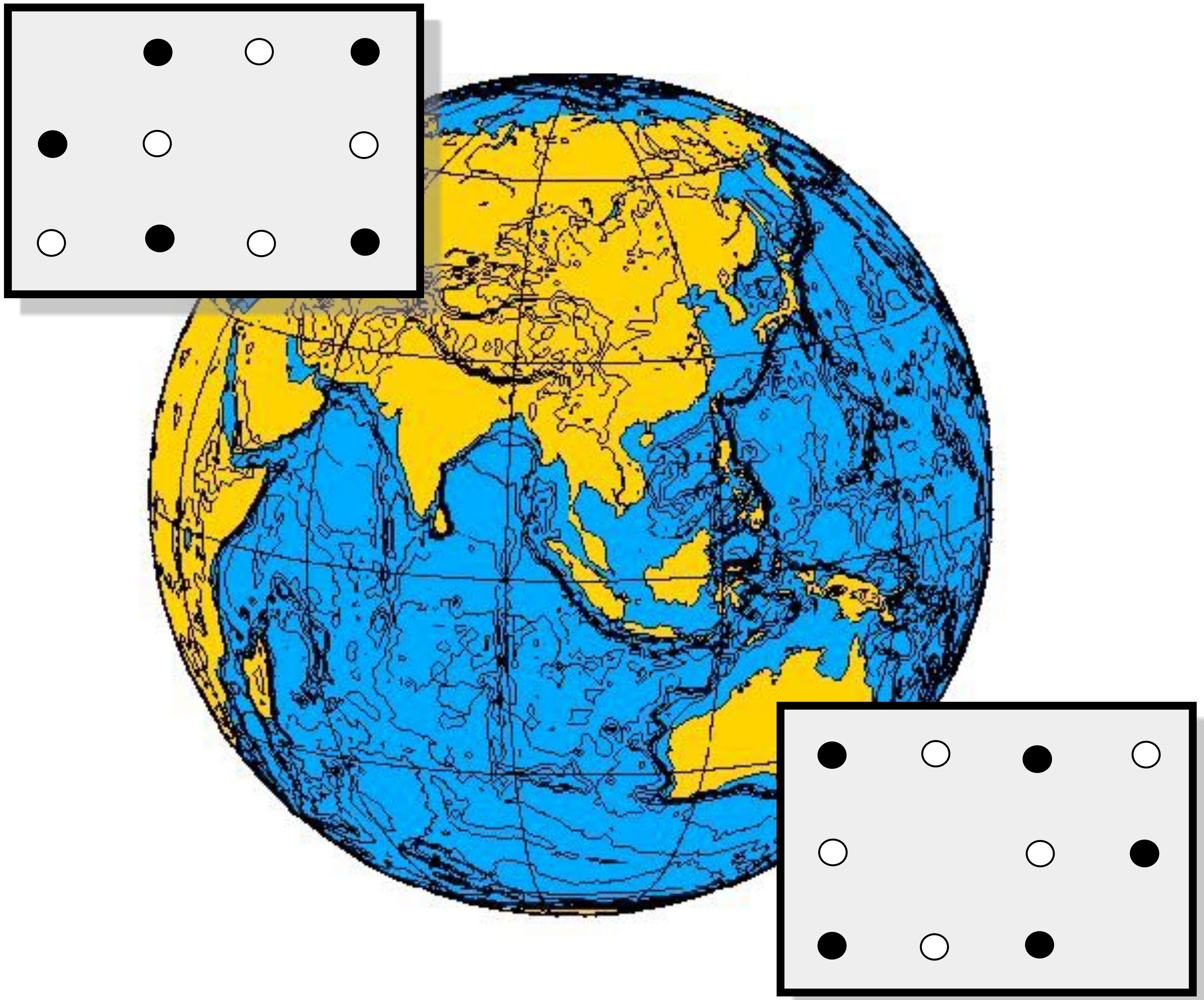

- Consider a grid graph embedded on a sphere.
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- Longest connections are between antipodal points.

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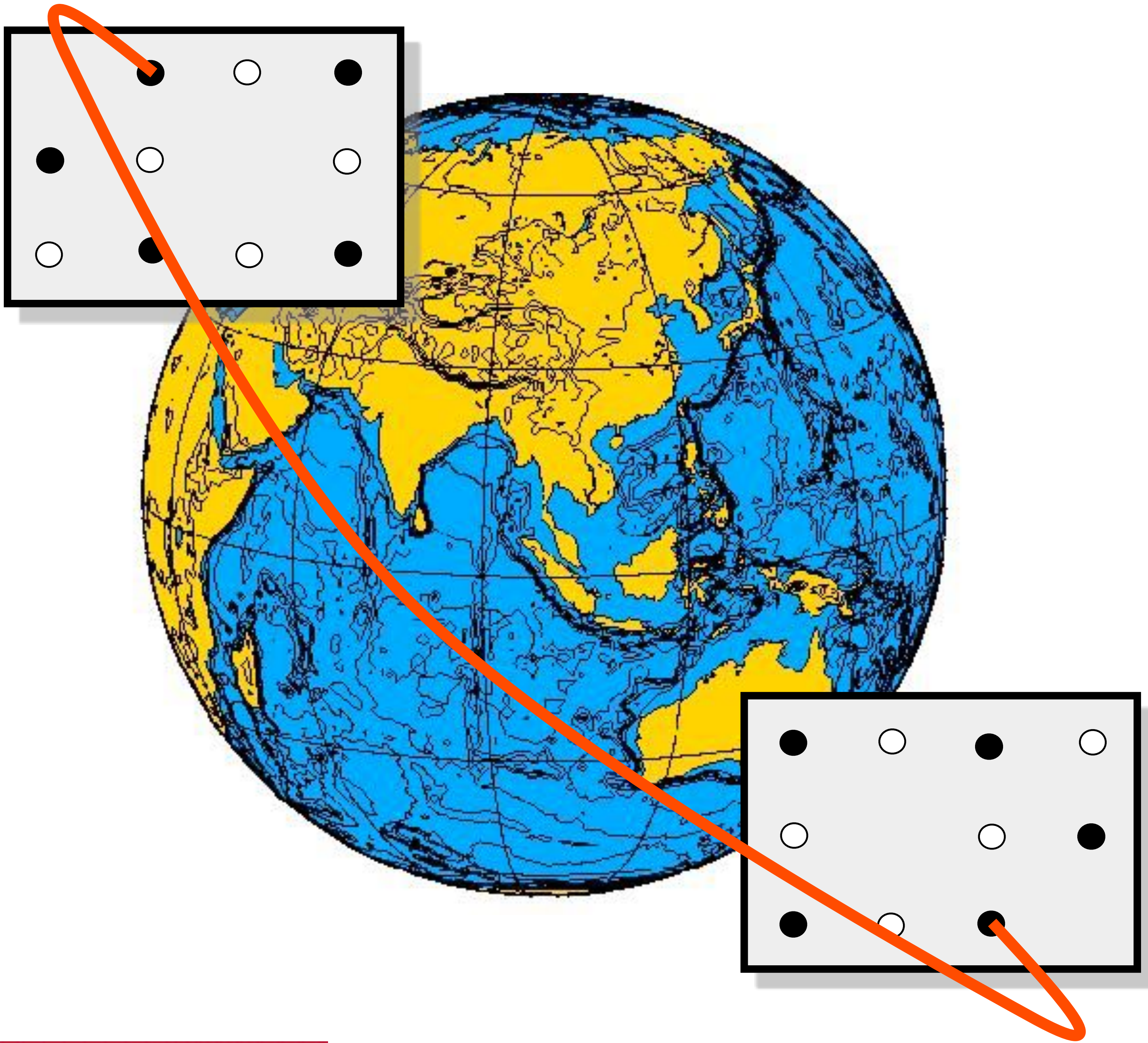

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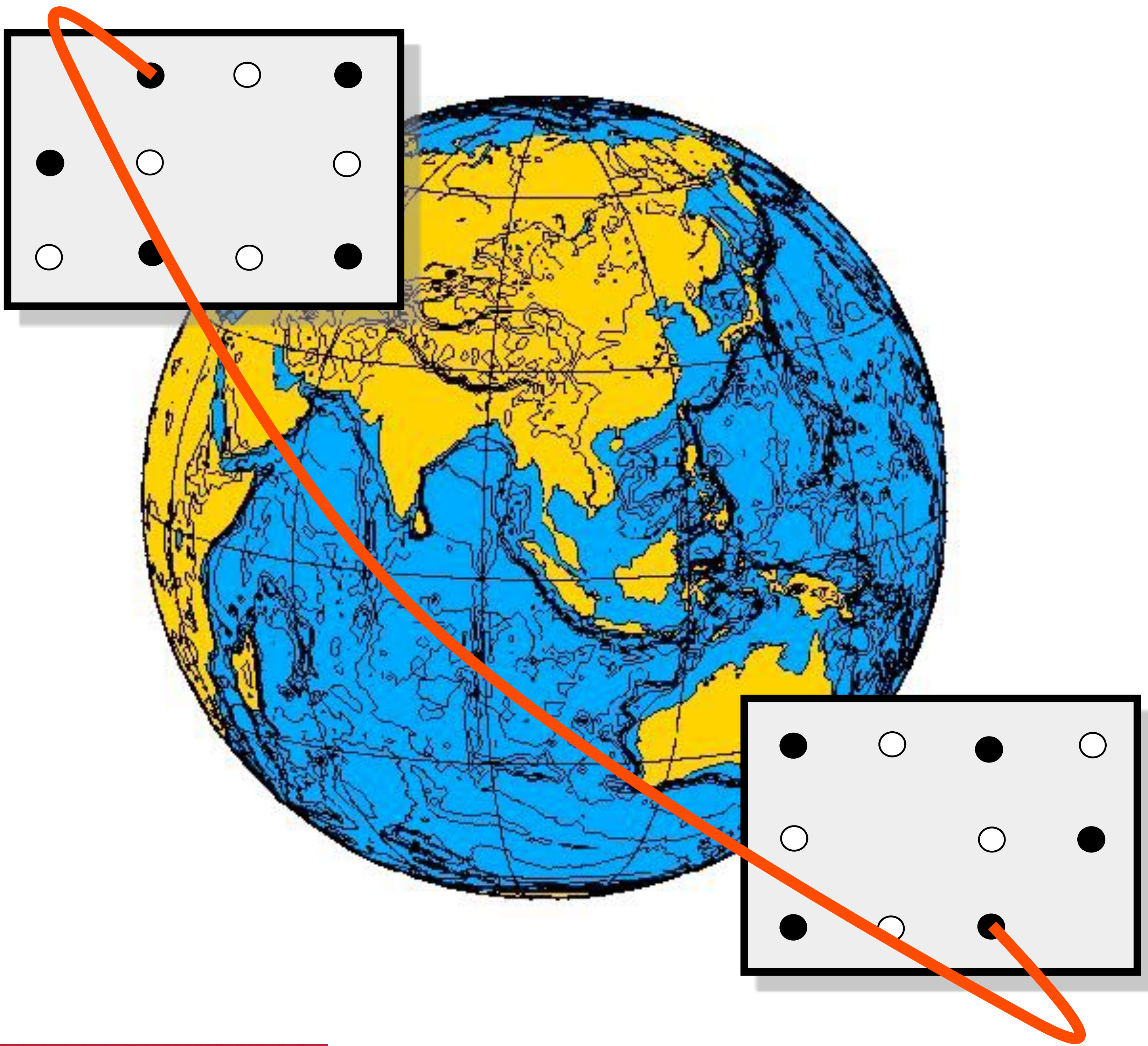

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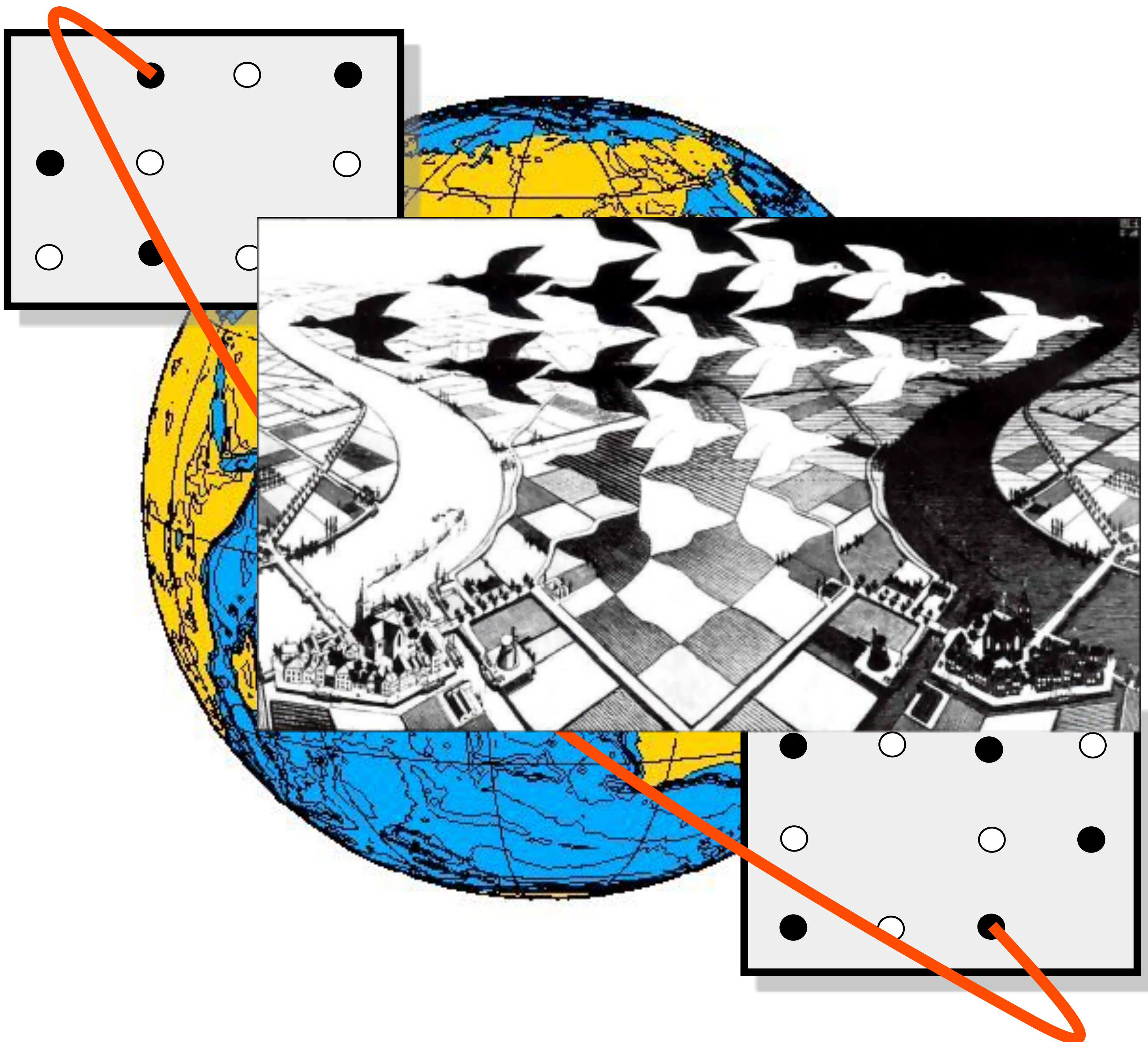
Think Globally



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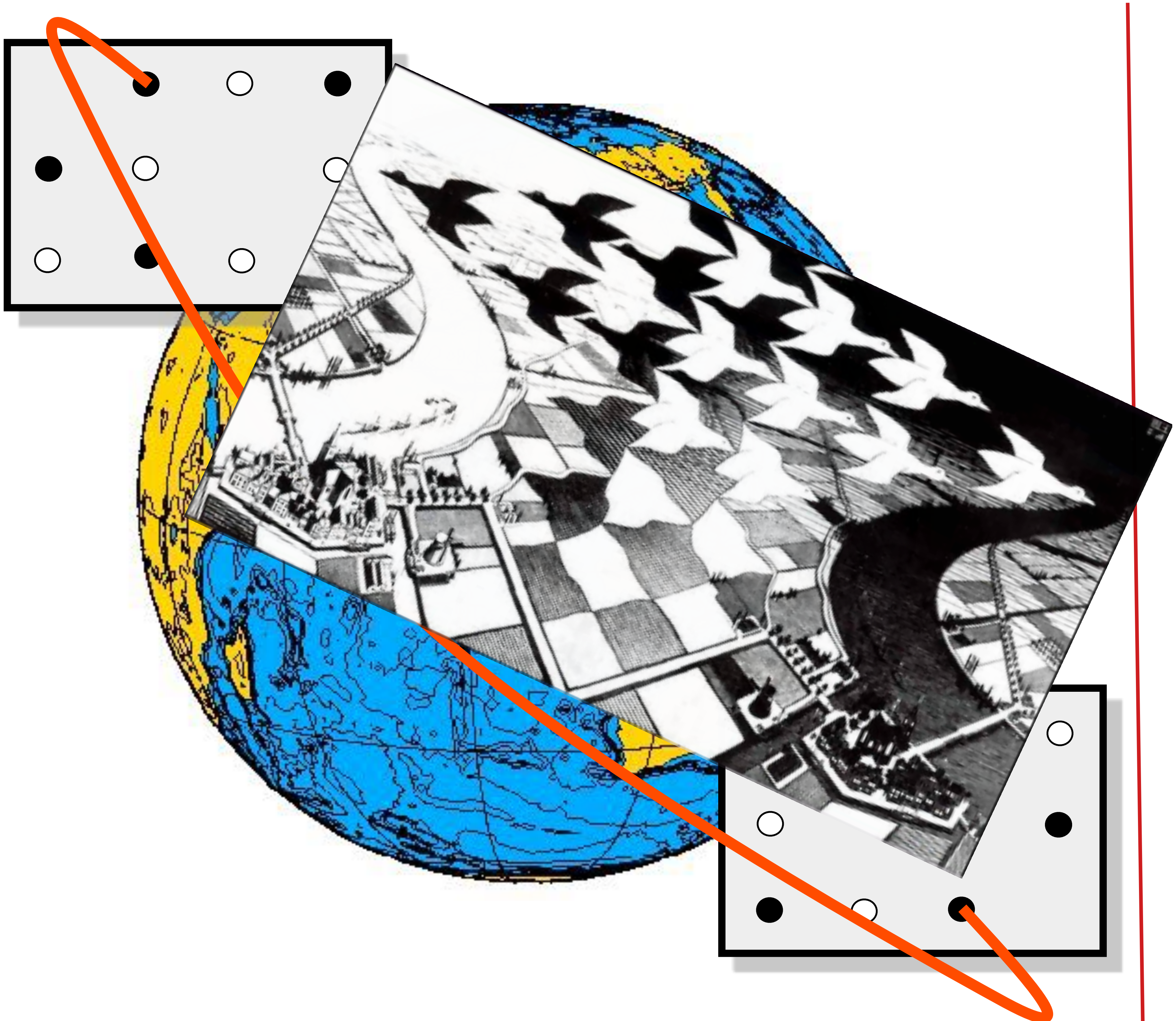

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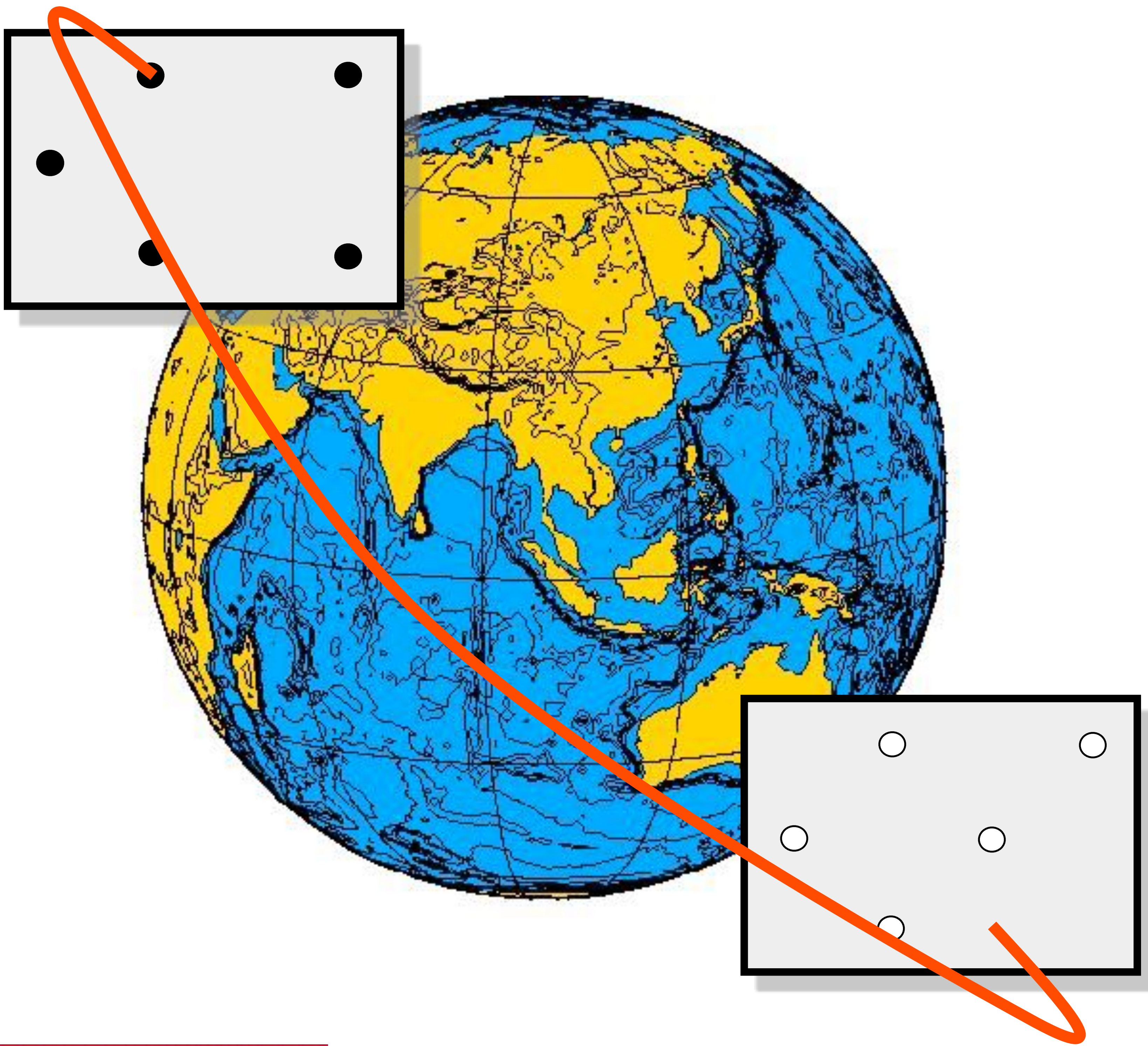

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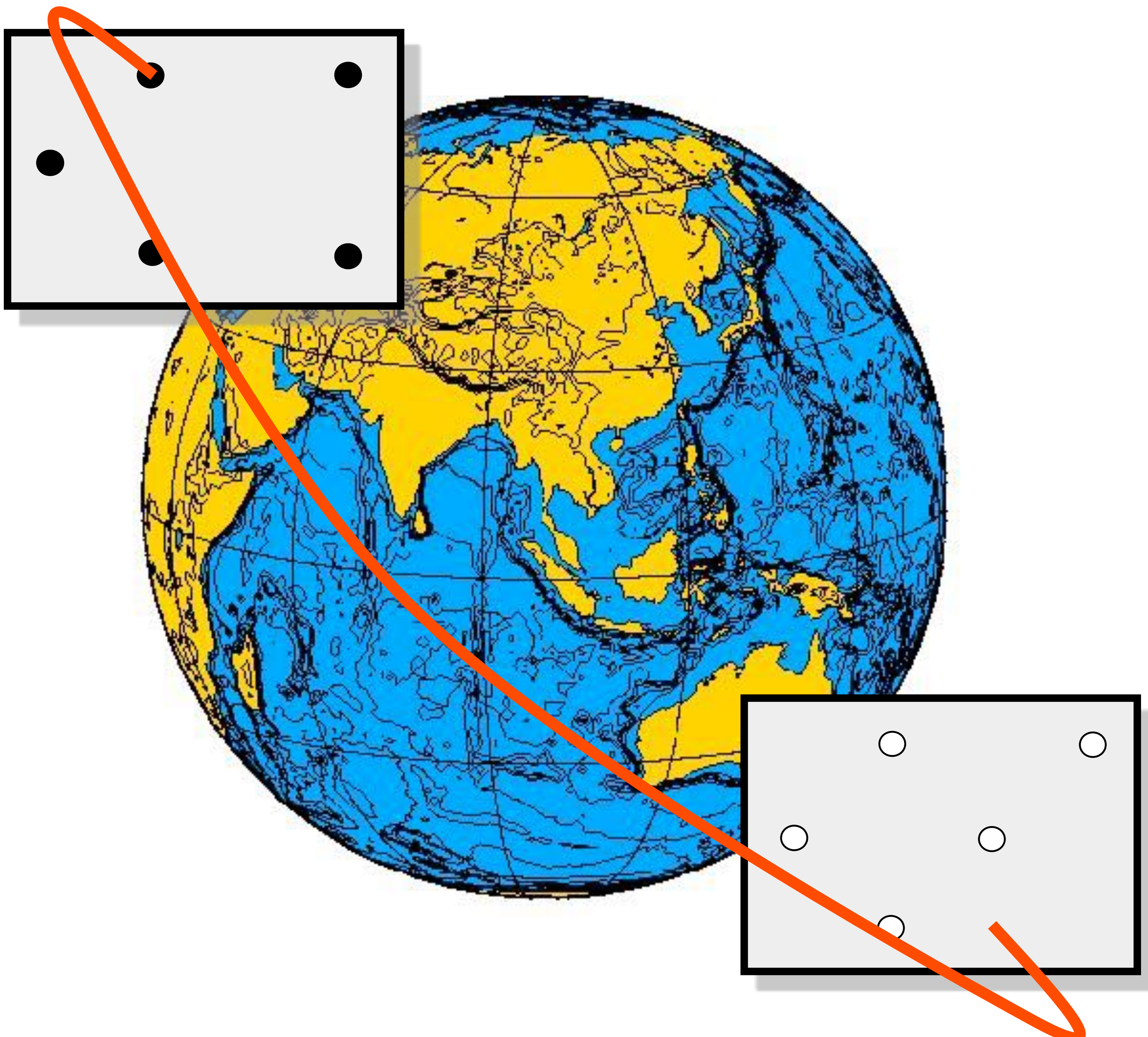

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- Keep black and white points in opposite hemispheres.

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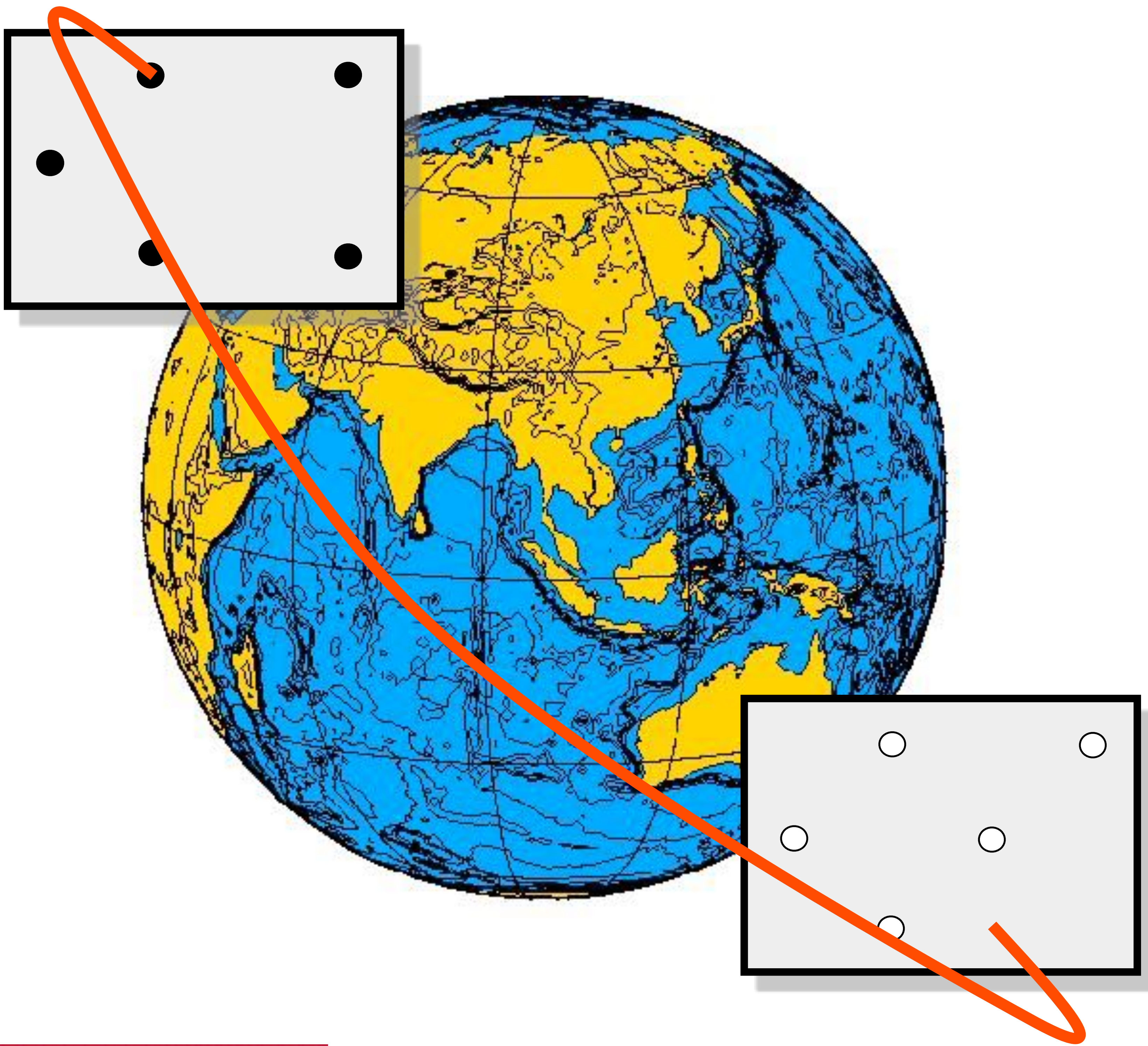

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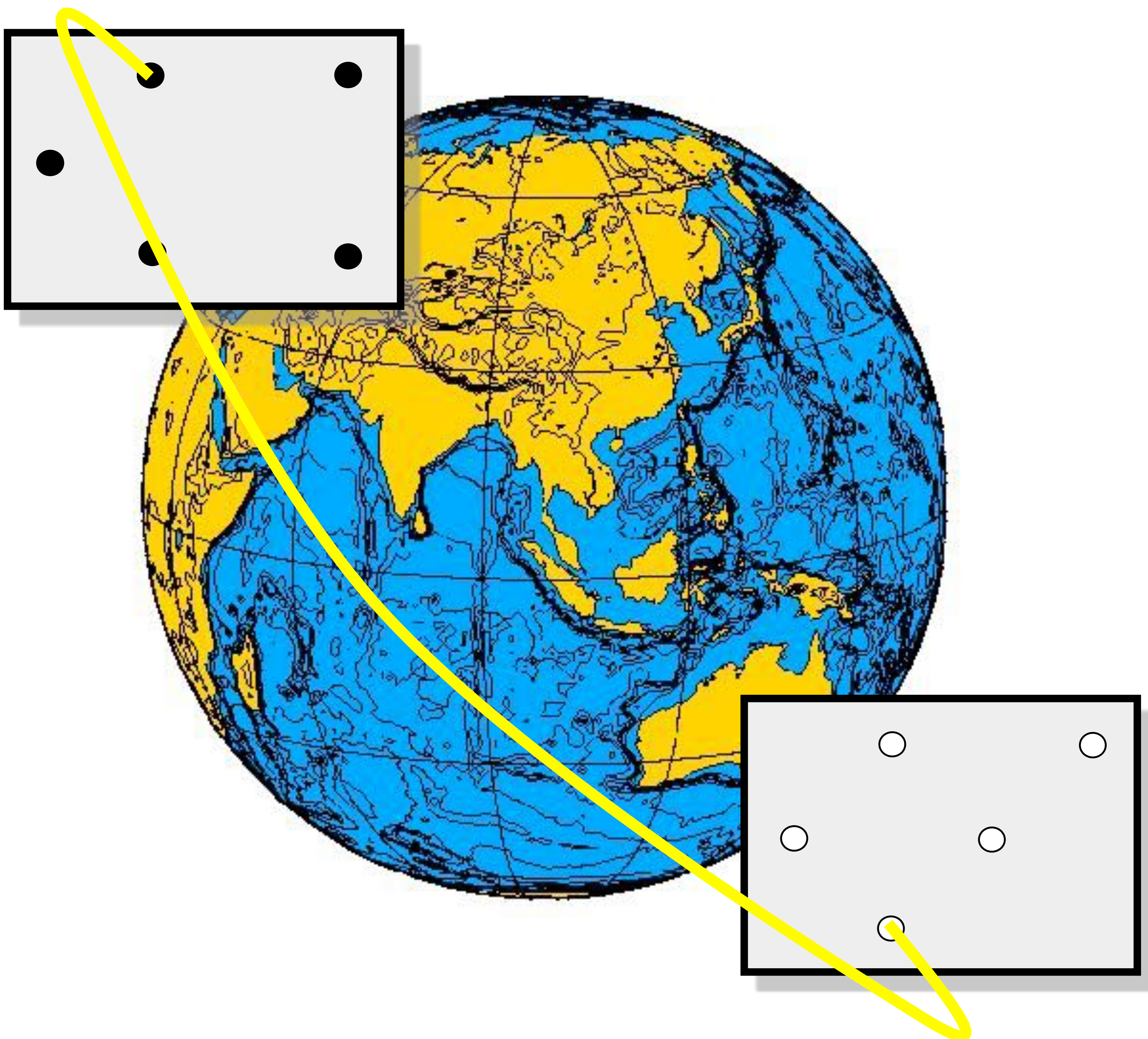

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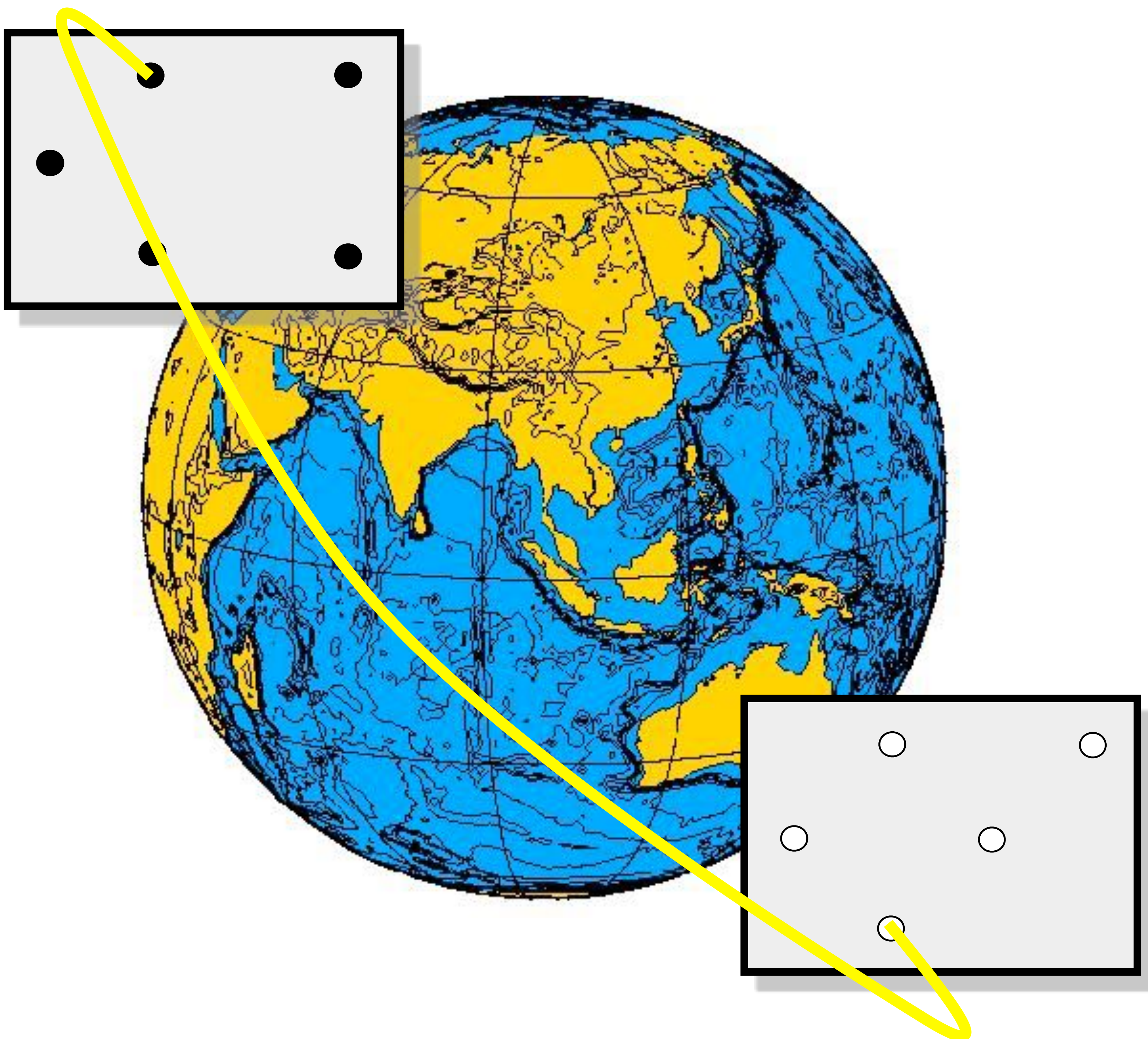
Think Globally

$$
\begin{aligned}
&L_2(S(v_i), S(v_j))]^2\\
&= [L_2((\cos(x_i\psi)\cos(y_i\psi), \sin(x_i\psi)\cos(y_i\psi), \sin(y_i\psi)),\\
&\qquad (\cos(\pi + x_j\psi)\cos(-y_j\psi), \sin(\pi + x_j\psi)\cos(-y_j\psi), \sin(-y_j\psi)))]^2\\
&= [\cos(x_i\psi)\cos(y_i\psi) + \cos(x_j\psi)\cos(y_j\psi)]^2\\
&\quad + [\sin(x_i\psi)\cos(y_i\psi) + \sin(x_j\psi)\cos(y_j\psi)]^2 + [\sin(y_i\psi) + \sin(y_j\psi)]^2\\
&= \left[\left(1 - \frac{(x_i\psi)^2}{2} + O((x_i\psi)^4)\right)\left(1 - \frac{(y_i\psi)^2}{2} + O((y_i\psi)^4)\right)\right.\\
&\quad \left. + \left(1 - \frac{(x_j\psi)^2}{2} + O((x_j\psi)^4)\right)\left(1 - \frac{(y_j\psi)^2}{2} + O((y_j\psi)^4)\right)\right]^2\\
&\quad + \left[(x_i\psi - O((x_i\psi)^3))\left(1 - \frac{(y_i\psi)^2}{2} + O((y_i\psi)^4)\right)\right.\\
&\quad \left. + (x_j\psi - O((x_j\psi)^3))\left(1 - \frac{(y_j\psi)^2}{2} + O((y_j\psi)^4)\right)\right]^2\\
&\quad + [y_i\psi - O((y_i\psi)^3) + y_j\psi - O((y_j\psi)^3)]^2\\
&= \left[2 - \frac{(x_i\psi)^2}{2} - \frac{(y_i\psi)^2}{2} - \frac{(x_j\psi)^2}{2} - \frac{(y_j\psi)^2}{2} + O(n^{-8})\right]^2\\
&\quad + [x_i\psi + x_j\psi + O(n^{-6})]^2 + [y_i\psi + y_j\psi + O(n^{-6})]^2\\
&= [4 - 2(x_i\psi)^2 - 2(y_i\psi)^2 - 2(x_j\psi)^2 - 2(y_j\psi)^2 + O(n^{-8})]\\
&\quad + [(x_i\psi)^2 + (x_j\psi)^2 + 2x_ix_j\psi^2 + O(n^{-8})]\\
&\quad + [(y_i\psi)^2 + (y_j\psi)^2 + 2y_iy_j\psi^2 + O(n^{-8})]\\
&= 4 - (x_i - x_j)^2\psi^2 - (y_i - y_j)^2\psi^2 + O(n^{-8}).
\end{aligned}
$$

- Consider a grid graph embedded on a sphere.
- Add second copy at the antipodes.
- Longest connections are between antipodal points.
- Grid graphs are bipartite.
- Keep black and white points in opposite hemispheres.
- Long connections correspond to short ones in the original grid graph.

Think Globally



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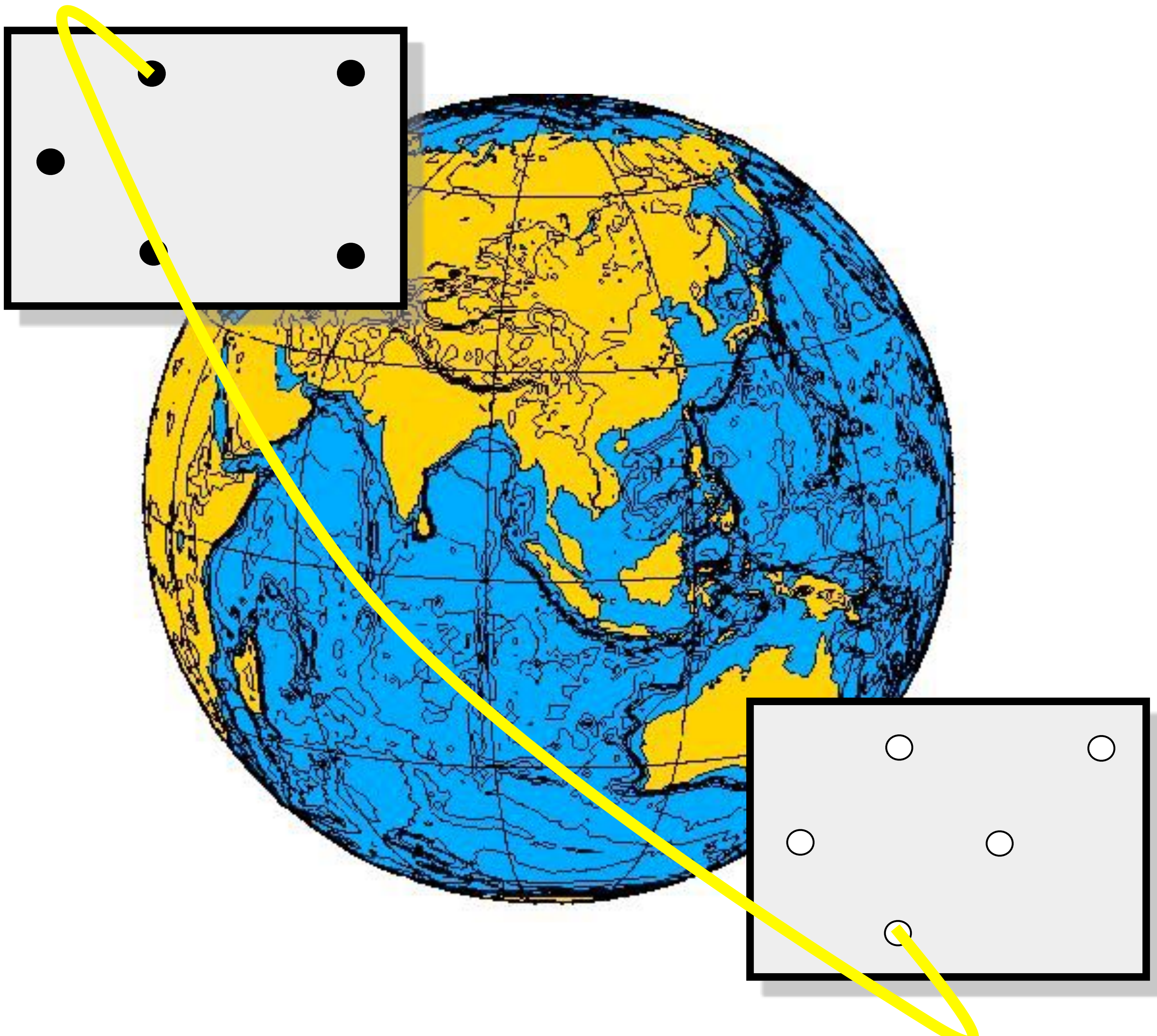

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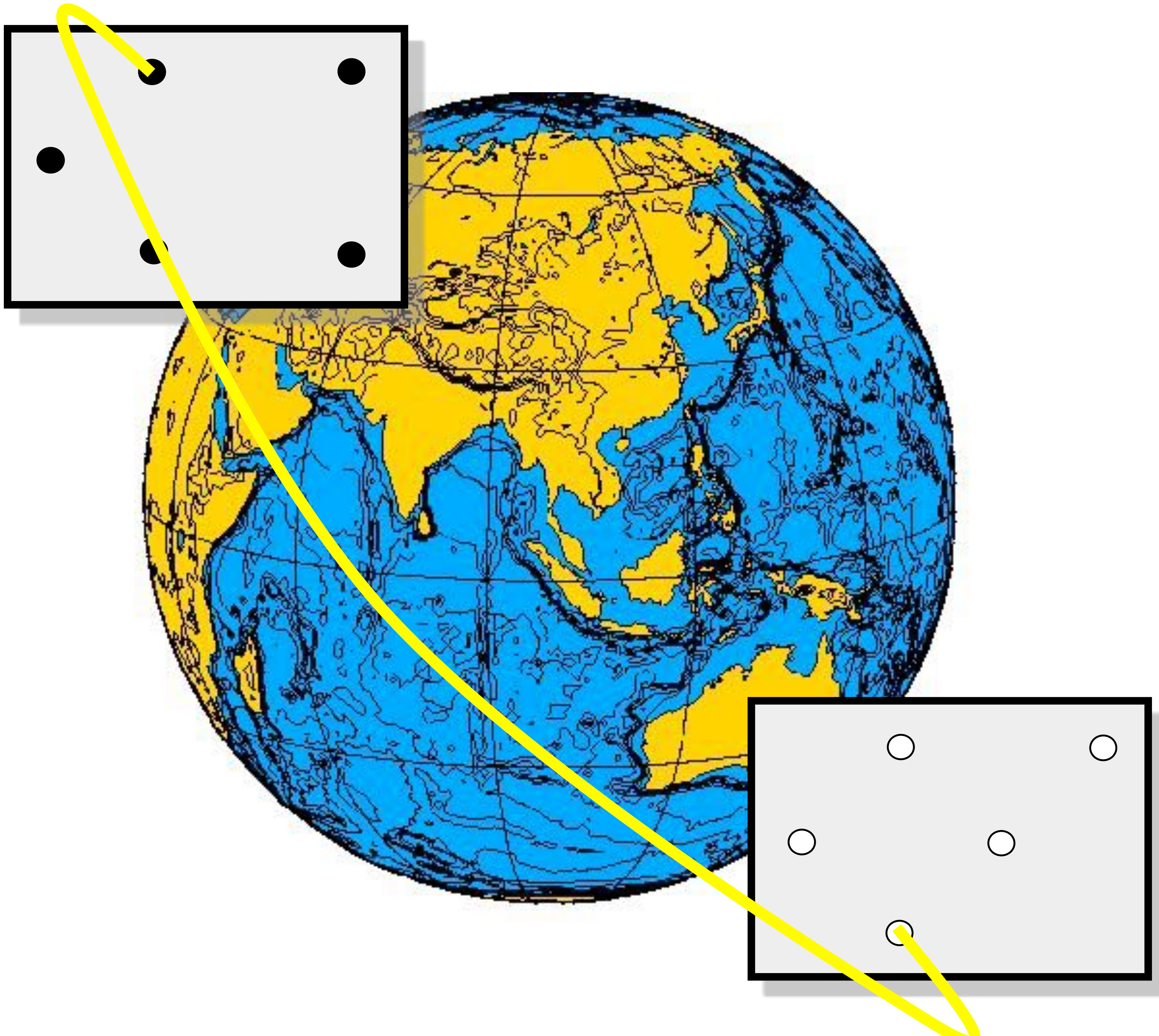

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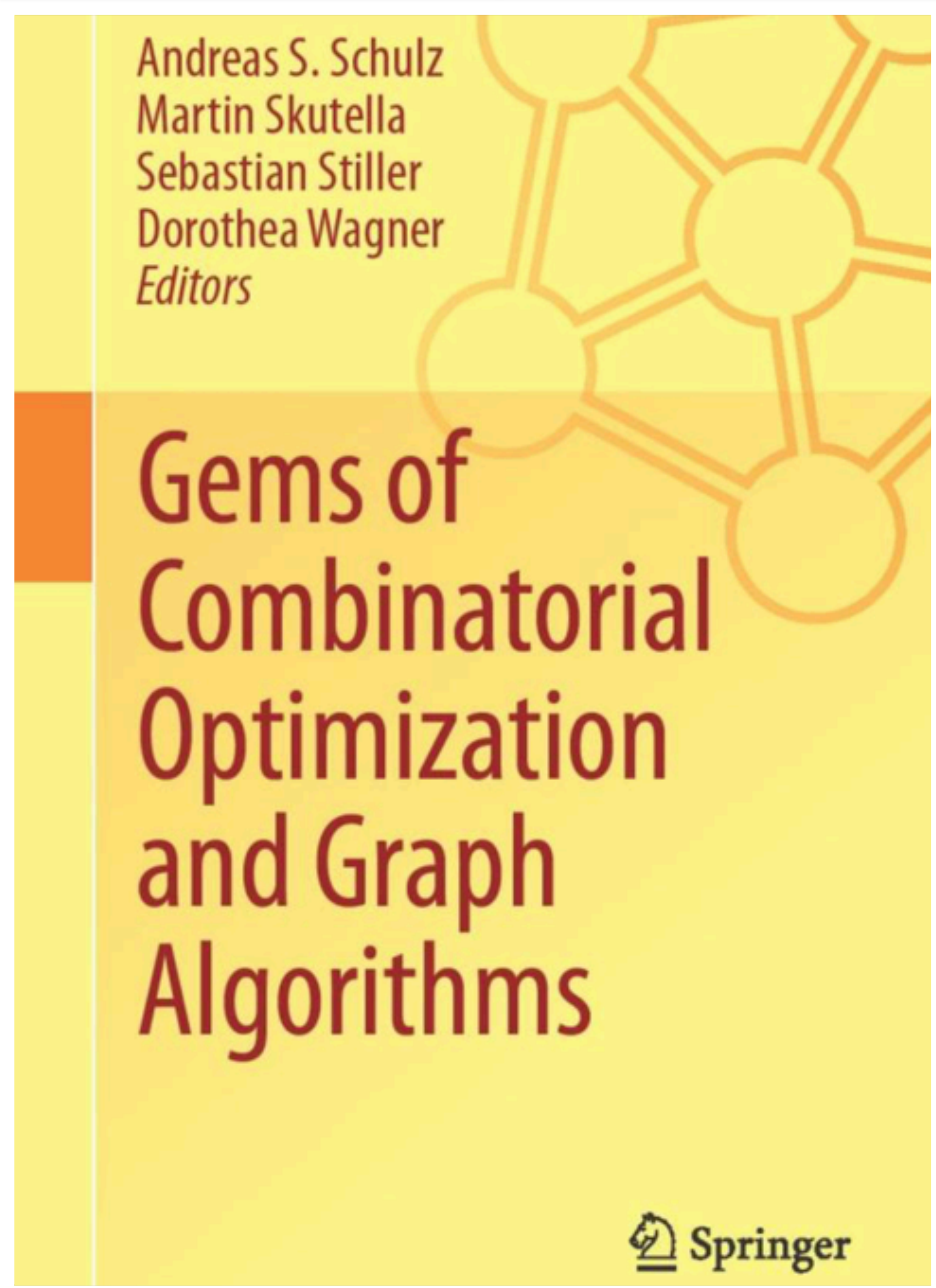
Finding a longest tour for Euclidean distances in 3D is an NP-hard problem.



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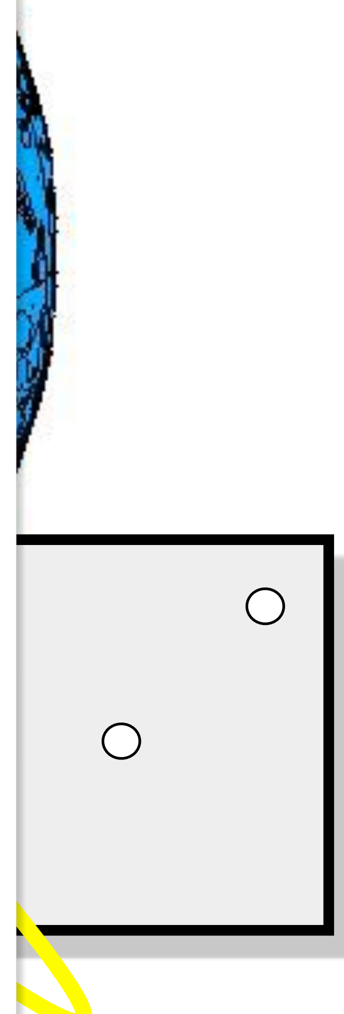

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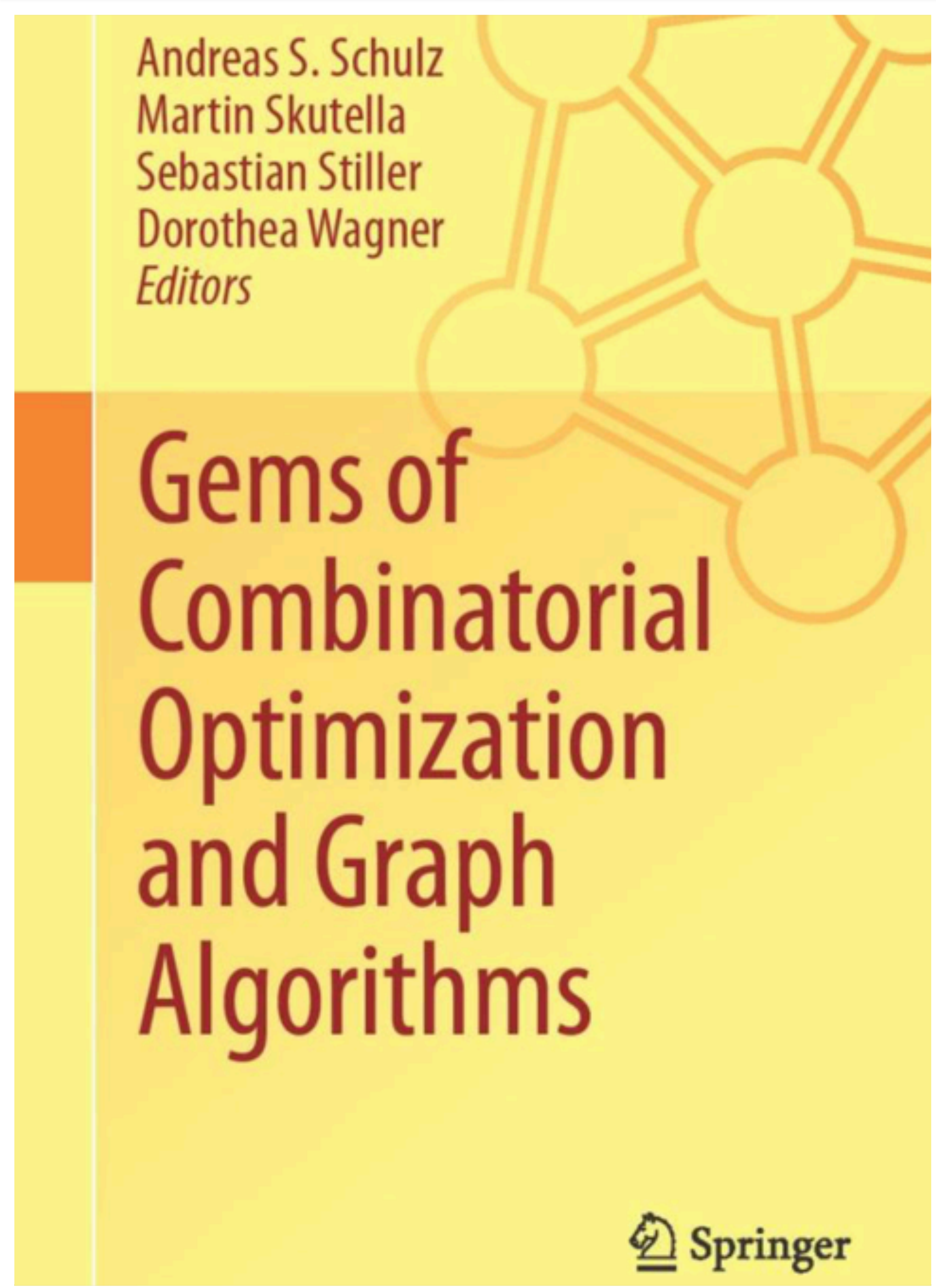

Finding Longest Geometric Tours

Sándor P. Fekete

Abstract We discuss the problem of finding a longest tour for a set of points in a geometric space. In particular, We show that a longest tour for a set of n points in the plane can be computed in time $O(n)$ if distances are determined by the Manhattan metric, while the same problem is NP-hard for points on a sphere under Euclidean distances.

1 Introduction: Short and Long Roundtrips

The Traveling Salesman Problem (TSP) is one of the classic problems of combinatorial optimization. Given a complete graph $G = (V, E)$ with edge weights $c(e)$ for all edges $e \in E$, find a shortest roundtrip through all vertices, i.e., a cyclic permutation π from the symmetric group S_n of all n vertices $v_1, \ldots, v_n$, such that the total tour length $\sum_{i=1}^{n} c(\{v_i, v_{\pi(i)}\})$ is minimized.

The difficulties of finding a good roundtrip are well known. The classical Odyssey is illustrated in Figure 1: according to legend, it took Ulysses many years to complete his voyage. One justification is the computational complexity of the TSP: it is one of the most famous NP-hard problems, so it does indeed take many years of CPU time to find provably optimal solutions for non-trivial instances.

However, there is an even more convincing justification for Ulysses' failure to be home in a more timely fashion: it was not him who chose his route. Instead, malevolent gods caused a deliberately long voyage—so the real objective was to *maximize* the traveled distance. This motivates the MaxTSP: Find a roundtrip that visits all vertices in a weighted graph, such that the total tour length is maximized.

In this chapter, we study longest tours in a geometric setting, in which the vertices are points in two- or three-dimensional space, and the edge weights are induced

Sándor P. Fekete
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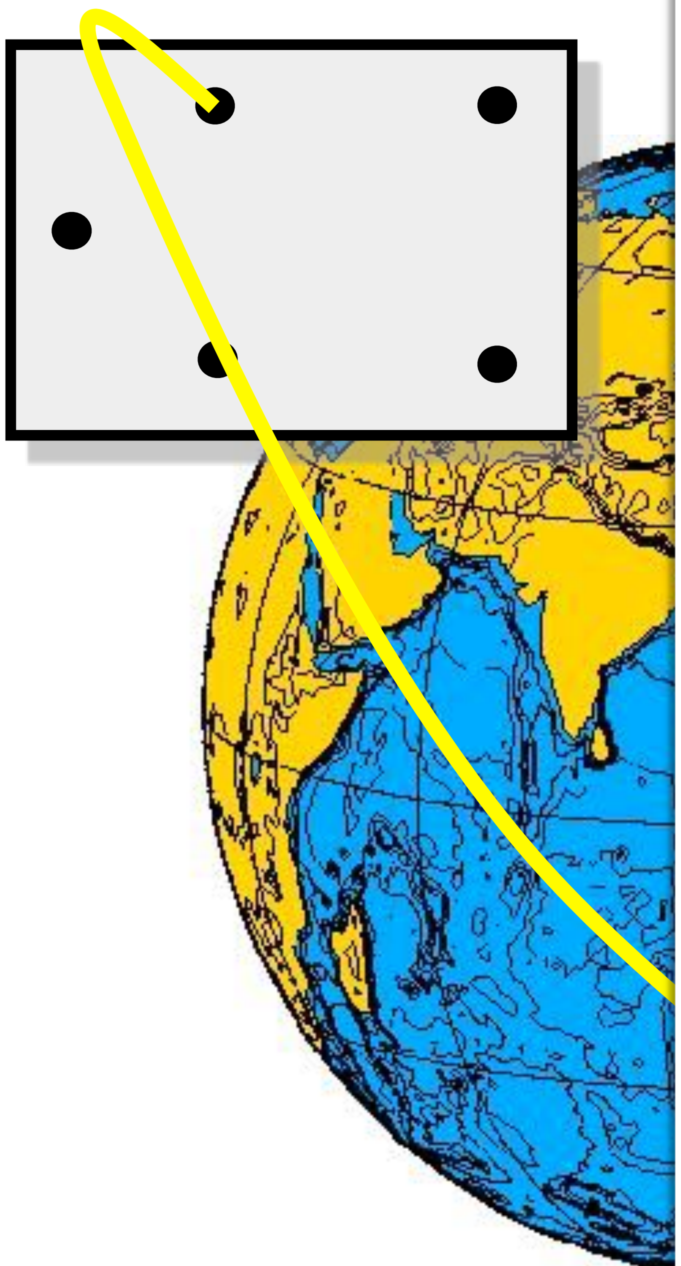

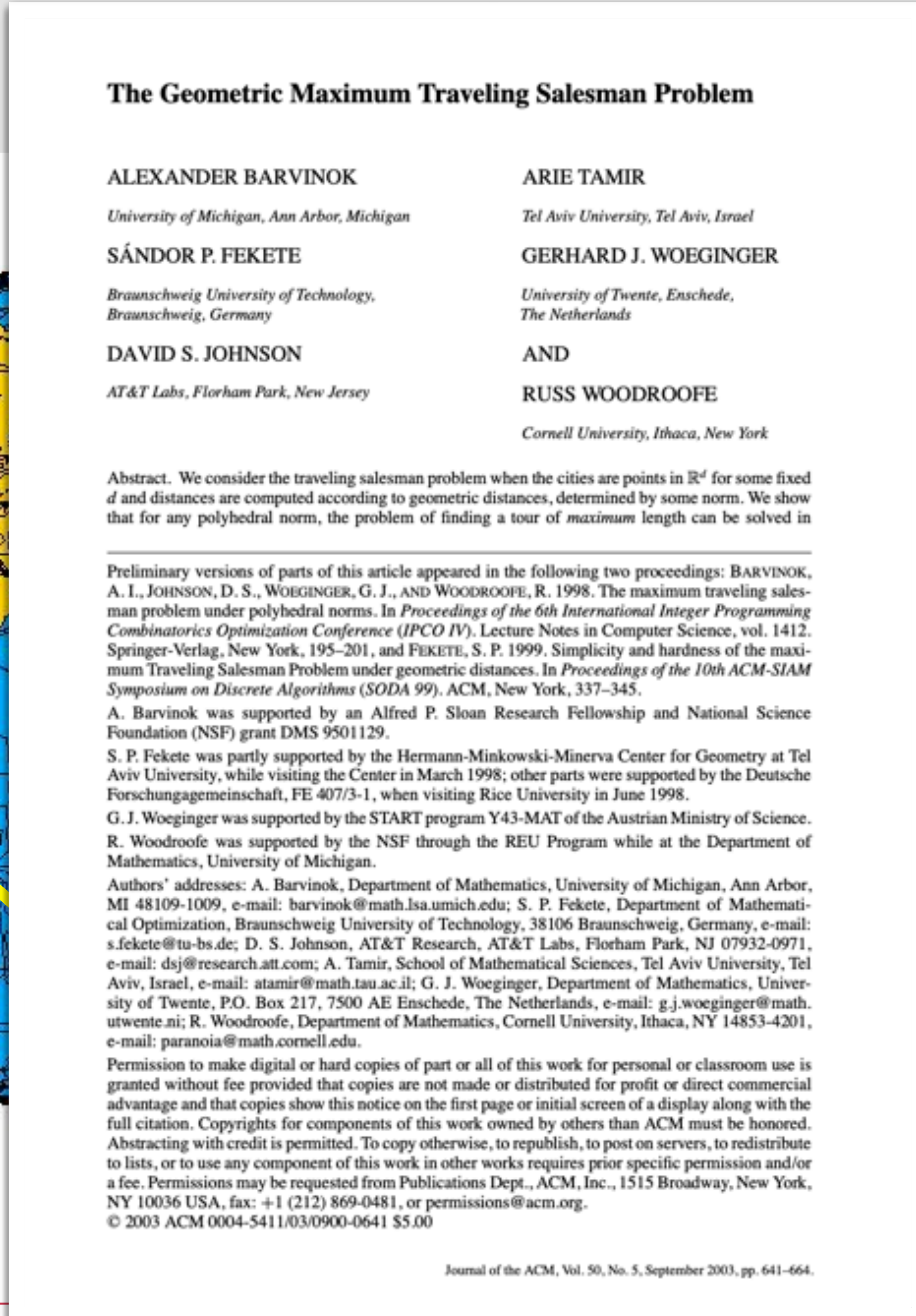

The Geometric Maximum Traveling Salesman Problem

ALEXANDER BARVINOK

University of Michigan, Ann Arbor, Michigan

SÁNDOR P. FEKETE

Braunschweig University of Technology, Braunschweig, Germany

DAVID S. JOHNSON

AT&T Labs, Florham Park, New Jersey

ARIE TAMIR

Tel Aviv University, Tel Aviv, Israel

GERHARD J. WOEGINGER

University of Twente, Enschede, The Netherlands

AND

RUSS WOODROOFE

Cornell University, Ithaca, New York

Abstract. We consider the traveling salesman problem when the cities are points in $\mathbb{R}^d$ for some fixed d and distances are computed according to geometric distances, determined by some norm. We show that for any polyhedral norm, the problem of finding a tour of *maximum* length can be solved in

Preliminary versions of parts of this article appeared in the following two proceedings: BARVINOK, A. I., JOHNSON, D. S., WOEGINGER, G. J., AND WOODROOFE, R. 1998. The maximum traveling salesman problem under polyhedral norms. In *Proceedings of the 6th International Integer Programming Combinatorics Optimization Conference* (*IPCO IV*). Lecture Notes in Computer Science, vol. 1412. Springer-Verlag, New York, 195–201, and FEKETE, S. P. 1999. Simplicity and hardness of the maximum Traveling Salesman Problem under geometric distances. In *Proceedings of the 10th ACM-SIAM Symposium on Discrete Algorithms* (*SODA 99*). ACM, New York, 337–345.

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S. P. Fekete was partly supported by the Hermann-Minkowski-Minerva Center for Geometry at Tel Aviv University, while visiting the Center in March 1998; other parts were supported by the Deutsche Forschungagemeinschaft, FE 407/3-1, when visiting Rice University in June 1998.

G. J. Woeginger was supported by the START program Y43-MAT of the Austrian Ministry of Science.

R. Woodroofe was supported by the NSF through the REU Program while at the Department of Mathematics, University of Michigan.

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Journal of the ACM, Vol. 50, No. 5, September 2003, pp. 641–664.

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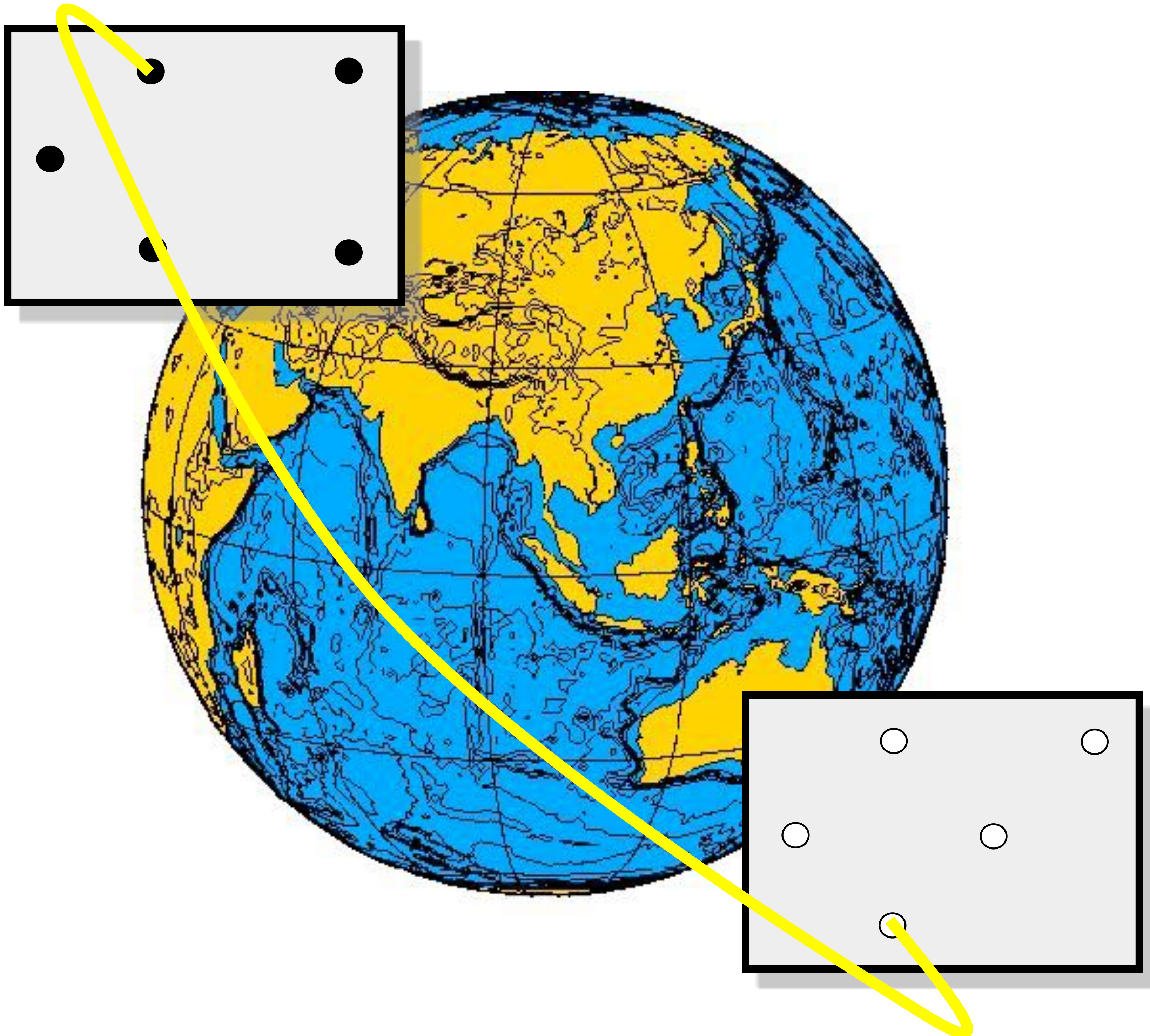

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Think Globally

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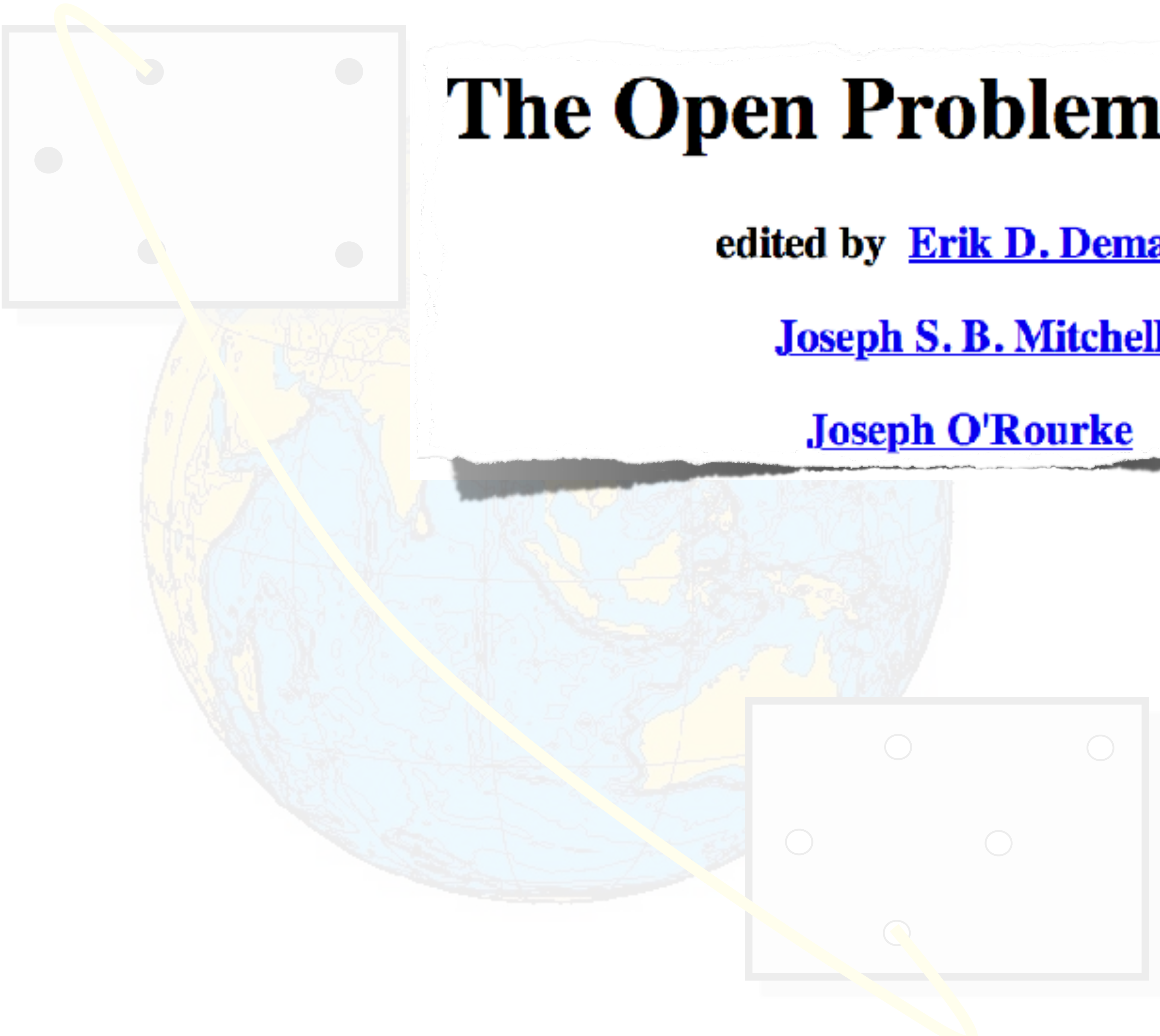

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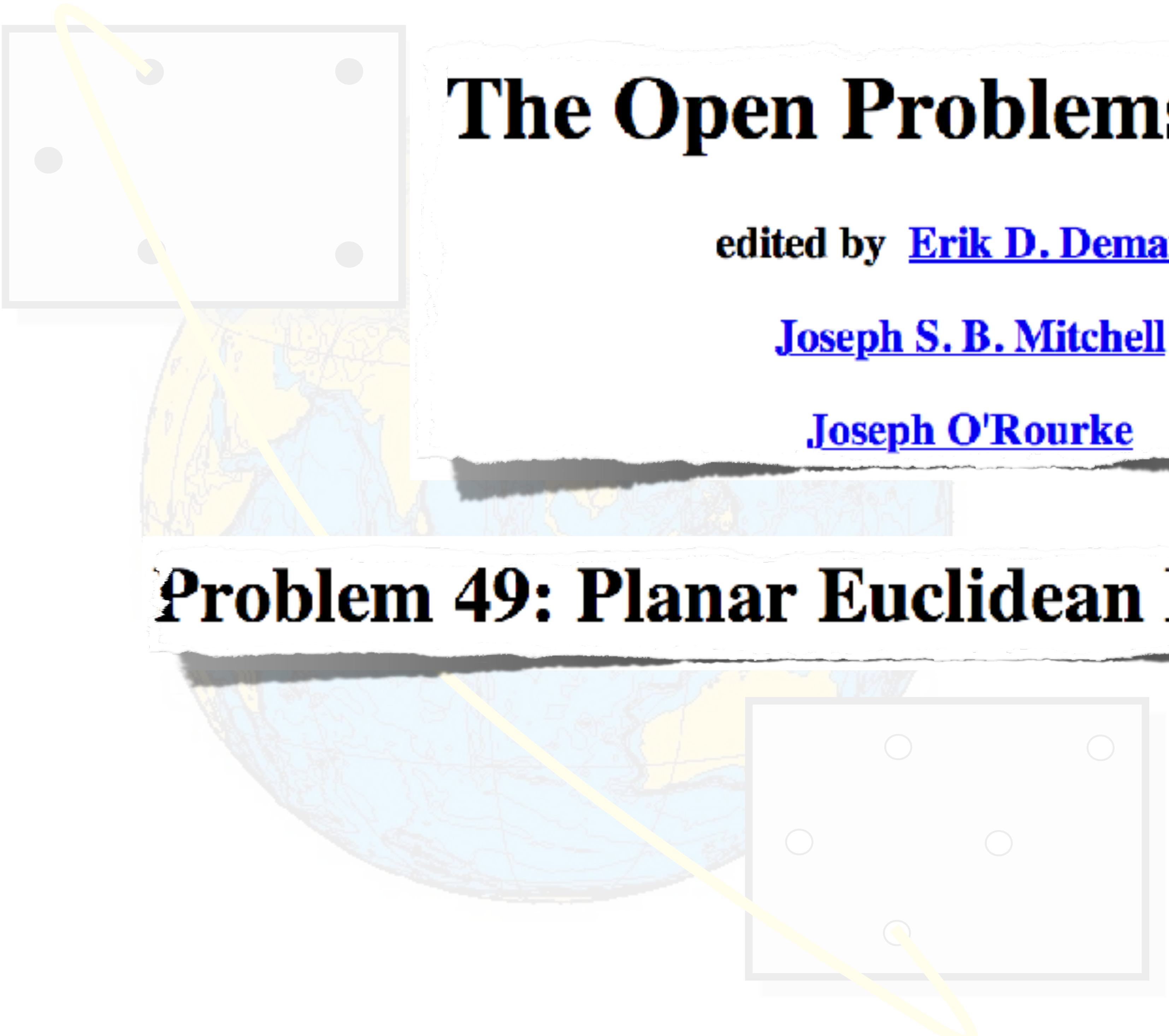

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Problem 49: Planar Euclidean Maximum TSP

Theorem:

Finding a longest tour for Euclidean distances in 3D is an NP-hard problem.

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Problem 49: Planar Euclidean Maximum TSP

CONJECTURE 4.1. *The Maximum TSP for Euclidean distances in the plane is an NP-hard problem.*

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1. Introduction
2. Longest Tours
3. **Stars and Matchings**
4. Nonsimple Polygons
5. Optimal Area
6. Turn Cost

3

3

Problem 7.1:

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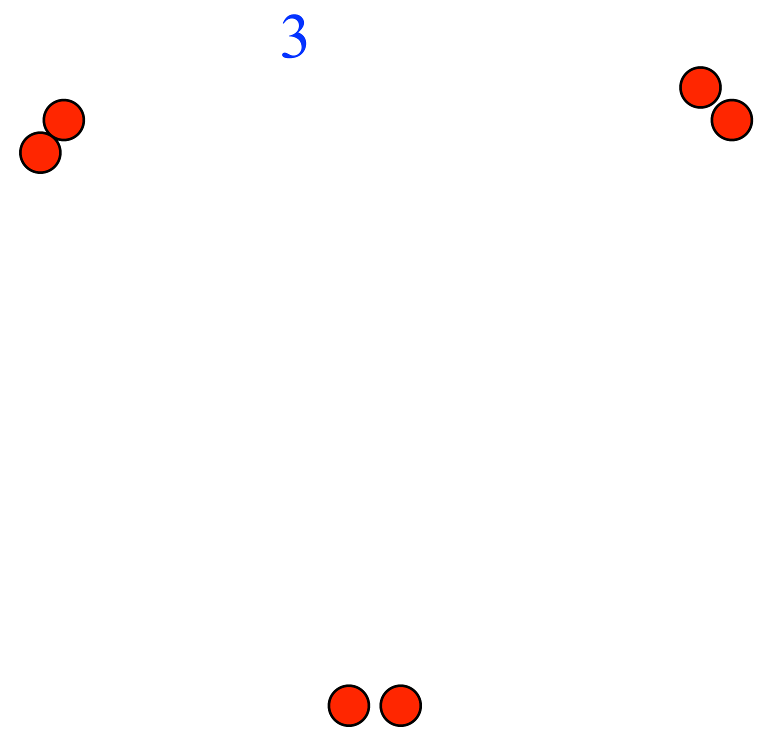

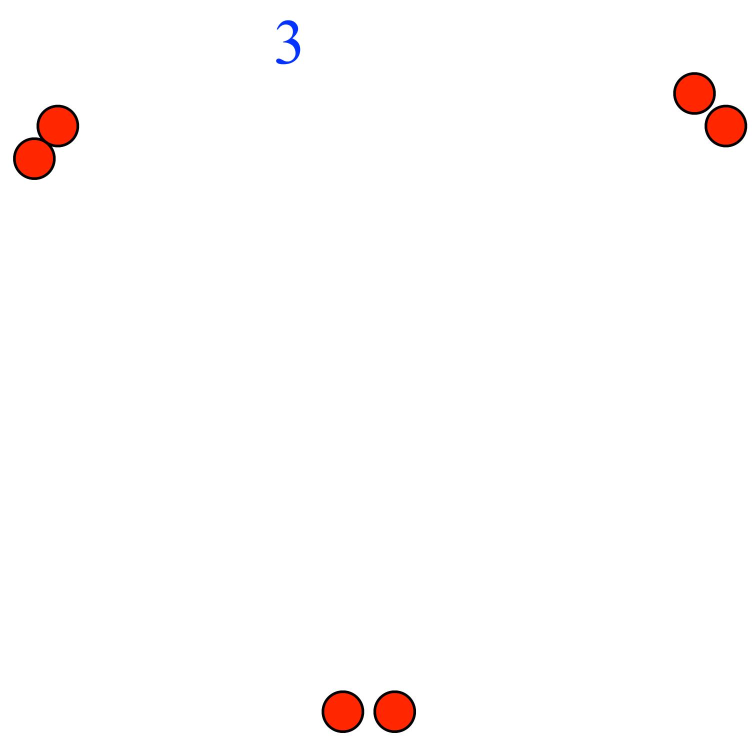

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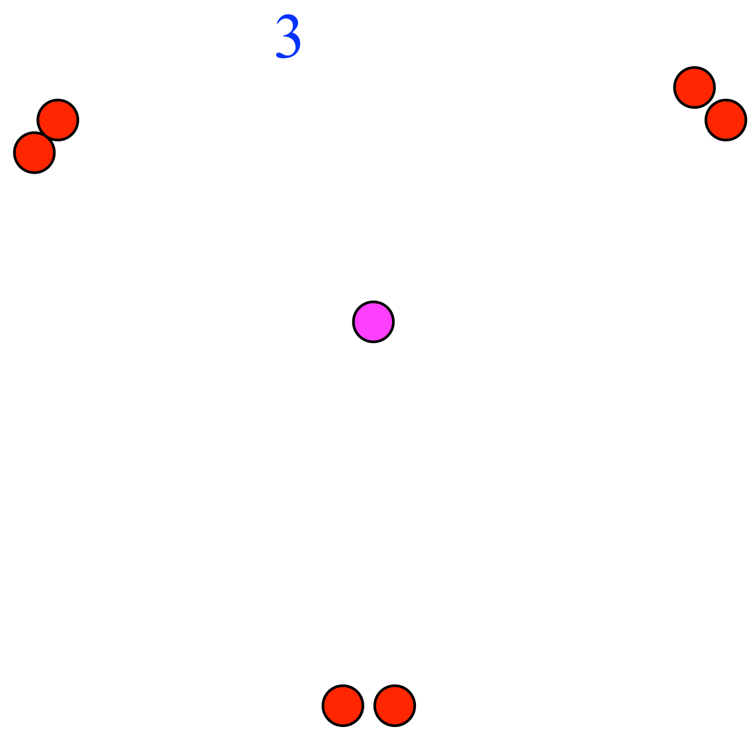

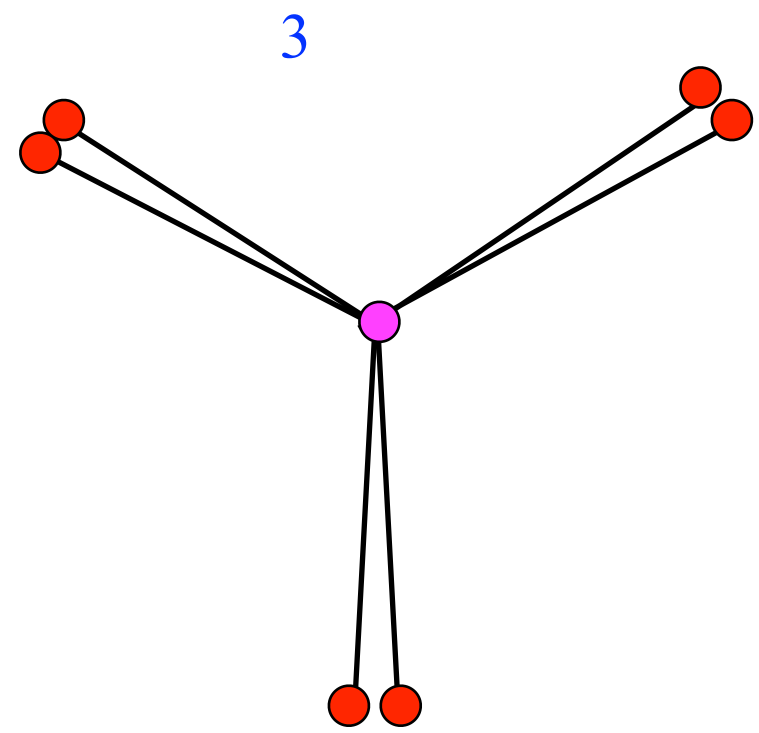

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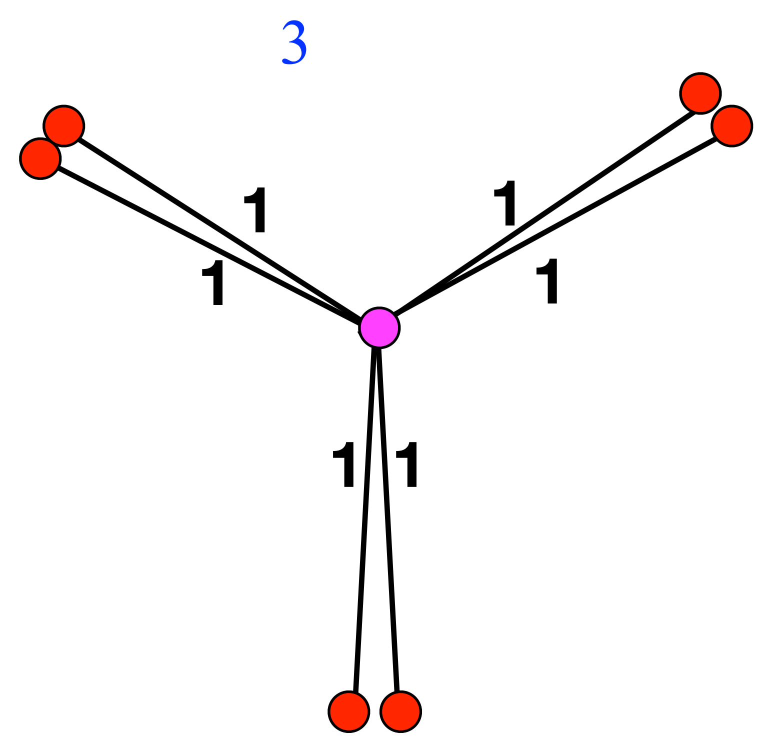

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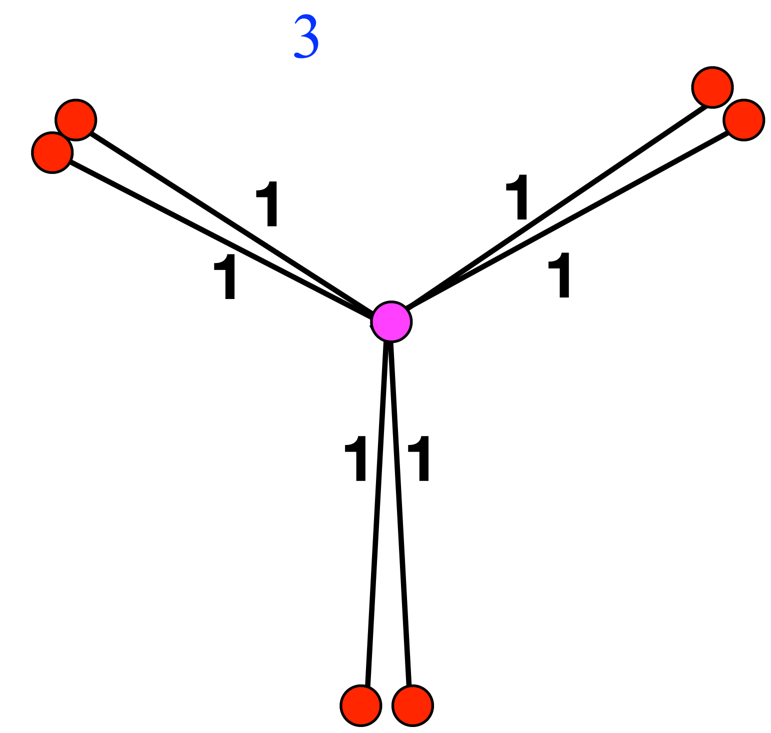

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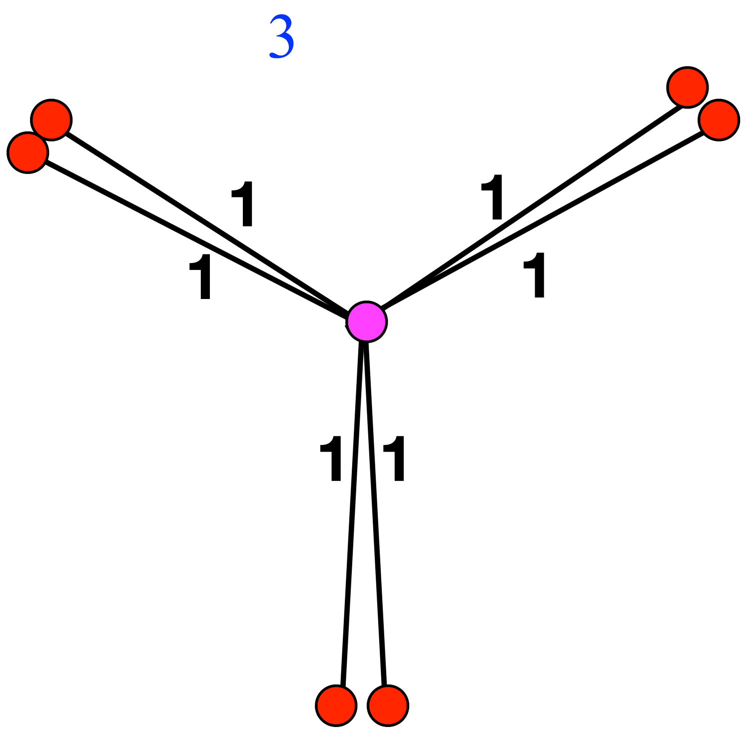

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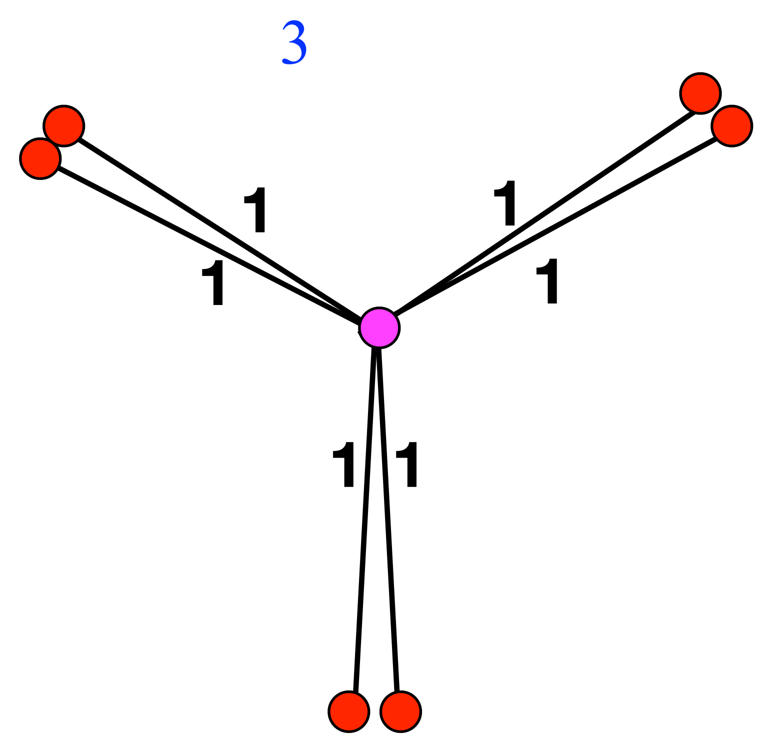

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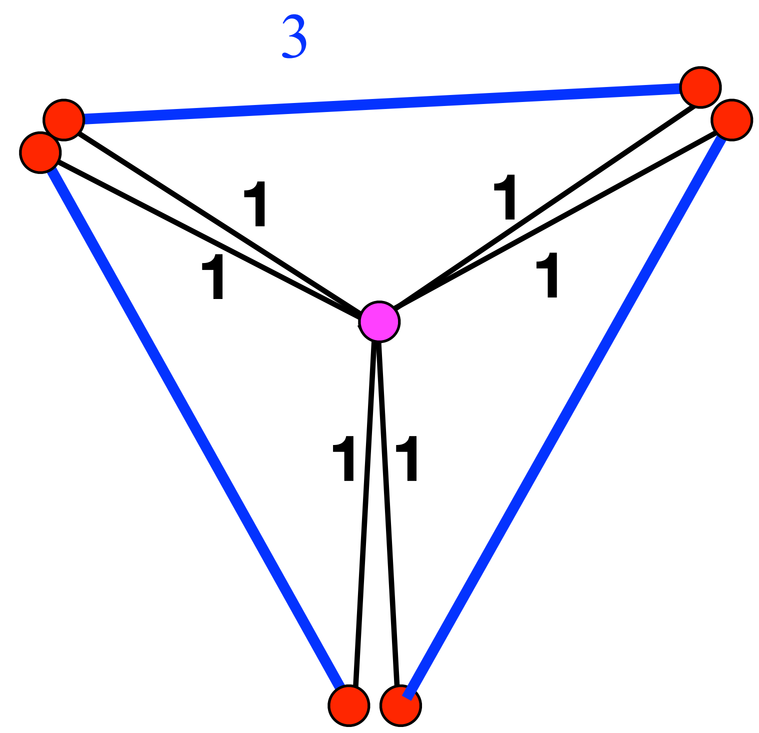

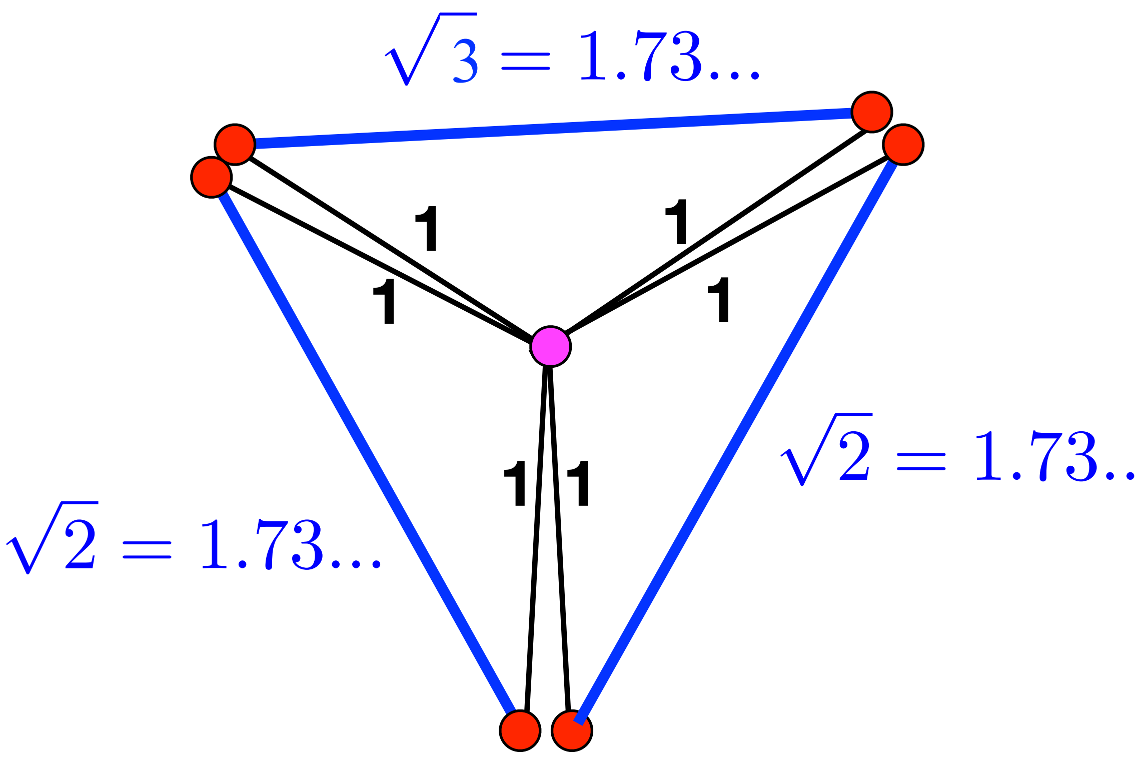

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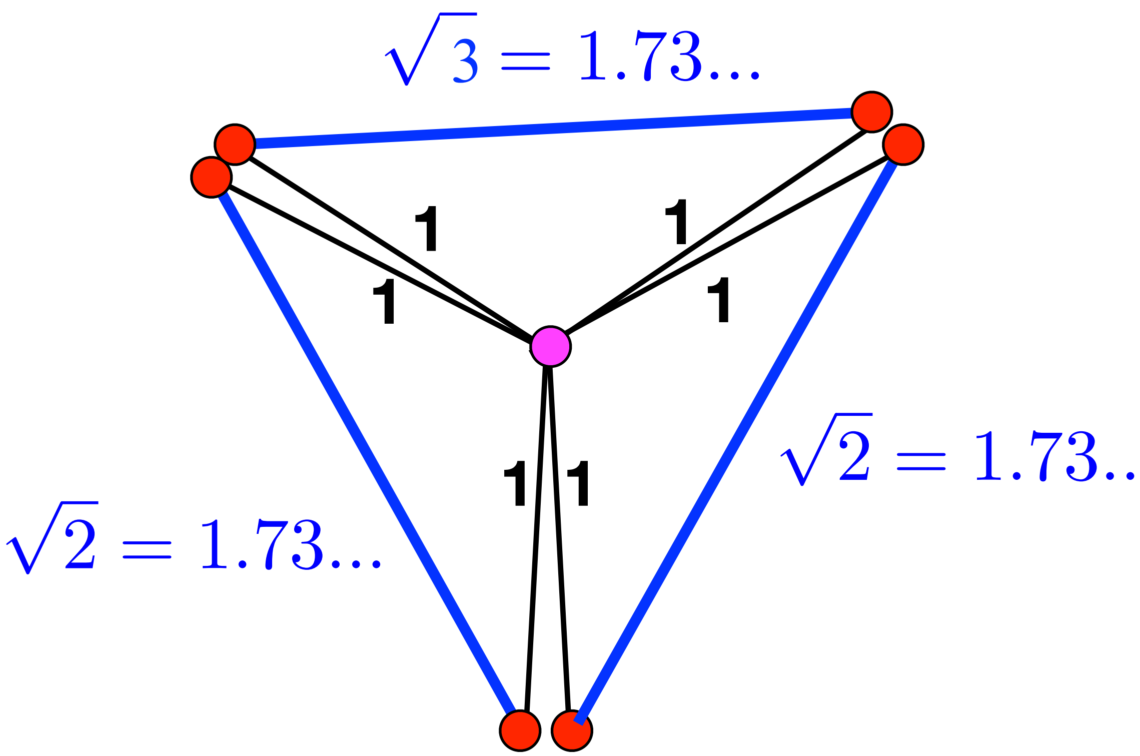

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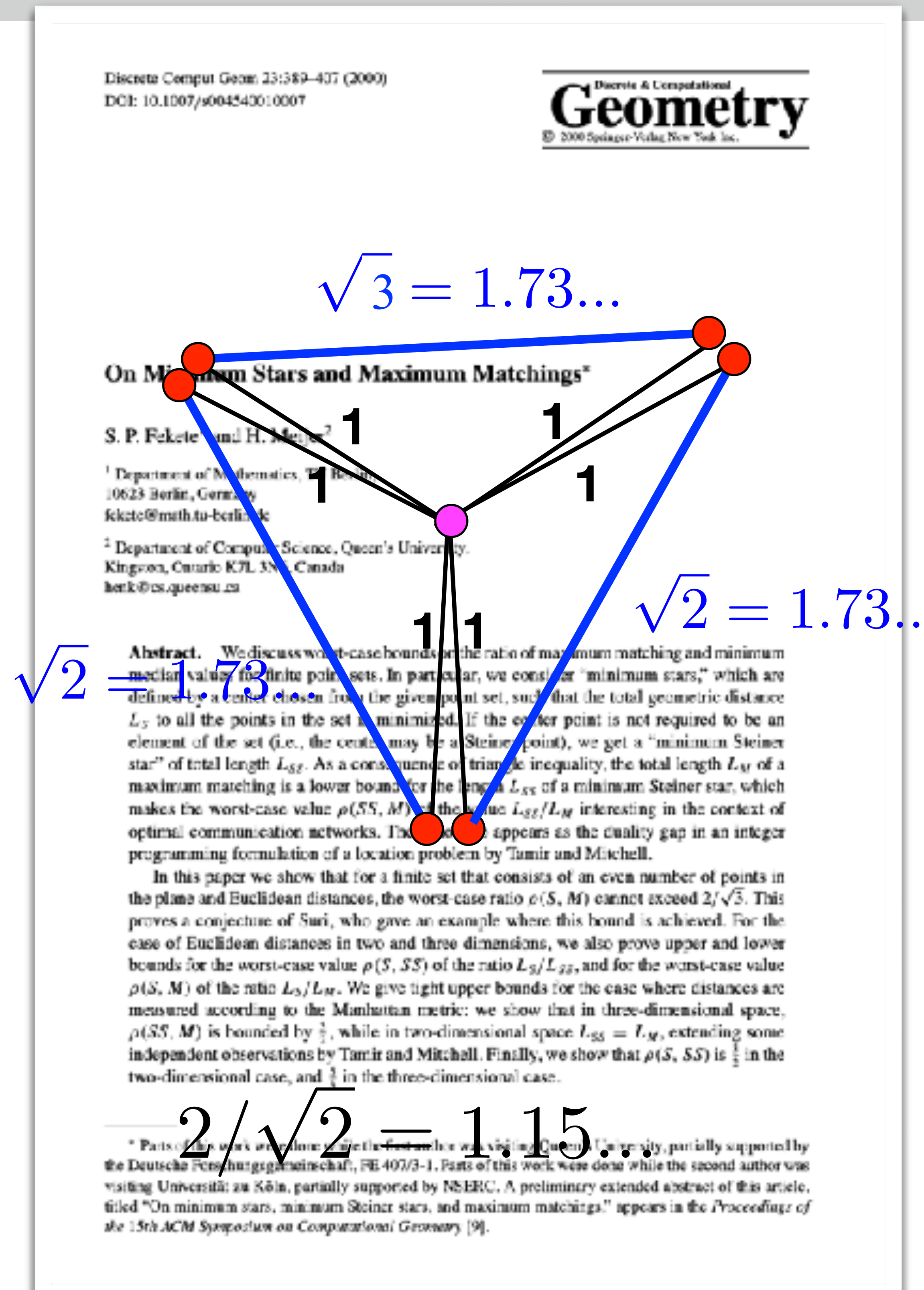

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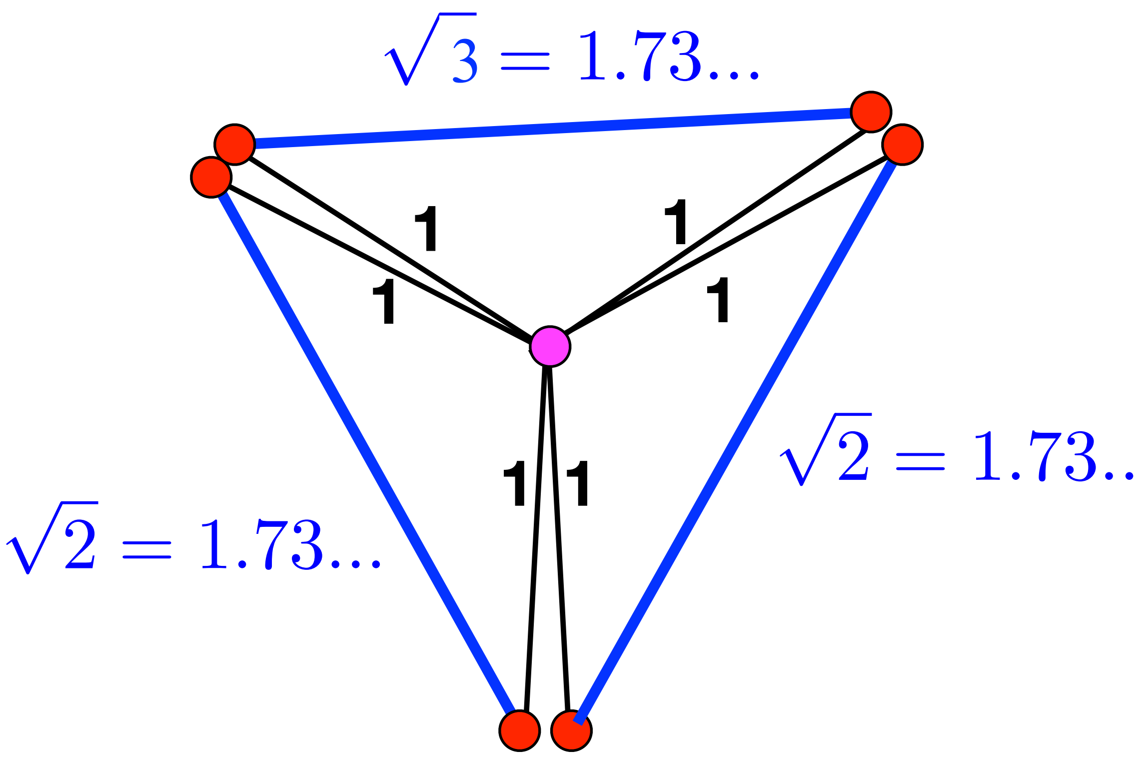

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Stars and Matchings [Fekete and Meijer 2000]

Proposition 1. *For point sets P of even cardinality in two-dimensional space with Manhattan distances, we have* $L_M = L_{SS}$.

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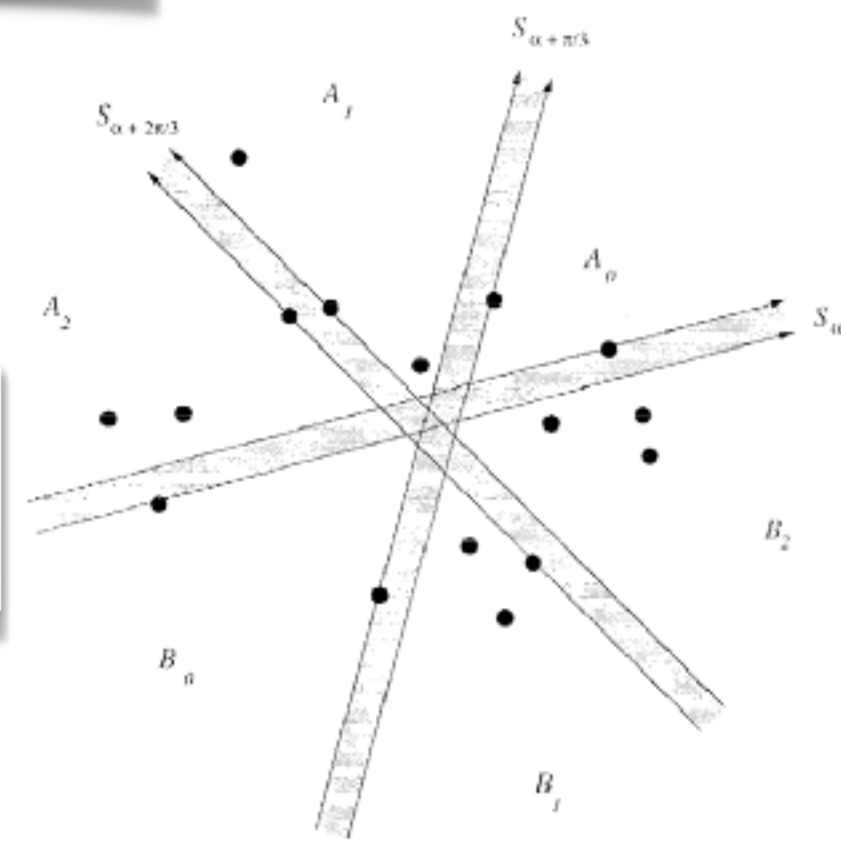

Fig. 1. Finding a small Steiner star and a large matching.

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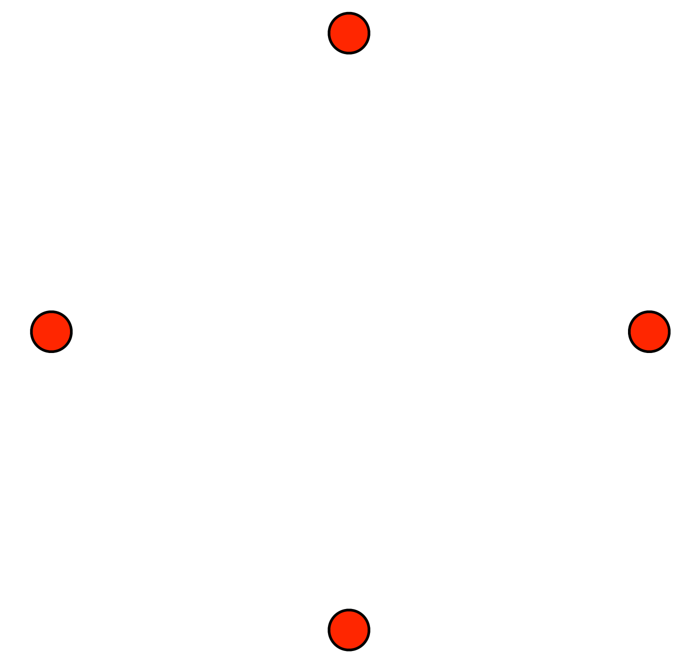

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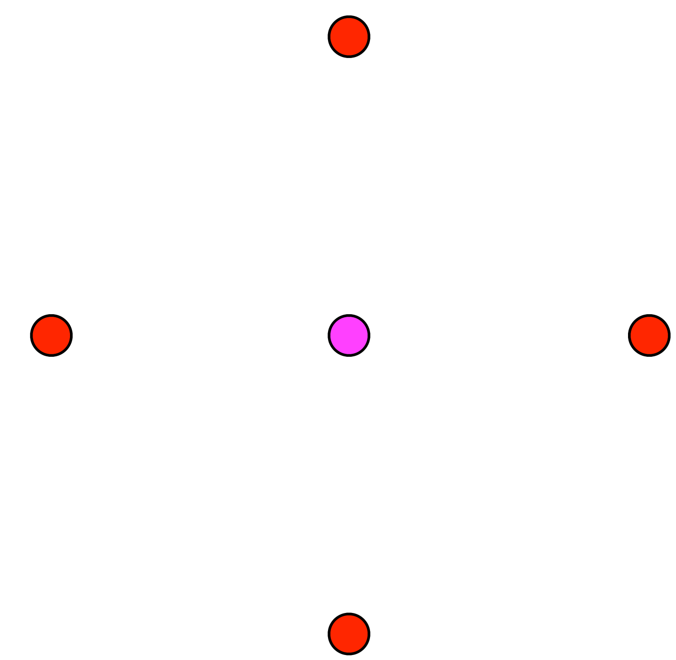

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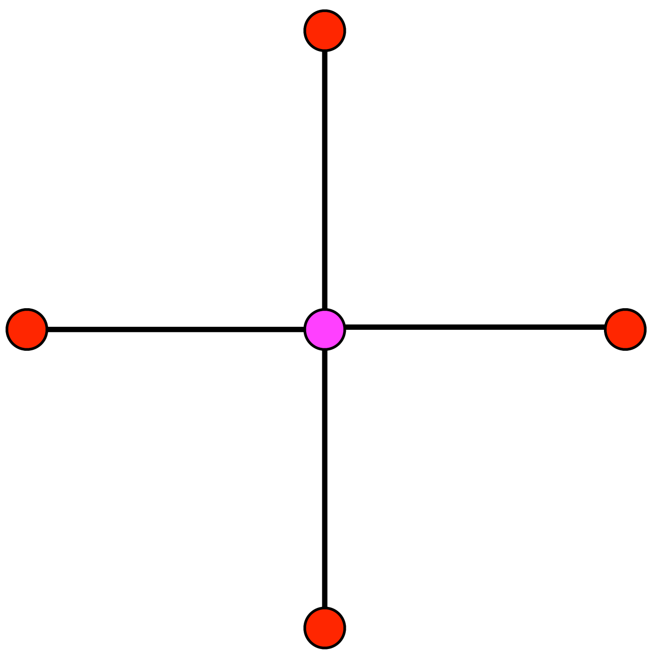

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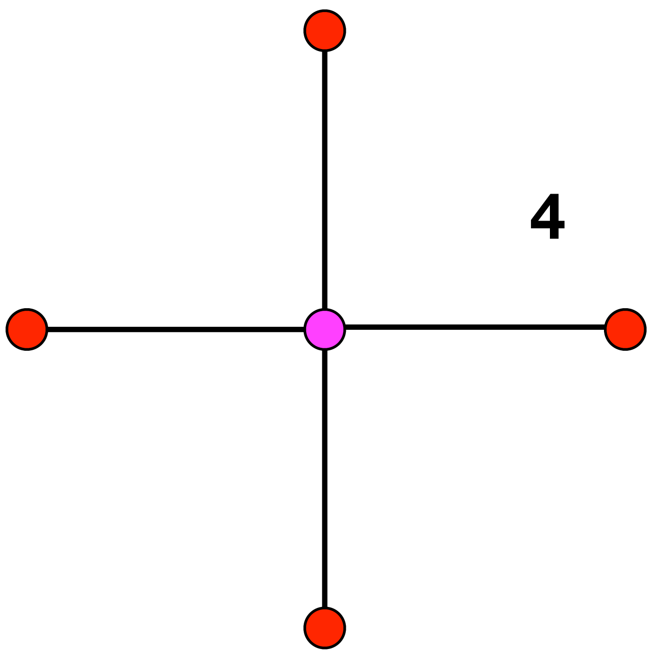

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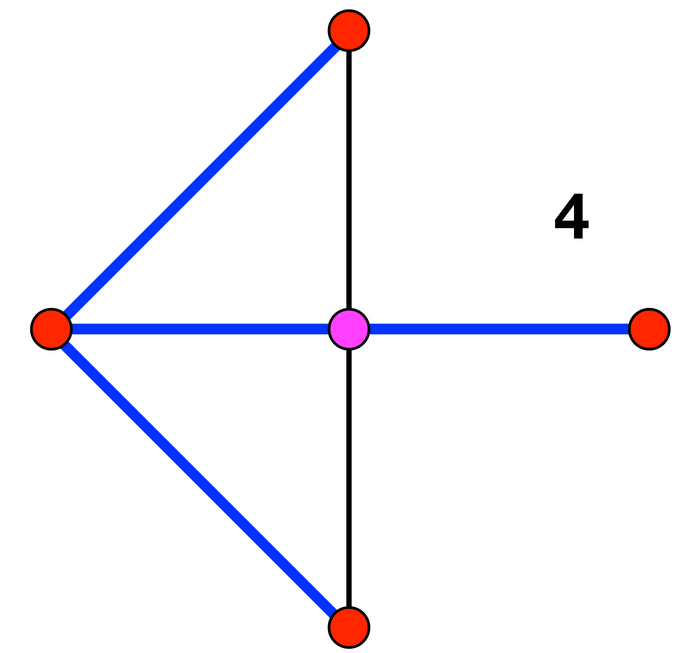

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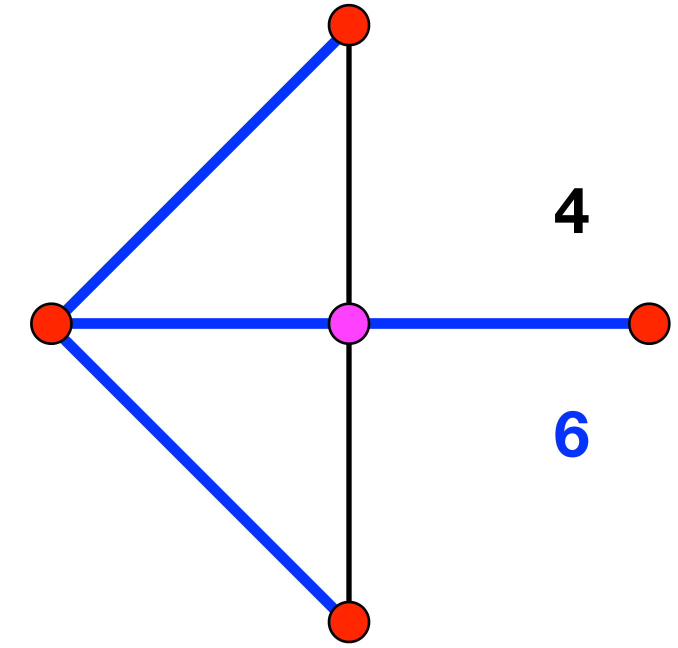

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Conjecture 1. *For point sets P in two-dimensional space with Euclidean distances, we have $\rho(S, SS) = 4/\pi$.*

Conjecture 2. *For point sets P in three-dimensional space with Euclidean distances, we have $\rho(S, SS) = \frac{4}{3}$.*

Table 1. Lower and upper bounds for worst-case ratios.

Distances	Dimension	Ratio	Lower bound	Upper bound
Euclidean	Two	$\rho(SS, M)$	$\frac{2}{\sqrt{3}} = 1.15\ldots$	$\frac{2}{\sqrt{3}} = 1.15\ldots$
		$\rho(S, SS)$	$\frac{4}{\pi} = 1.27\ldots$	$\sqrt{2} = 1.41\ldots$
		$\rho(S, M)$	$\frac{4}{3} = 1.33\ldots$	$\frac{2\sqrt{2}}{\sqrt{3}} = 1.63\ldots$
	Three	$\rho(SS, M)$	$\frac{\sqrt{3}}{\sqrt{2}} = 1.22\ldots$	$\sqrt{2} = 1.41\ldots$
		$\rho(S, SS)$	$\frac{4}{3} = 1.33\ldots$	$\sqrt{2} = 1.41\ldots$
		$\rho(S, M)$	$\frac{3}{2} = 1.5$	2
Manhattan	Two	$\rho(SS, M)$	1	1
		$\rho(S, SS)$	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$
		$\rho(S, M)$	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$
	Three	$\rho(SS, M)$	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$
		$\rho(S, SS)$	$\frac{5}{3} = 1.66\ldots$	$\frac{5}{3} = 1.66\ldots$
		$\rho(S, M)$	$\frac{5}{3} = 1.66\ldots$	2

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Solving a „hard“ problem to approximate an „easy“ one
[Fekete, Meijer, Rohe, Tietze 2002]

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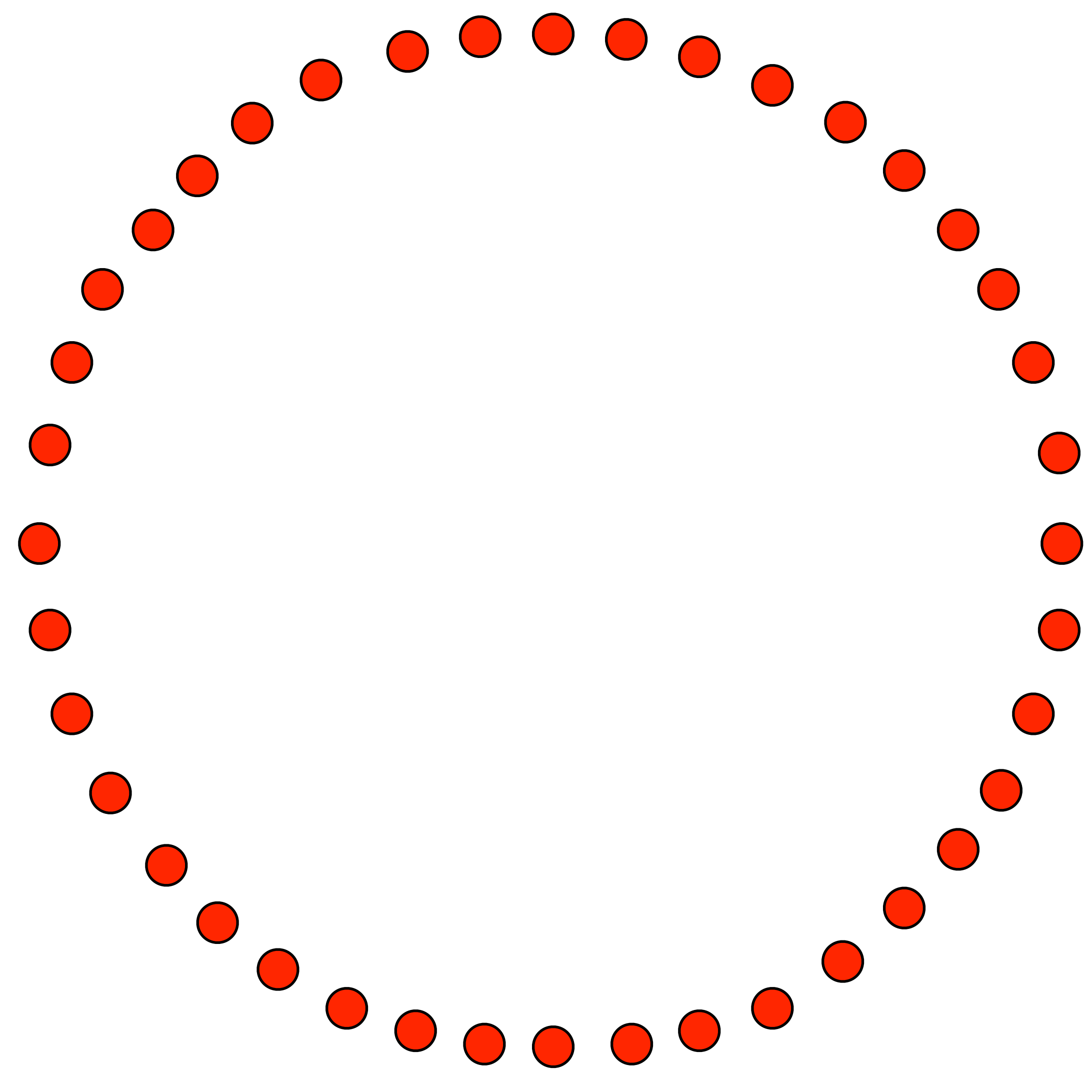

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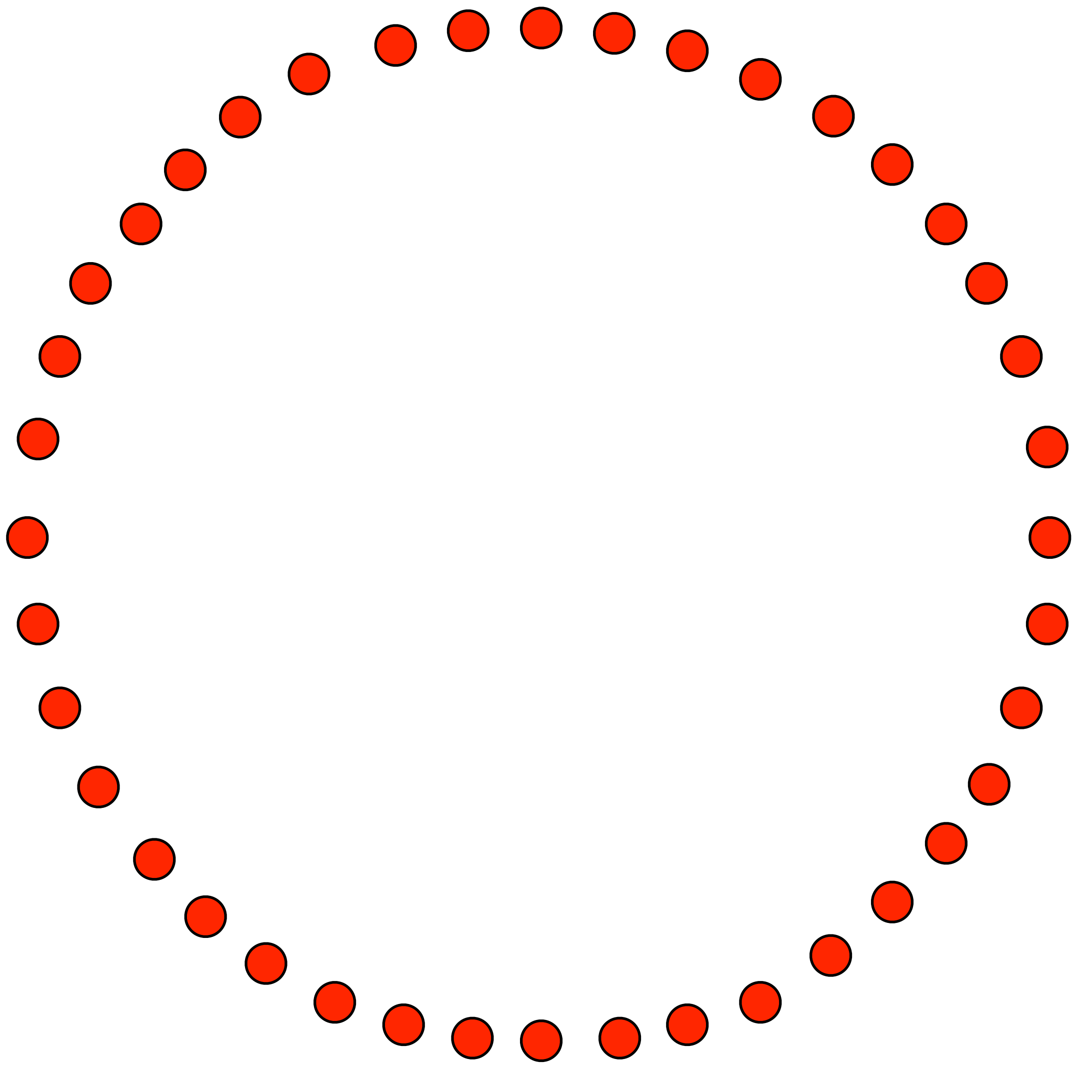

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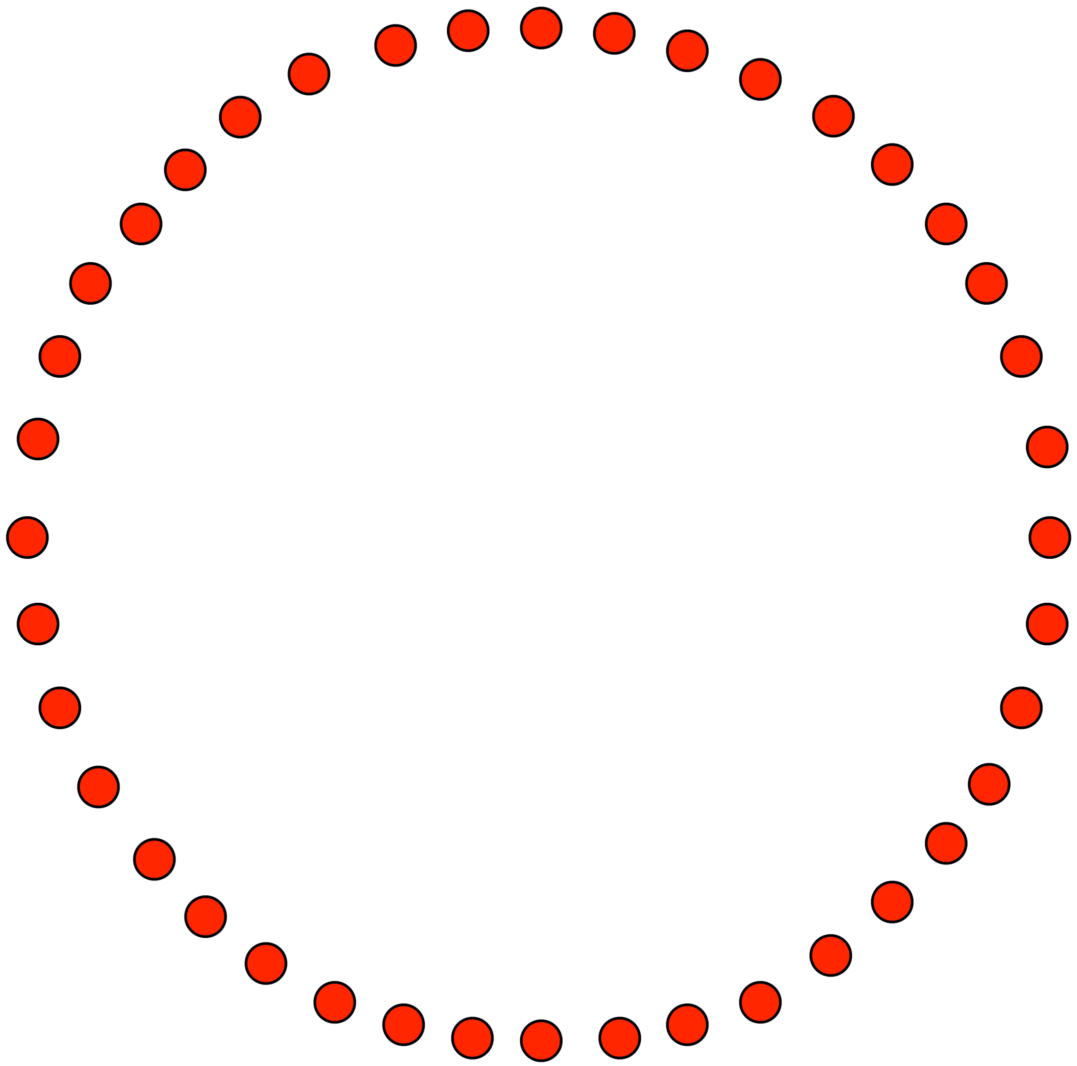

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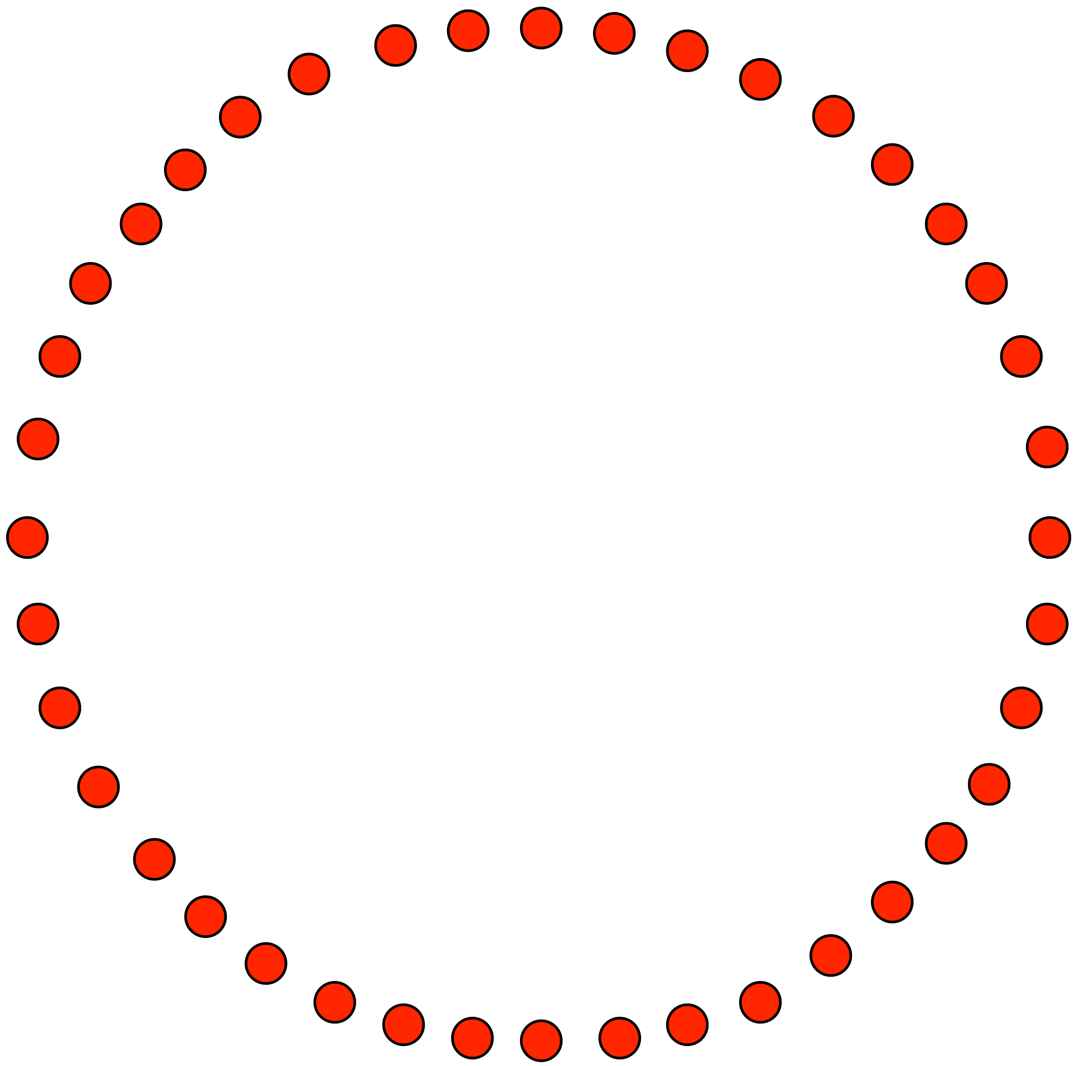

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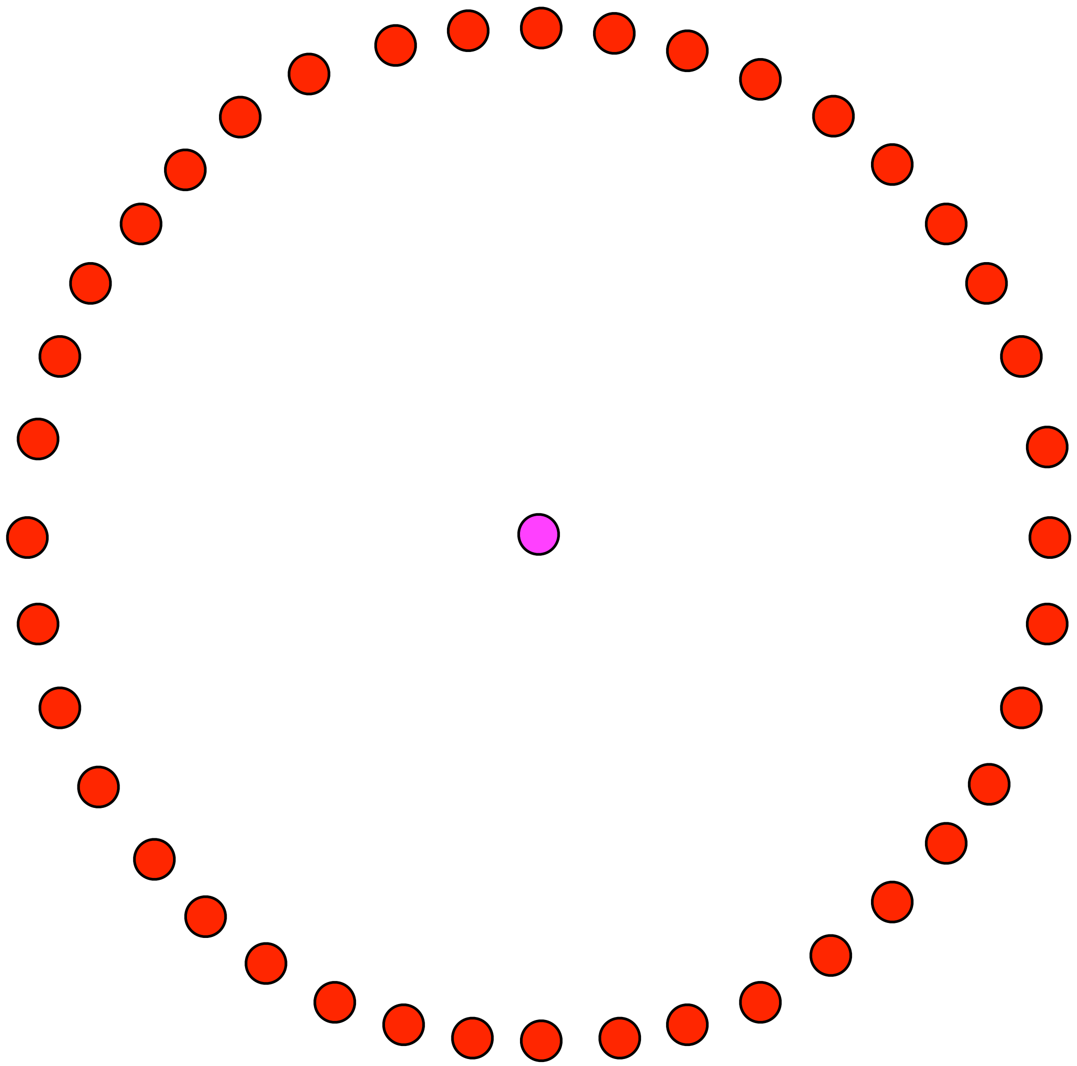

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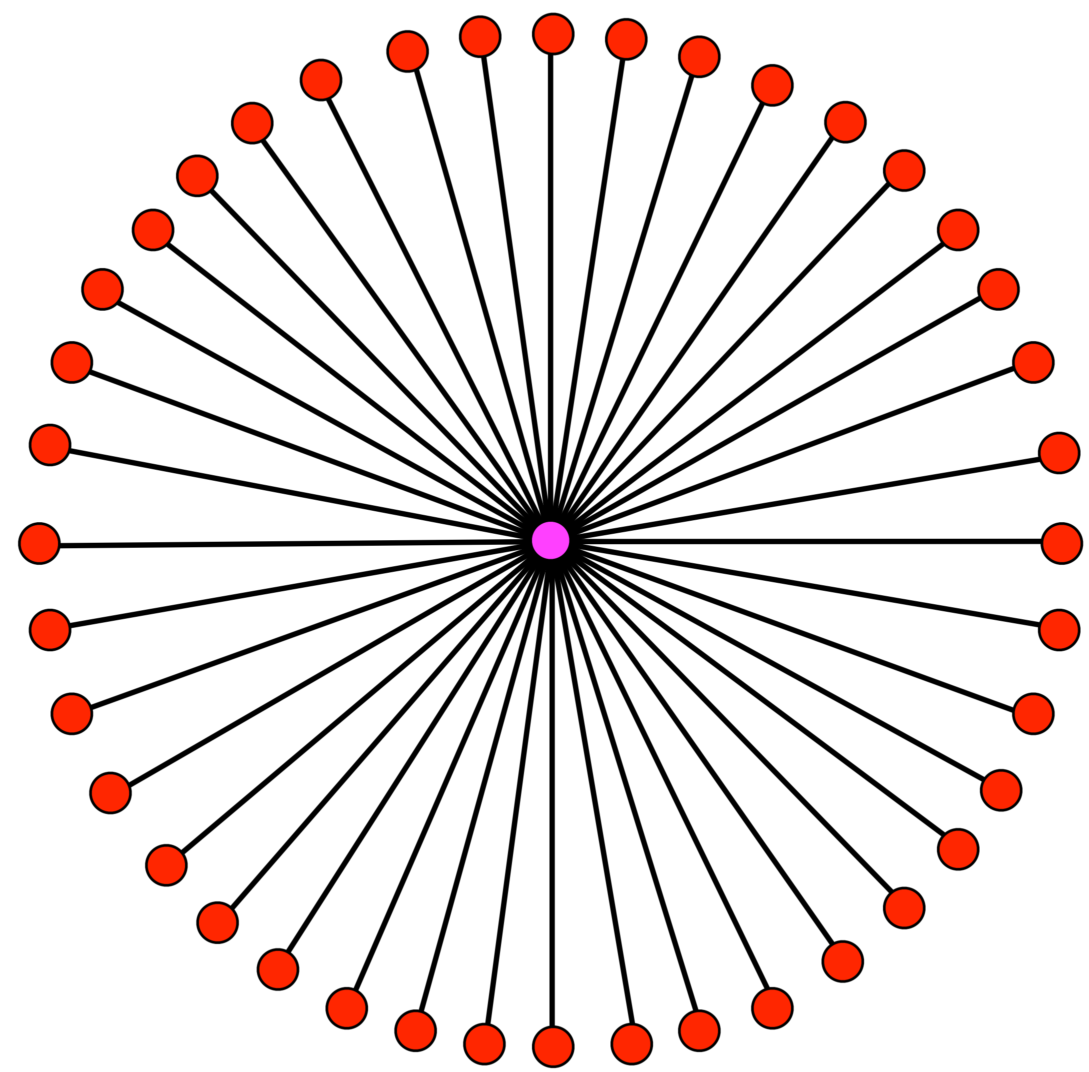

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Solving a „hard“ problem to approximate an „easy“ one [Fekete, Meijer, Rohe, Tietze 2002]

Discrete Comput Geom 3:177–191 (1988)

Discrete & Computational Geometry

The Algebraic Degree of Geometric Optimization Problems

Chanderjit Bajaj

Department of Computer Science, Purdue University, West Lafayette, IN 47907, USA

Abstract. In this paper we apply Galois methods to certain fundamental *geometric optimization* problems whose exact computational complexity has been an open problem for a long time. In particular we show that the classic Weber problem, along with the *line-restricted* Weber problem and its *three-dimensional* version are in general not solvable by radicals over the field of rationals. One direct consequence of these results is that for these geometric optimization problems there exists *no exact* algorithm under models of computation where the root of an algebraic equation is obtained using arithmetic operations and the extraction of kth roots. This leaves only numerical or symbolic approximations to the solutions, where the complexity of the approximations is shown to be primarily a function of the algebraic degree of the optimum solution point.

1. Introduction

Geometric optimization problems are inherently not pure combinatorial problems since the optimal solution often belongs to an infinite feasible set, the entire real Euclidean space. Such problems frequently arise in computer-aided design and robotics. It has thus become increasingly important to devise appropriate methods to analyze the complexity of problems where combinatorial analysis methods seem to fail. Here we take a step in this direction by applying Galois algebraic methods to certain fundamental *geometric optimization* problems. These problems are noncombinatorial and have no known polynomial time solutions. Neither have these problems shown to be intractable (NP-hard, etc.). In fact, the recognition versions of these optimization problems are not even known to be in the class NP [10].

The use of algebraic methods for analyzing the complexity of geometric problems has been popular since the time of Descartes, Gauss, Abel, and Galois. The complexity of straight-edge and compass constructions has been known to

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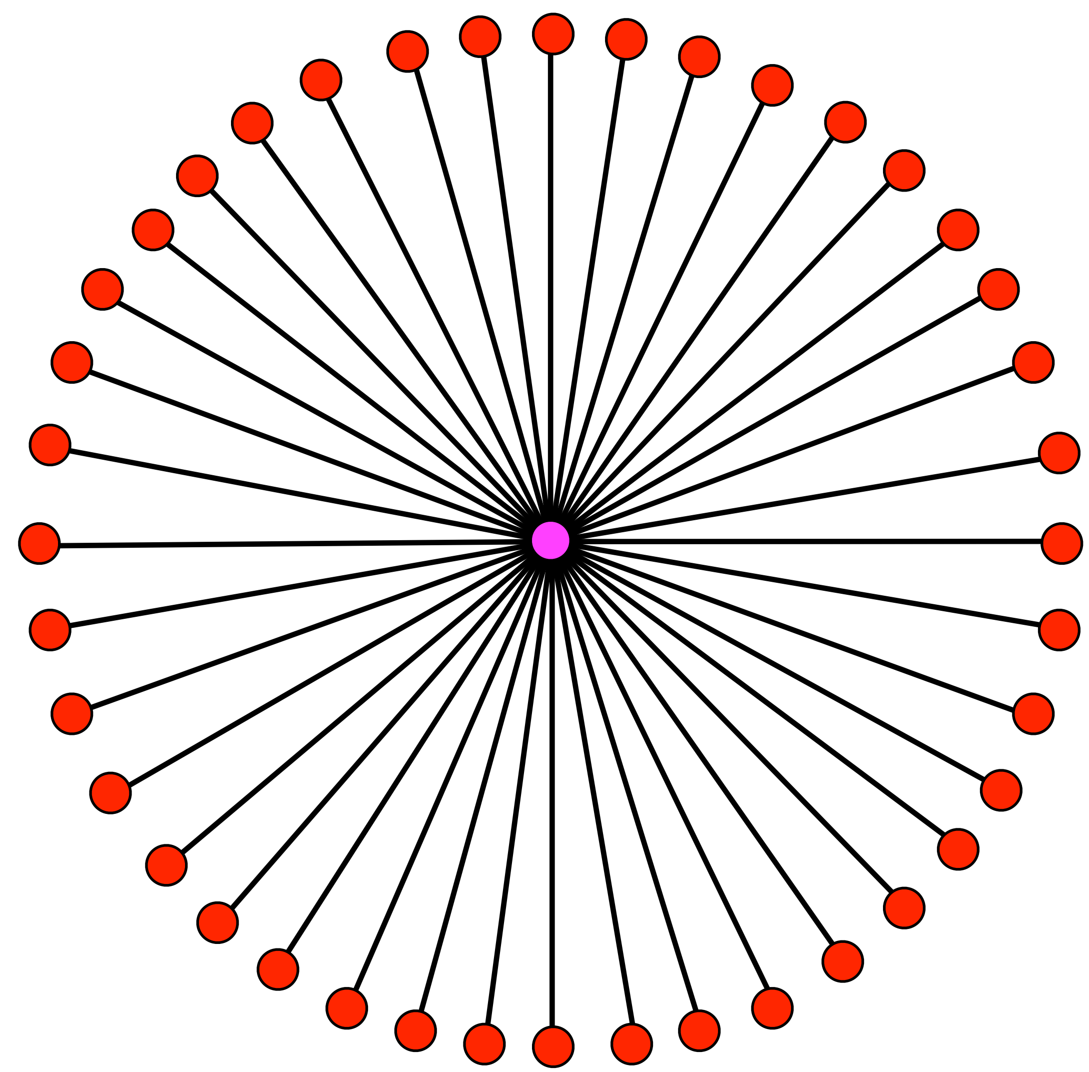

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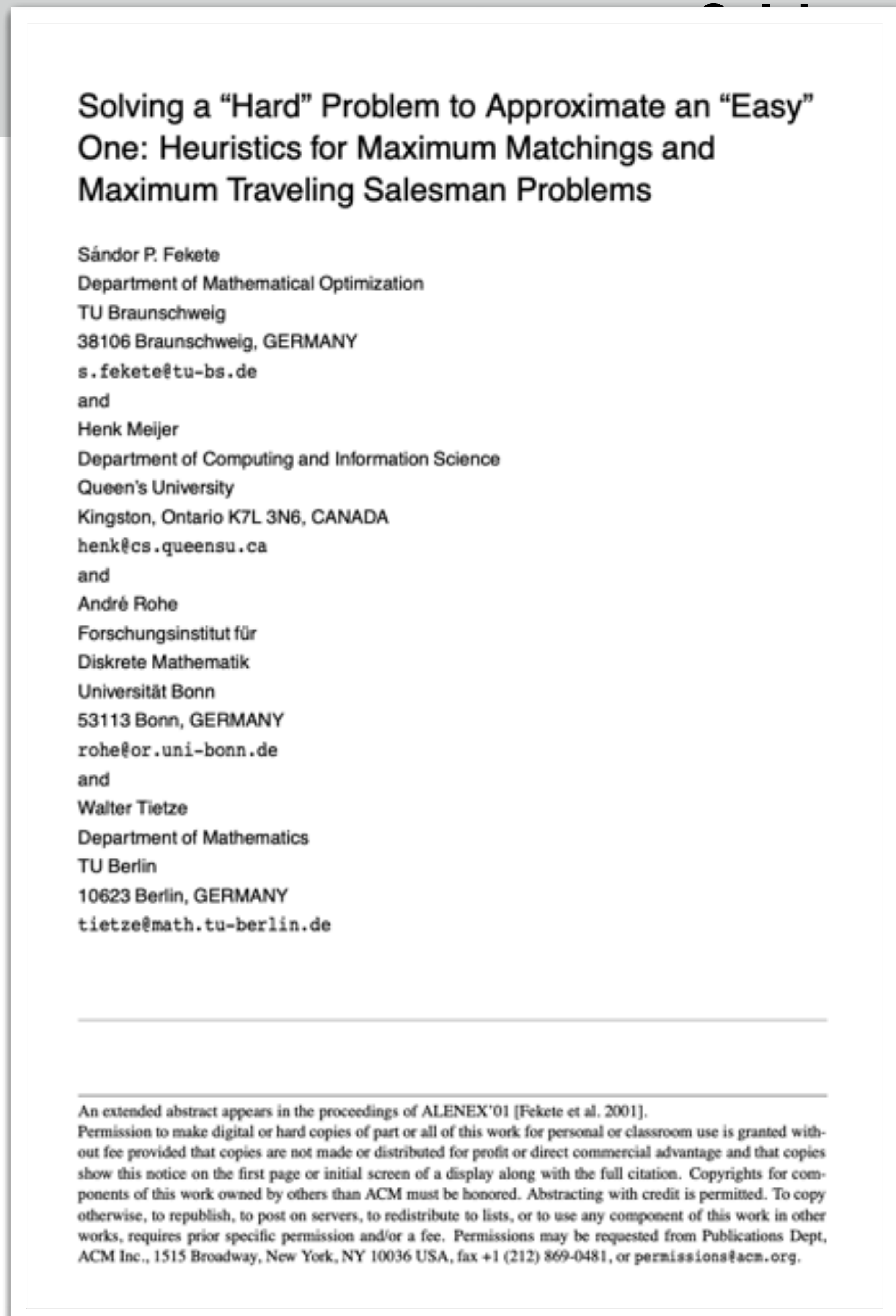

Solving a “Hard” Problem to Approximate an “Easy” One: Heuristics for Maximum Matchings and Maximum Traveling Salesman Problems

Sándor P. Fekete
Department of Mathematical Optimization
TU Braunschweig
38106 Braunschweig, GERMANY
s.fekete@tu-bs.de
and
Henk Meijer
Department of Computing and Information Science
Queen's University
Kingston, Ontario K7L 3N6, CANADA
henk@cs.queensu.ca
and
André Rohe
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53113 Bonn, GERMANY
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and
Walter Tietze
Department of Mathematics
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An extended abstract appears in the proceedings of ALENEX'01 [Fekete et al. 2001].

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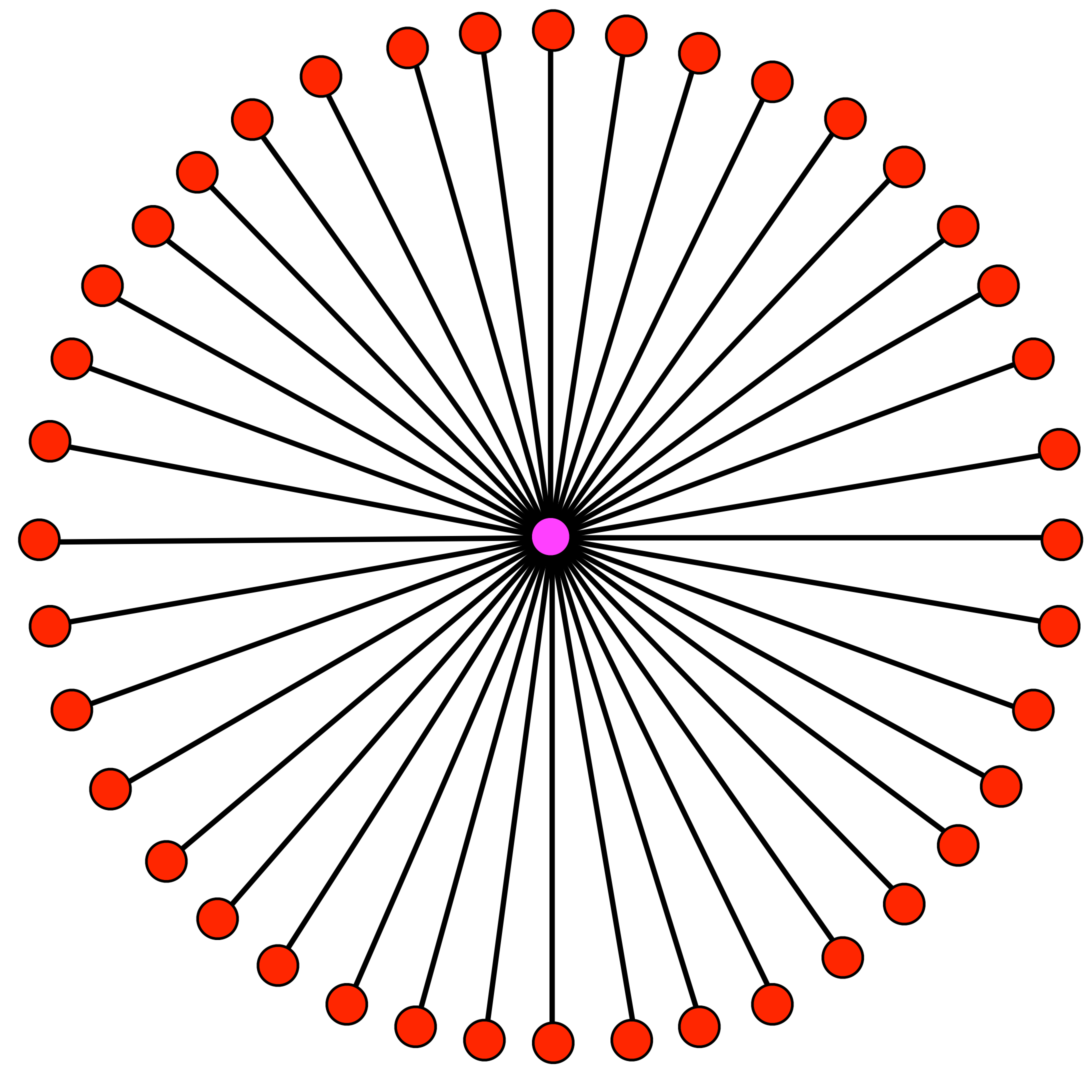

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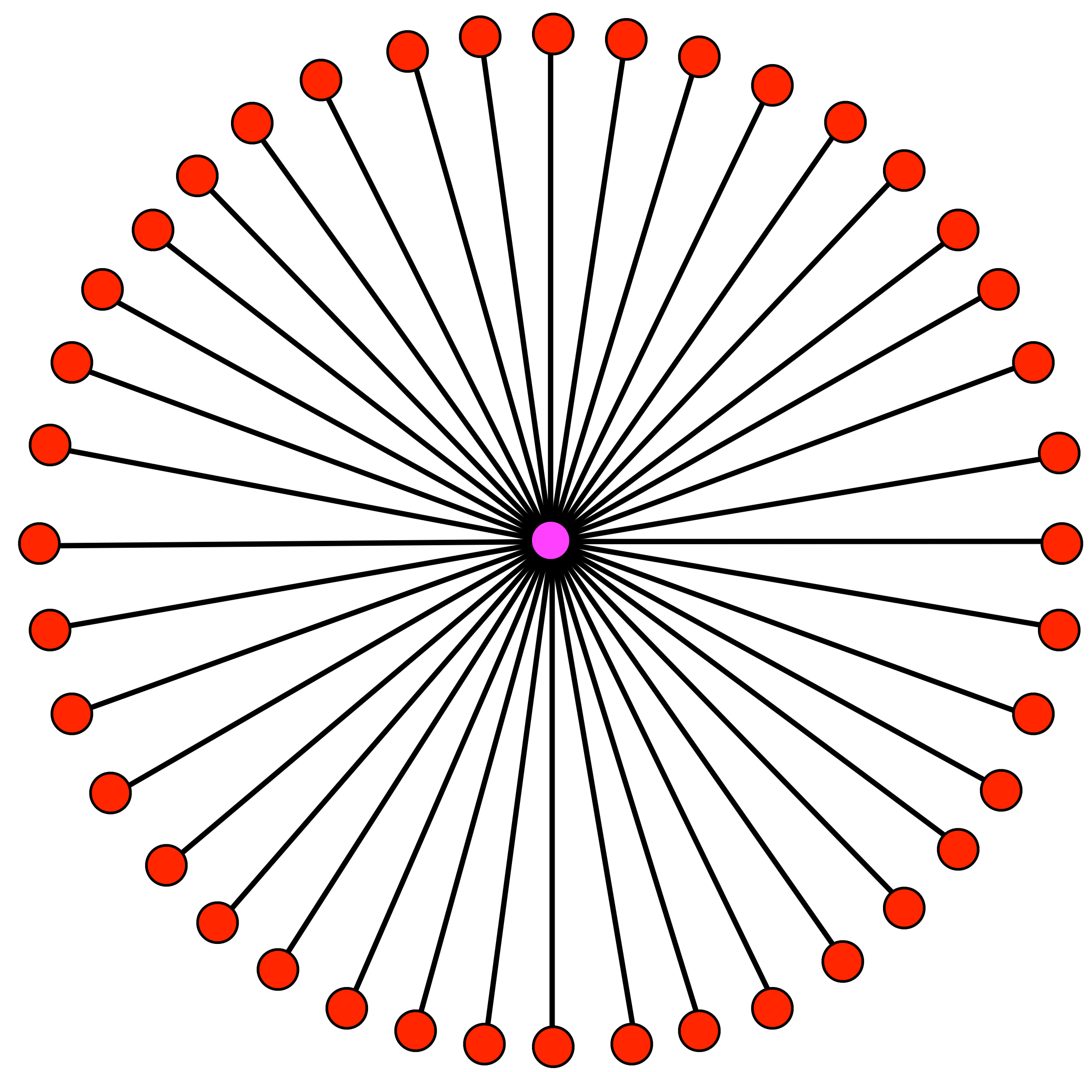

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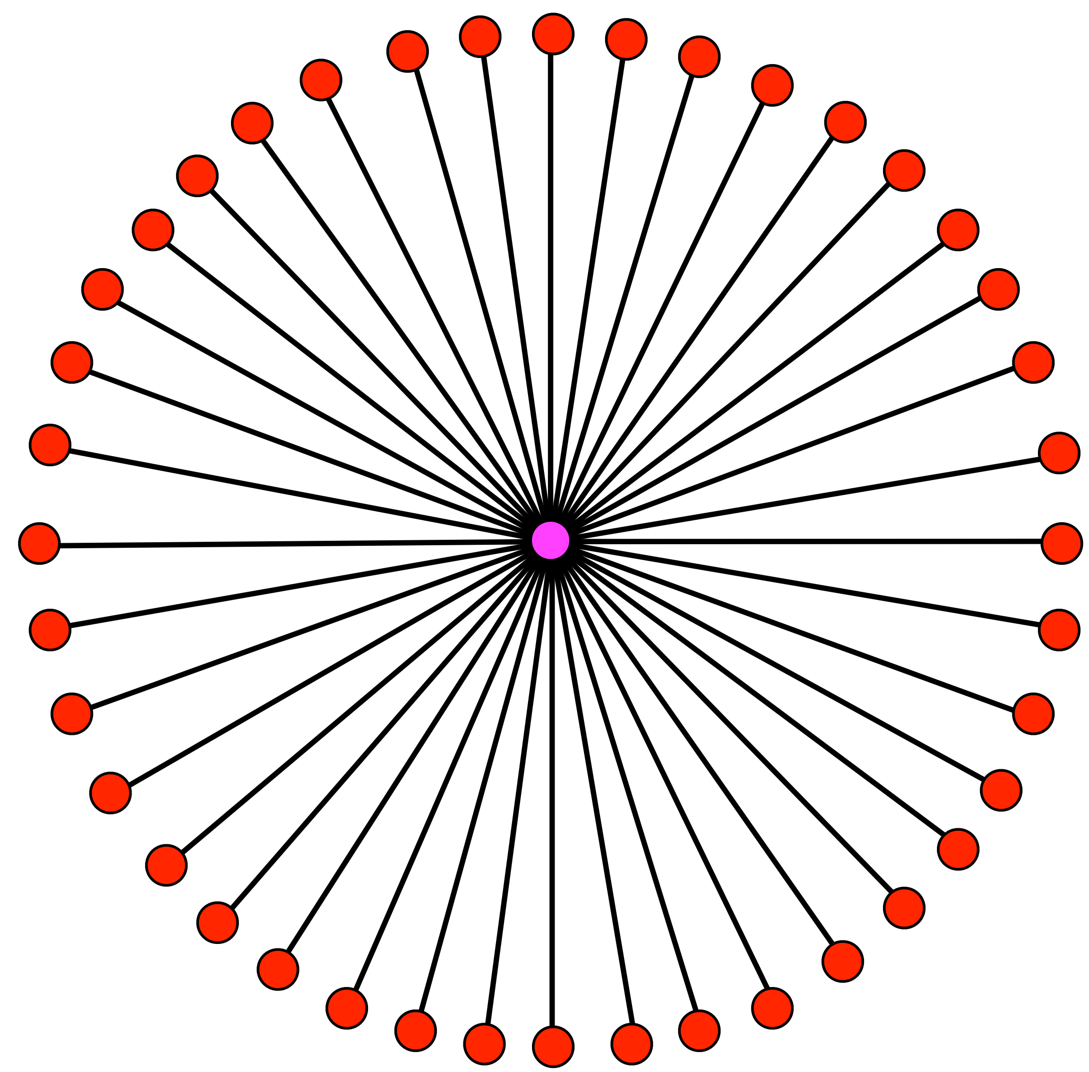

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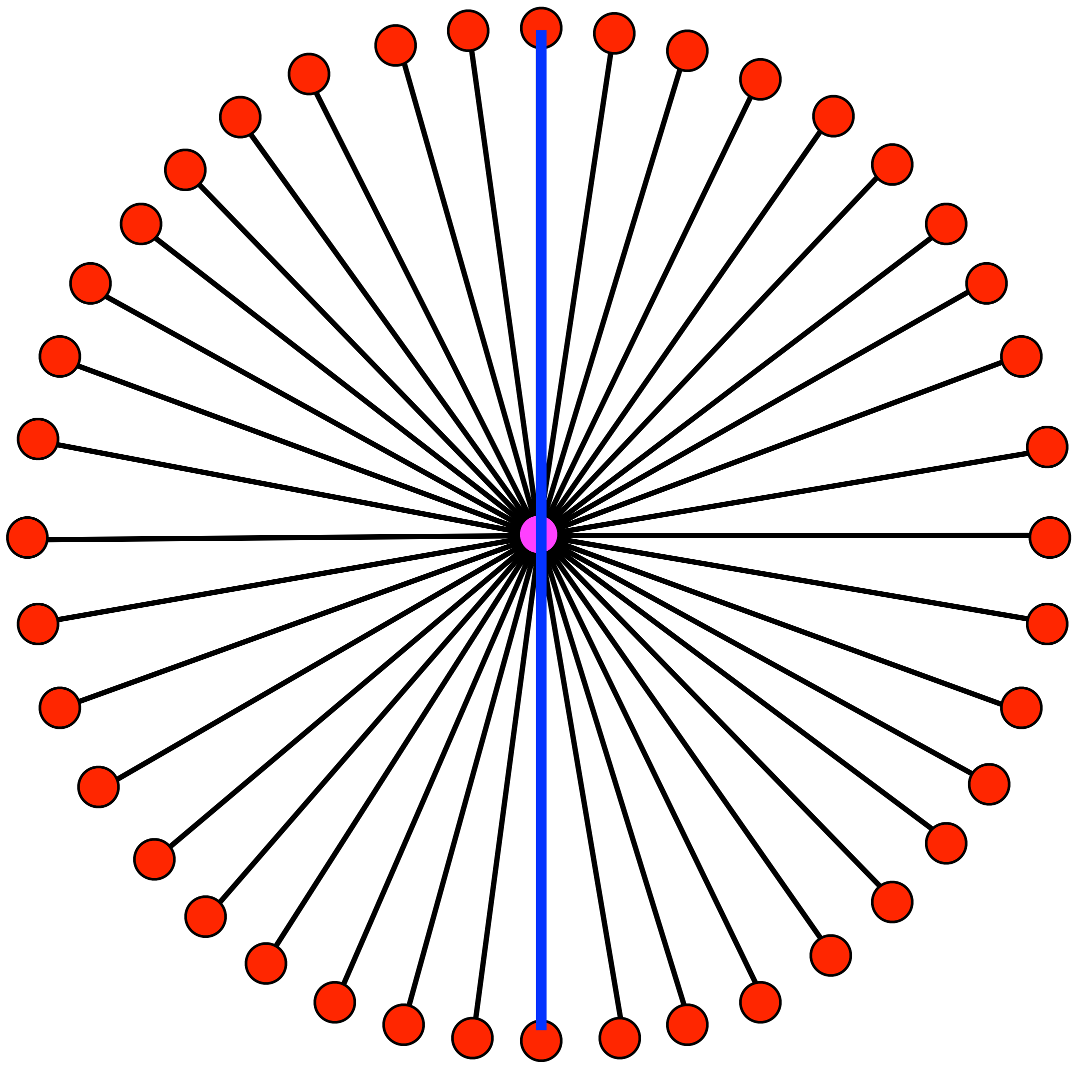

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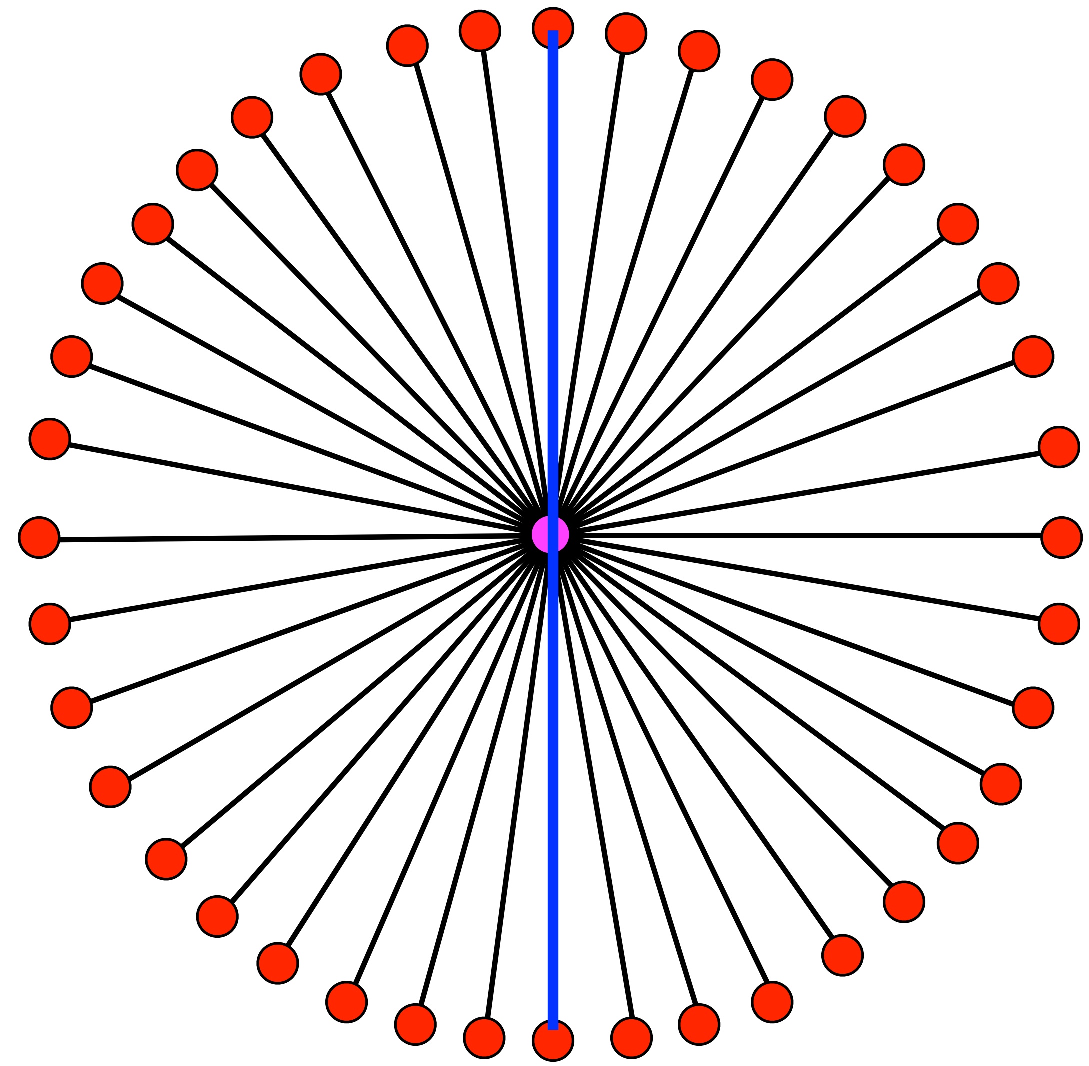

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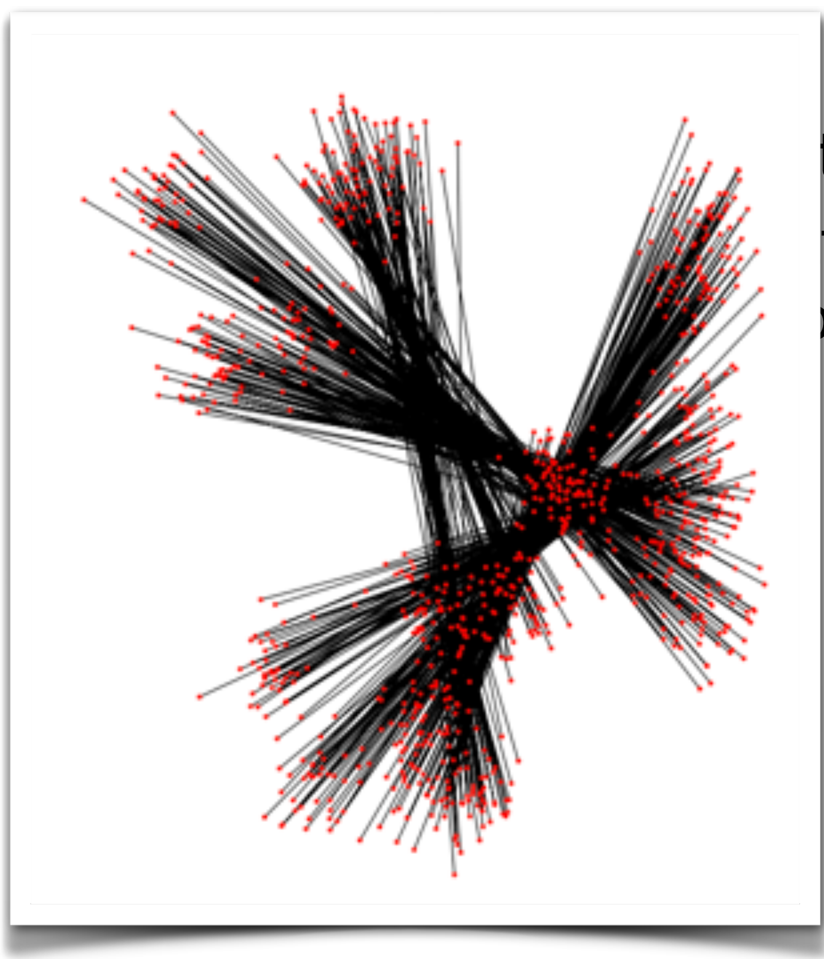

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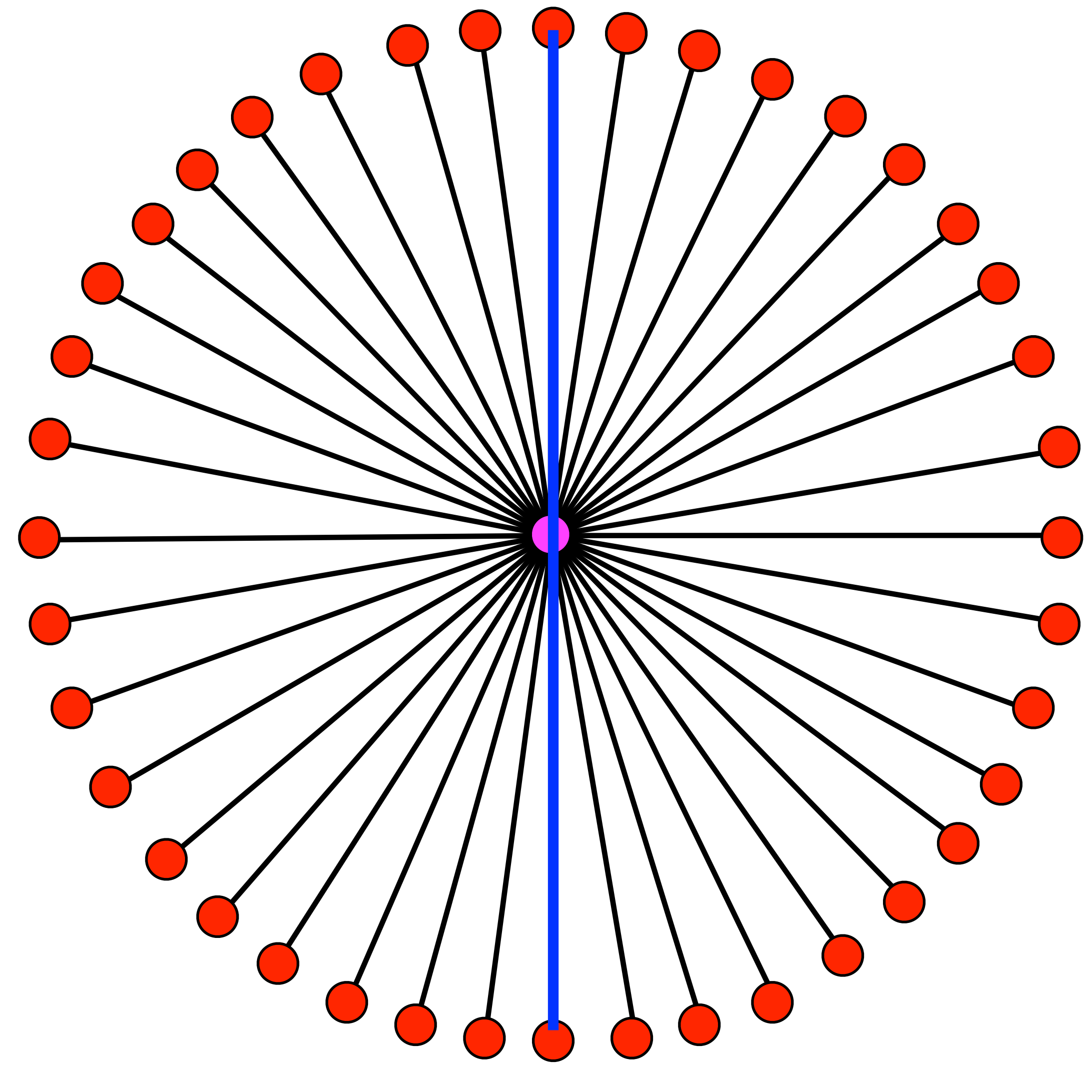

1. Introduction
2. Longest Tours
3. Stars and Matchings
4. Nonsimple Polygons
5. Optimal Area
6. Turn Cost

Shape Reconstruction

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Shape Reconstruction

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Shape Reconstruction

Given: *n* points in the plane sampled from a shape

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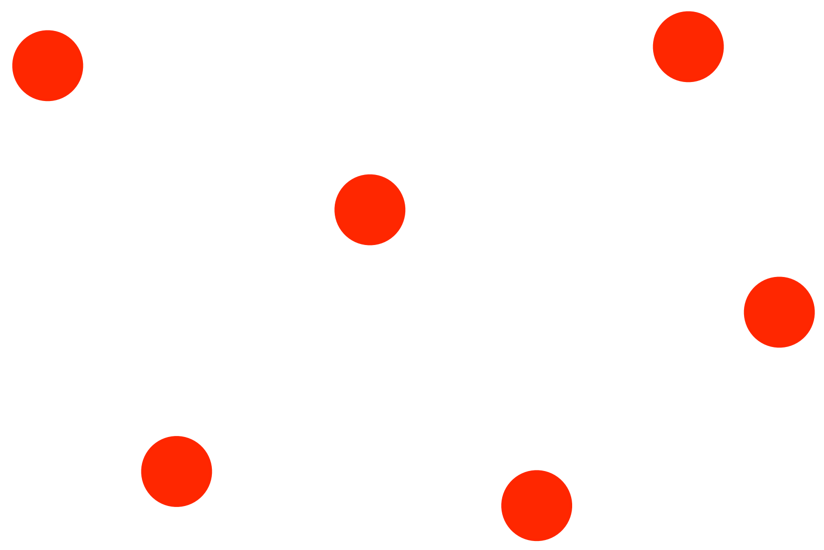

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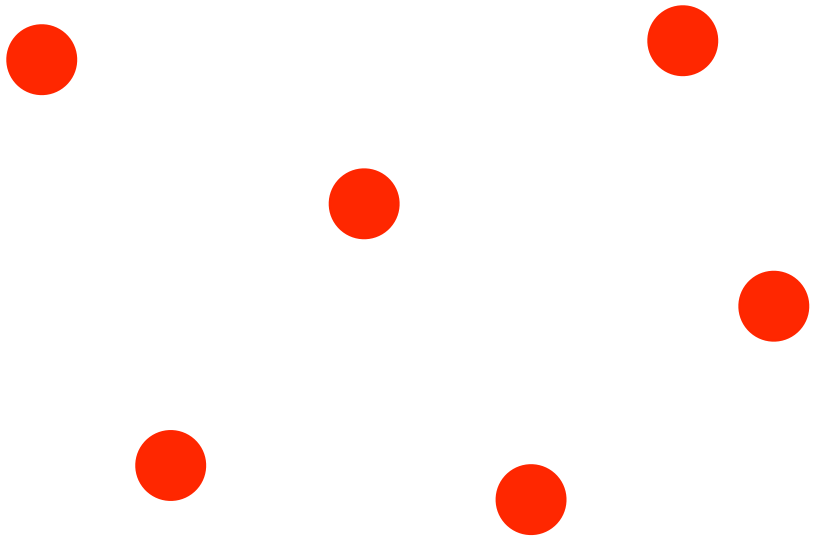

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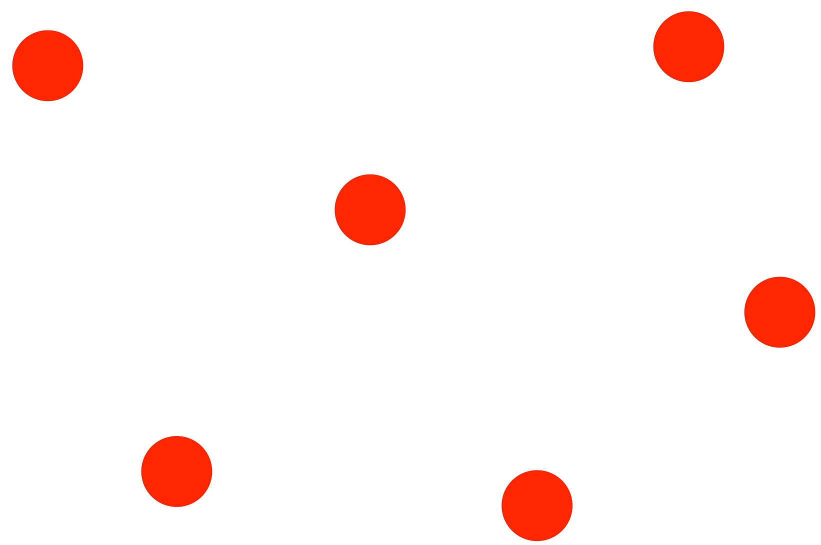

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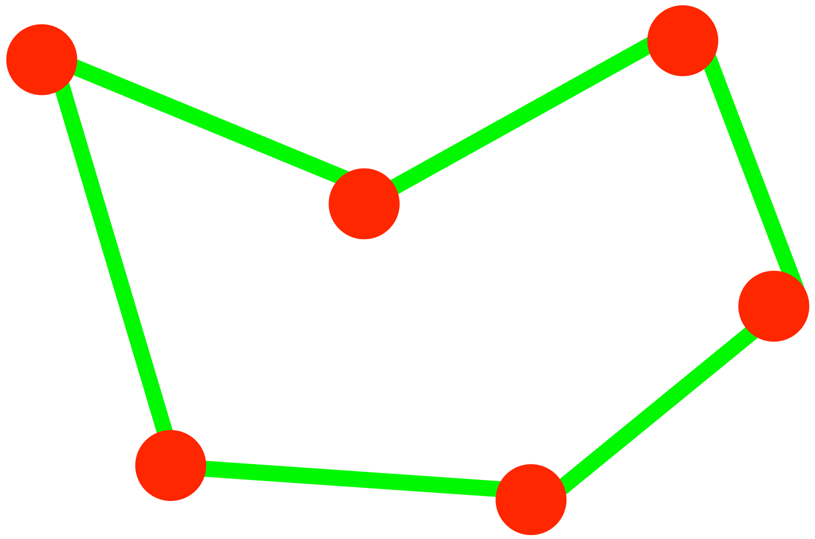

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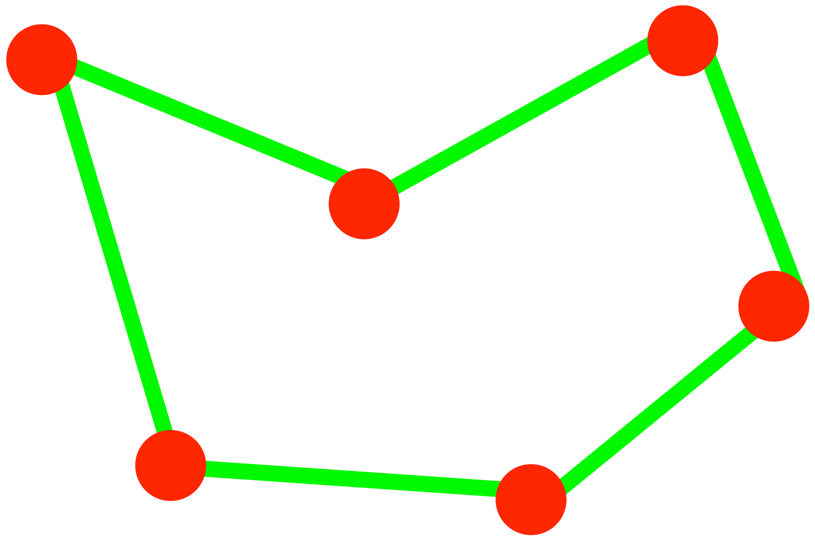

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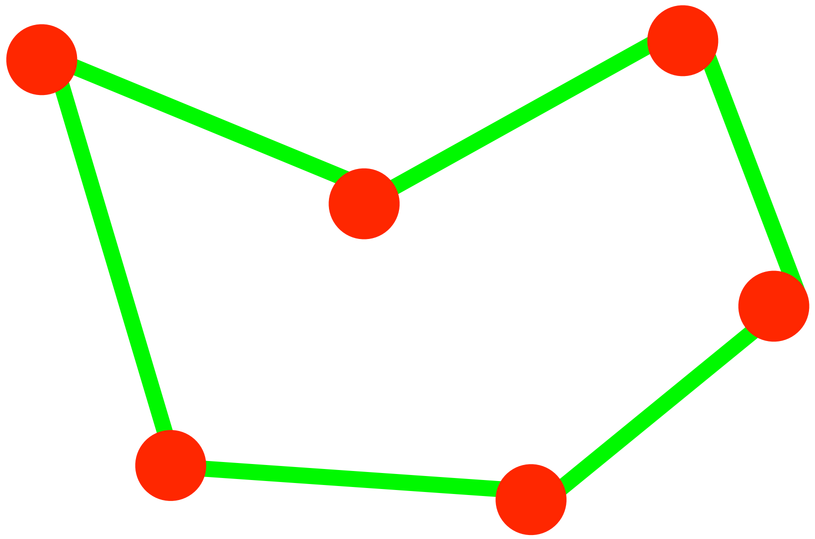

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Geometric: Why a simple curve?!

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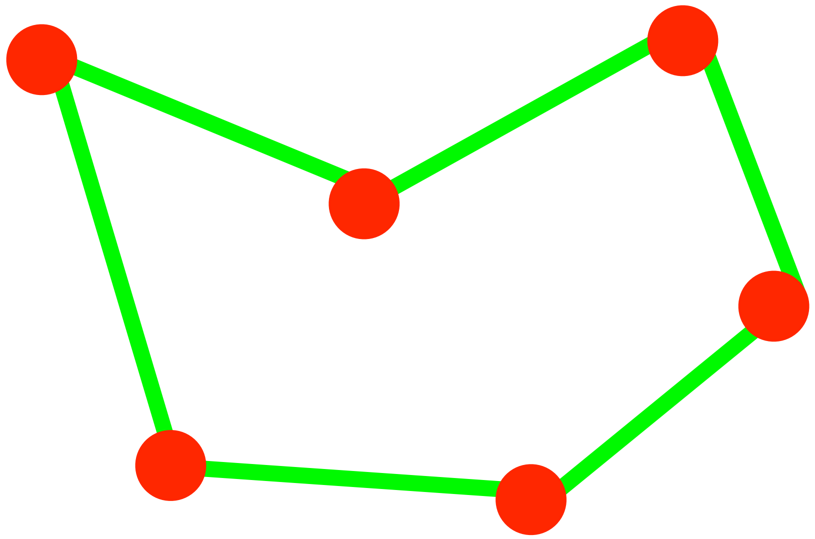

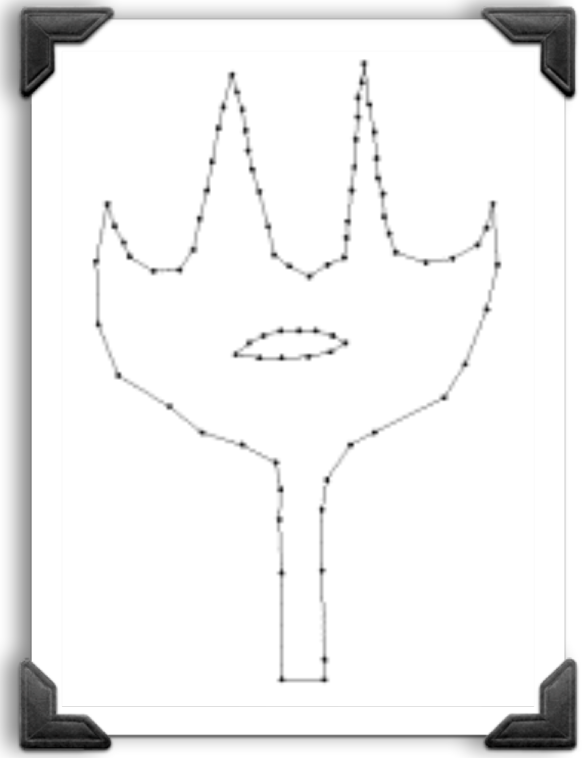

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Minimum Perimeter Polygon (MPP)

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Minim … (MPP)


Journal of Computational Geometry jocg.org

COMPUTING NONSIMPLE POLYGONS OF MINIMUM PERIMETER*

Sándor P. Fekete,† *Andreas Haas,*† *Michael Hemmer,*† *Michael Hoffmann,*‡ *Irina Kostitsyna,*§ *Dominik Krupke,*† *Florian Maurer,*† *Joseph S. B. Mitchell,*¶ *Arne Schmidt,*† *Christiane Schmidt,*‖ *Julian Troegel*†

Abstract. We consider the Minimum Perimeter Polygon Problem (MP3): for a given set V of points in the plane, find a polygon P with holes that has vertex set V, such that the total boundary length is smallest possible. The MP3 can be considered a natural geometric generalization of the Traveling Salesman Problem (TSP), which asks for a *simple* polygon with minimum perimeter. Just like the TSP, the MP3 occurs naturally in the context of curve reconstruction.

Even though the closely related problem of finding a minimum cycle cover is polynomially solvable by matching techniques, we prove how the topological structure of a polygon leads to NP-hardness of the MP3. On the positive side, we provide constant-factor approximation algorithms.

In addition to algorithms with theoretical worst-case guarantess, we provide *practical* methods for computing provably optimal solutions for relatively large instances, based on integer programming. An additional difficulty compared to the TSP is the fact that only a subset of subtour constraints is valid, depending not on combinatorics, but on geometry. We overcome this difficulty by establishing and exploiting geometric properties. This allows us to reliably solve a wide range of benchmark instances with up to 600 vertices within reasonable time on a standard machine. We also show that restricting the set of connections between points to edges of the Delaunay triangulation yields results that are on average within 0.5% of the optimum for large classes of benchmark instances.

1 Introduction

Two of the most fundamental structures in Computational Geometry are planar point sets and polygons. In this paper we study a natural algorithmic connection between them. For

*A preliminary extended abstract [13] appeared in the Proceedings of the 15th Symposium on Experimental and Efficient Algorithms (SEA 2016).

†*Department of Computer Science, TU Braunschweig, Mühlenpfordtstr. 23, 38106 Braunschweig, Germany.* {s.fekete,a.haas,m.hemmer,d.krupke,arne.schmidt,j.troegel}@tu-bs.de

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§*Department of Mathematics and Computer Science, TU Eindhoven, 5600 MB Eindhoven, The Netherlands.* i.kostitsyna@tue.nl

¶*Department of Applied Mathematics and Statistics, Stony Brook University, Stony Brook, NY 11794-3600, USA.* joseph.mitchell@stonybrook.edu

‖*Department of Science and Technology, Linköping University, SE 60174 Norrköping, Sweden.* christiane.schmidt@liu.se

JoCG 8(1), 340–365, 2017 340

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Minim… (MPP)

Journal of Computational Geometry jocg.org

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Minimum Perimeter Polygon (MPP)

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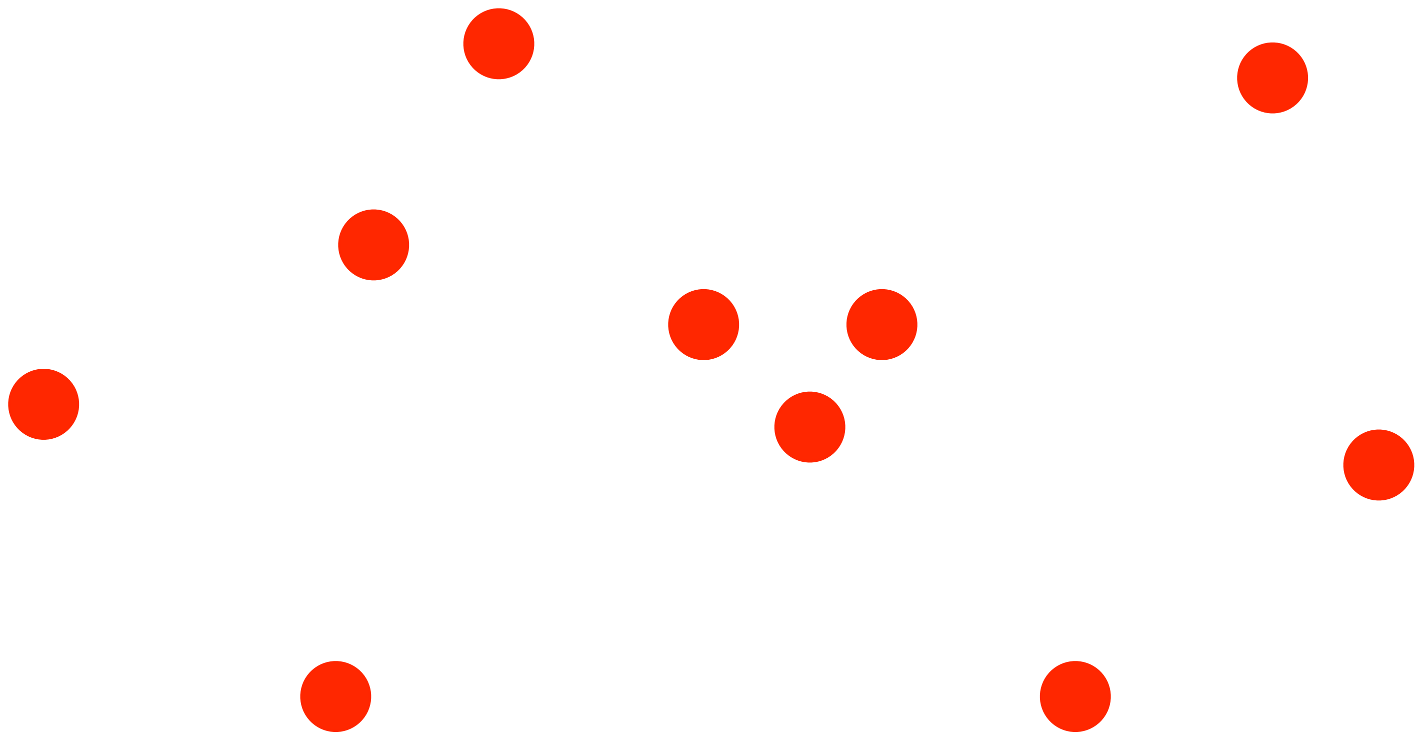
Minimum Perimeter Polygon (MPP)

Given: n points in the plane

Minimum Perimeter Polygon (MPP)

Given: *n* points in the plane



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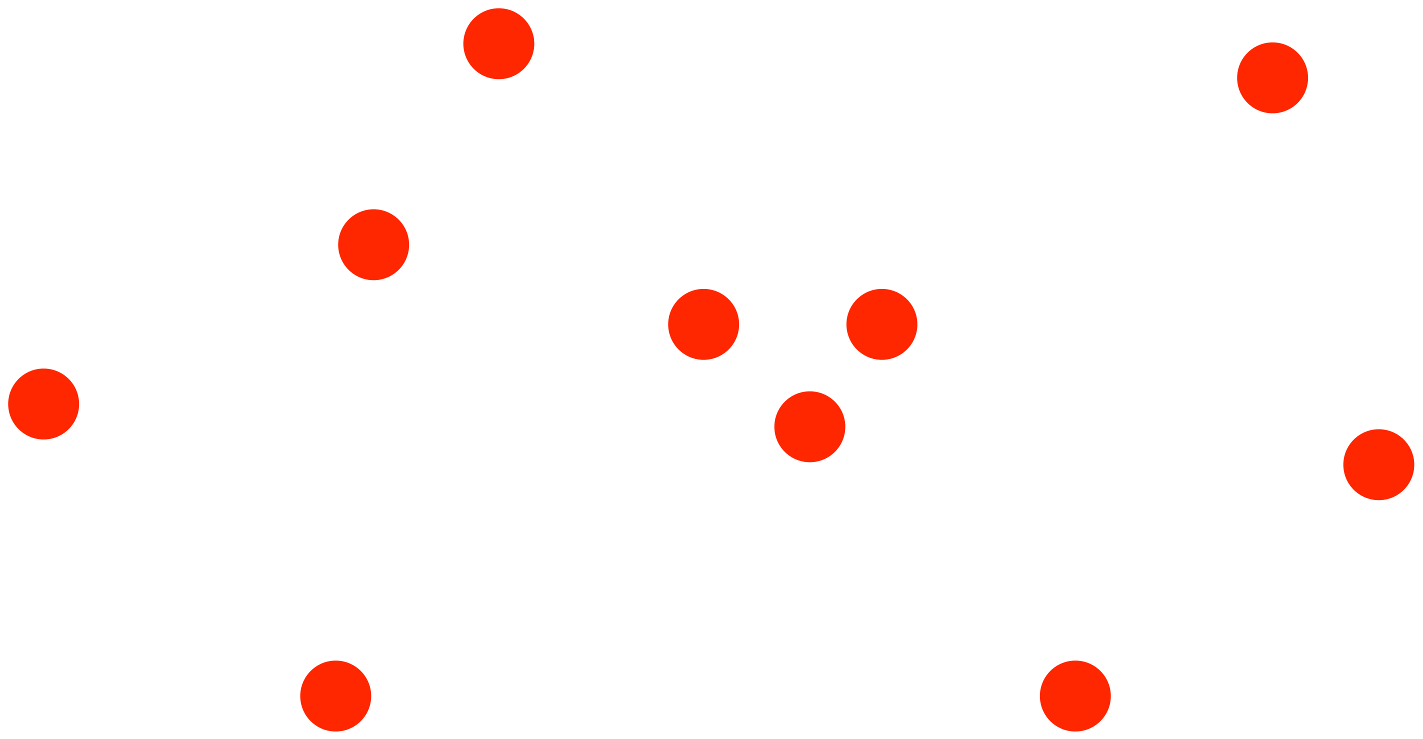

Wanted:

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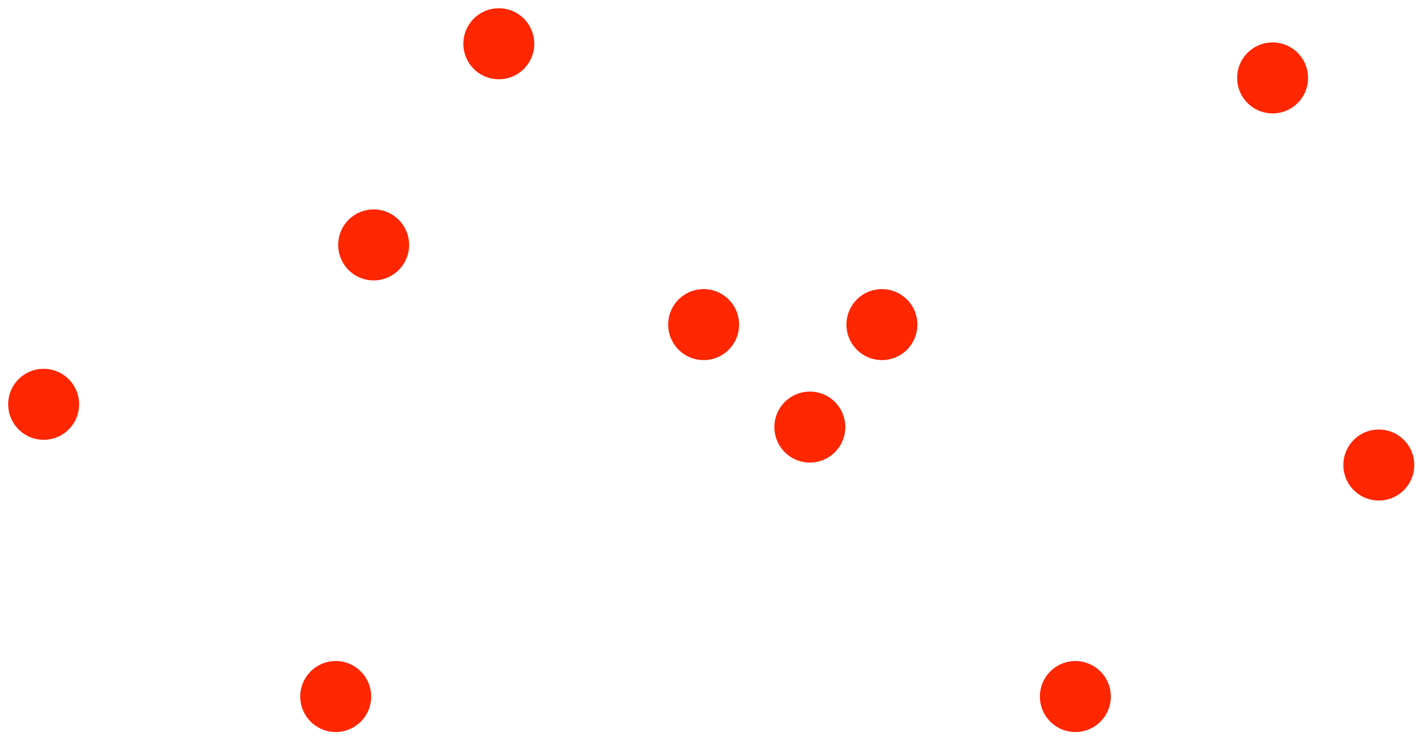
Minimum Perimeter Polygon (MPP)

Given: *n* points in the plane



Wanted: A valid polygon of minimum perimeter length

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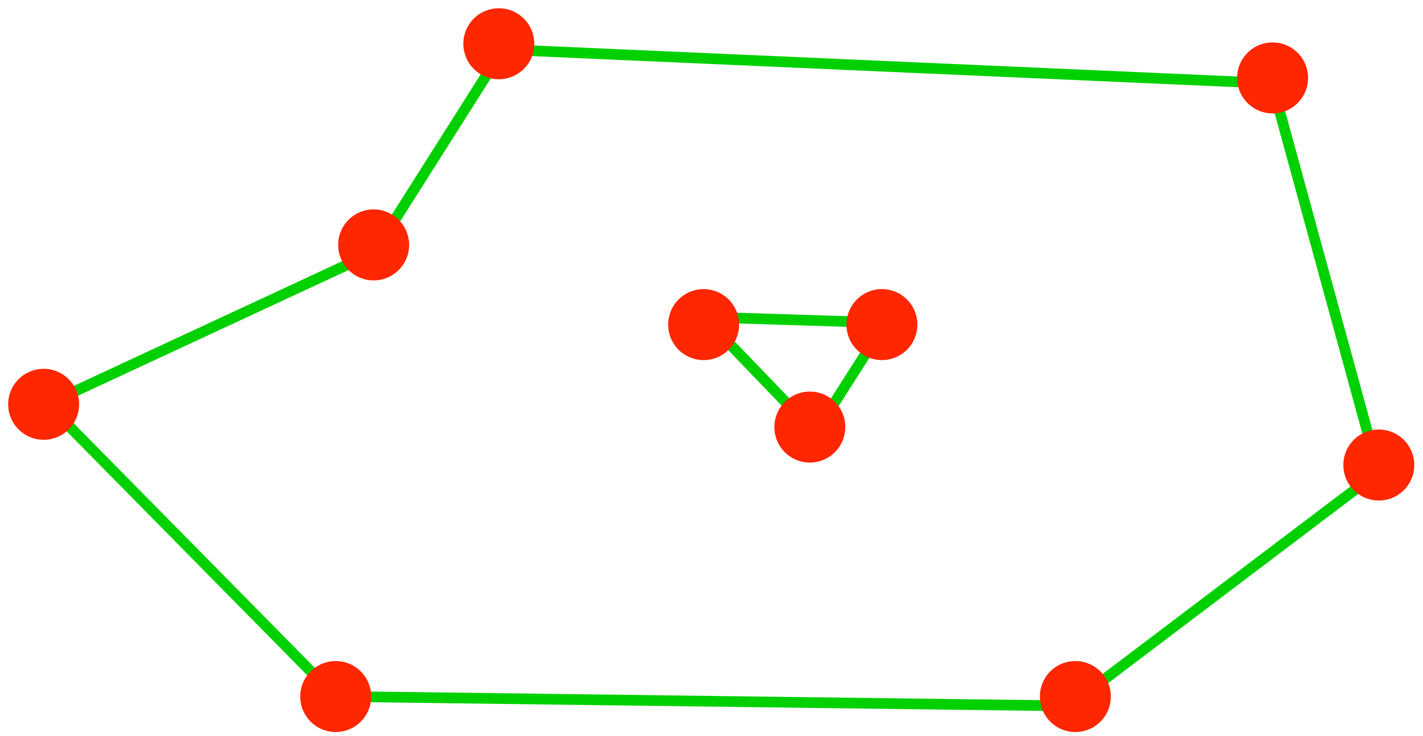

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Minimum Perimeter Polygon (MPP)

Figure 1: A Minimum Perimeter Polygon for an instance with 960 vertices.

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An Open Problem Resolved

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Theorem 1. *The MPP problem is NP-hard.*

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Constant-Factor Approximation

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Constant-Factor Approximation

Theorem 2. *There exists a polynomial time 3-approximation for the MPP.*

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3-Approximation

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Idea:

3-Approximation

Idea: • Compute outer hull

3-Approximation

Idea: • Compute outer hull $\leq$ **OPT**

3-Approximation

Idea:

- Compute outer hull $\leq$ **OPT**
- Compute 2-factor of interior points

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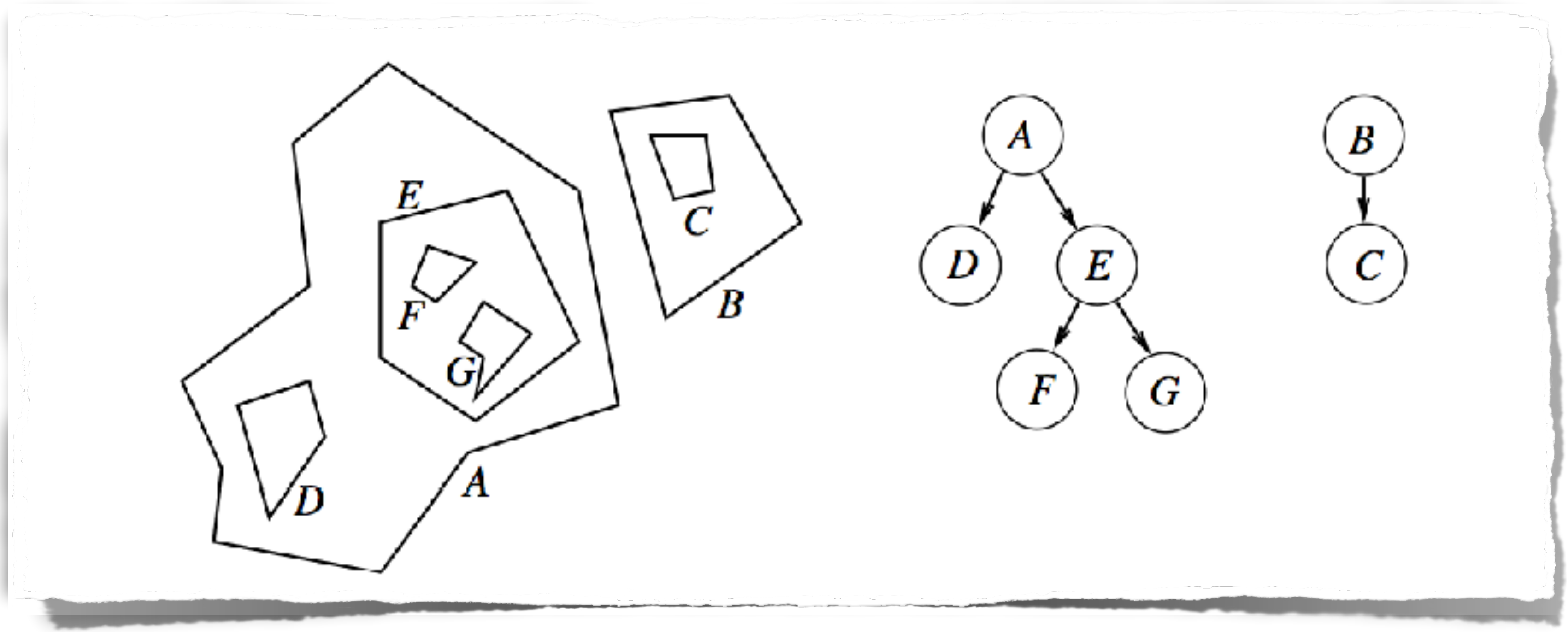
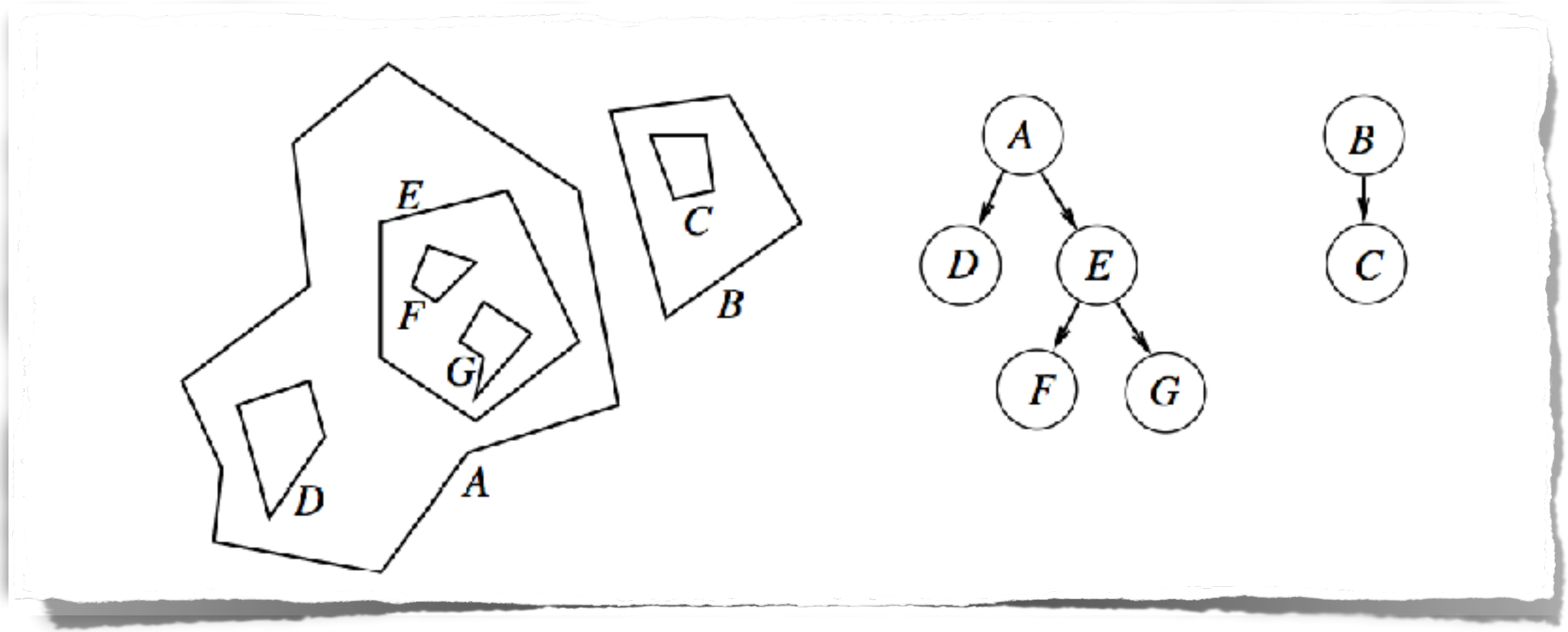


3-Approximation

Idea:

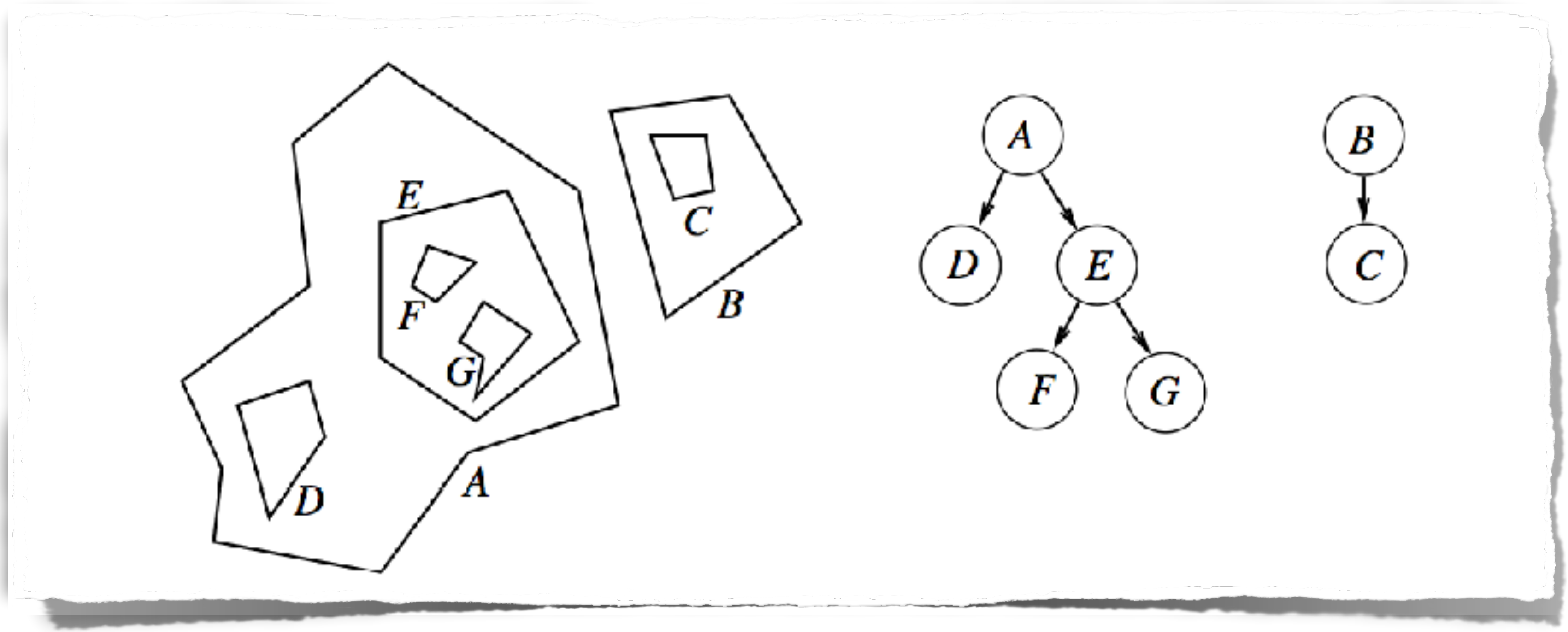
- Compute outer hull ≤ OPT
- Compute 2-factor of interior points ≤ OPT



3-Approximation

Idea:

- Compute outer hull ≤ OPT
- Compute 2-factor of interior points ≤ OPT
- Merge cycles



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Idea:

- Compute outer hull ≤ OPT
- Compute 2-factor of interior points ≤ OPT
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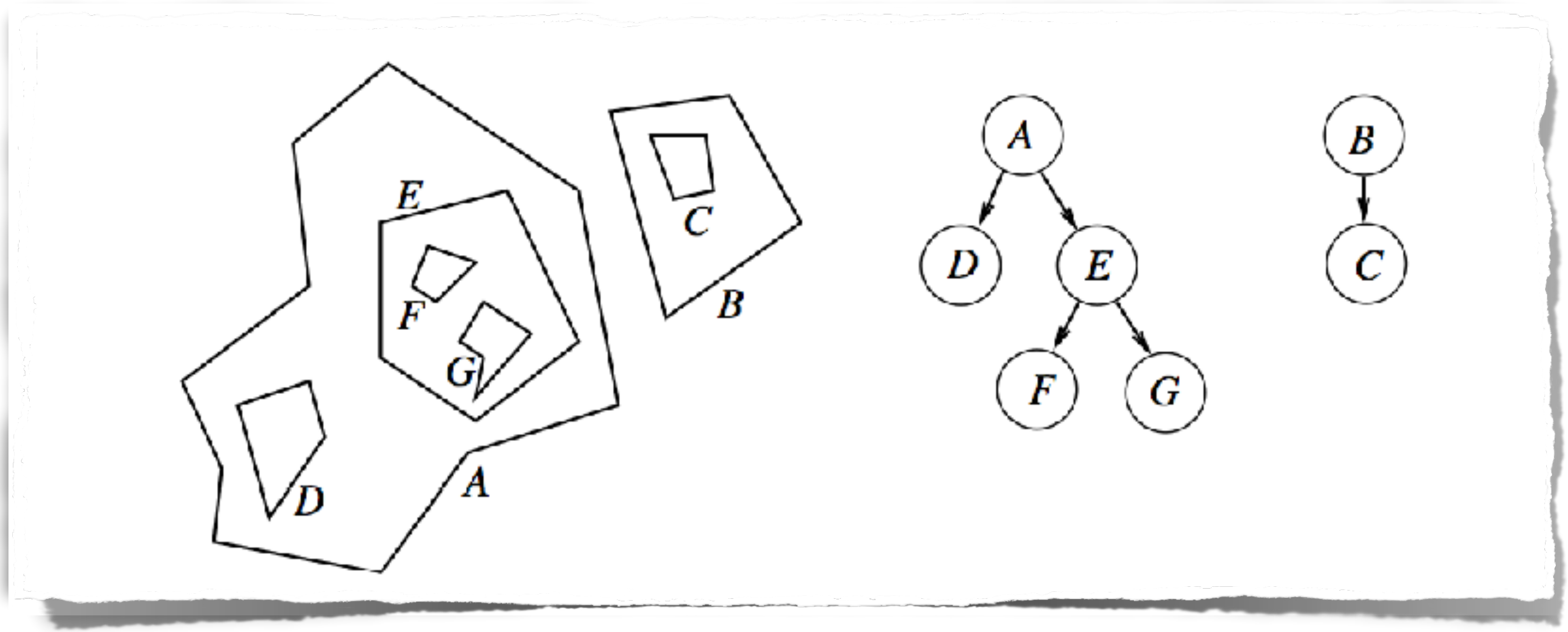

Technical Details: Merge Cycles (1)

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We now describe phase 1. Let T be a nontrivial tree of F. Associated with T are a set of cycles, one per node. A node u of T that has no outgoing edge of T (i.e., U has no children) is a sink node; it corresponds to a cycle that has no cycle contained within it. Let v be a node of T that has at least one child, but no grandchildren. (Such a node must exist in a nontrivial tree T.) Then, v corresponds to a cycle (simple polygon) P_v, within which there is one or more disjoint simple polygonal cycles, $P_{u_1}, P_{u_2}, \ldots, P_{u_k}$, one for each of the $k \geq 1$ children of v. We describe an operation that replaces P_v with a new weakly simple polygon, Q_v, whose interior is disjoint from those of $P_{u_1}, P_{u_2}, \ldots, P_{u_k}$. Let $e = pq$ ($p, q \in V$) be any edge of P_v; assume that pq is a counterclockwise edge, so that the interior of P_v lies to the left of the oriented segment pq. Let Γ be a shortest path within P_v, from p to q, that has all of the polygons $P_{u_1}, P_{u_2}, \ldots, P_{u_k}$ to its right; thus, Γ is a "taut string" path within P_v, homotopically equivalent to ∂P_v, from p to q. (Such a geodesic path is related to the "relative convex hull" of the polygons $P_{u_1}, P_{u_2}, \ldots, P_{u_k}$ within P_v, which is the shortest cycle within P_v that encloses all of the polygons; the difference is that Γ is "anchored" at the endpoints p and q.) Note that Γ is a polygonal path whose vertices are either (convex) vertices of the polygons P_{u_j} or (reflex) vertices of P_v. Consider the closed polygonal walk that starts at p, follows the path Γ to q, then continues counterclockwise around the boundary, ∂P_v, of P_v until it returns to p. This closed polygonal walk is the counterclockwise traversal of a weakly simple polygon, Q_v, whose interior is disjoint from the interiors of the polygons $P_{u_1}, P_{u_2}, \ldots, P_{u_k}$. Refer to Figure 8. The length of this closed walk (the counterclockwise traversal of the boundary of Q_v) is at most twice the perimeter of P_v, since the path Γ has length at most that of the counterclockwise boundary ∂P_v, from q to p (since Γ is a homotopically equivalent shortening of this boundary). We consider the boundary of P_v to be replaced with the cycle around the boundary of Q_v, and this process has reduced the degree of nesting in T: node v that used to have k children (leaves of T) is now replaced by a node v' corresponding to Q_v, and v' and the k children of v are now all siblings in the modified tree, T'. If v had a parent, w, in T, then v' and the k children of v are now children of W; if v had no parent in T (i.e., it was the root of T), then T has been transformed into a set of $k+1$ cycles, none of which are nested within another cycle of $\gamma(U)$. (Each is within the convex hull $CH(V)$, but there is no other surrounding cycle of $\gamma(U)$.) We continue this process of transforming a surrounding parent cycle (node v) into a sibling cycle (node v'), until each tree T of F becomes a set of isolated nodes, and finally F has no edges (there is no nesting).

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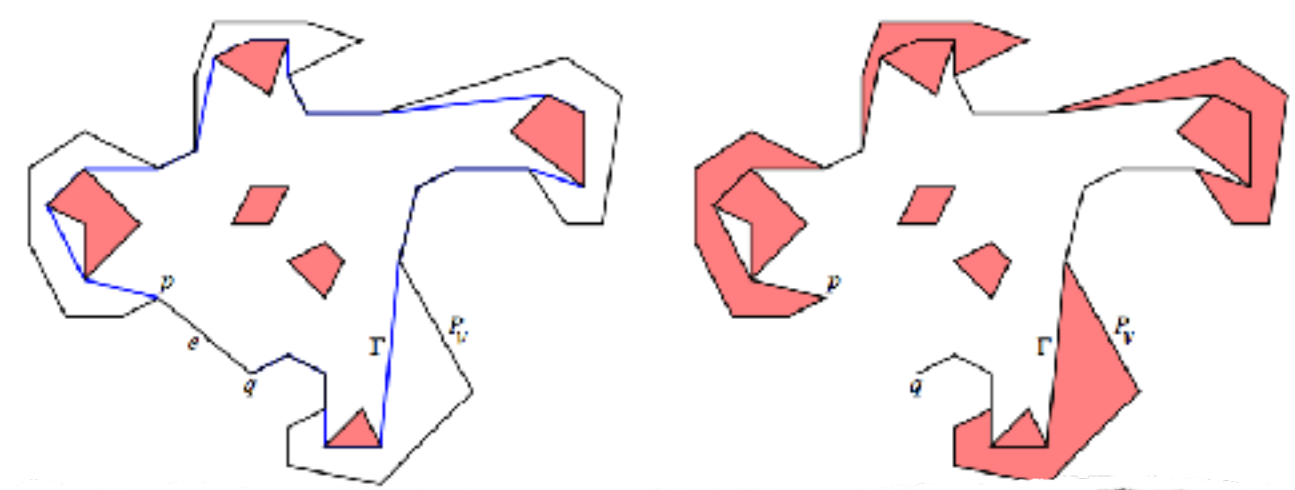

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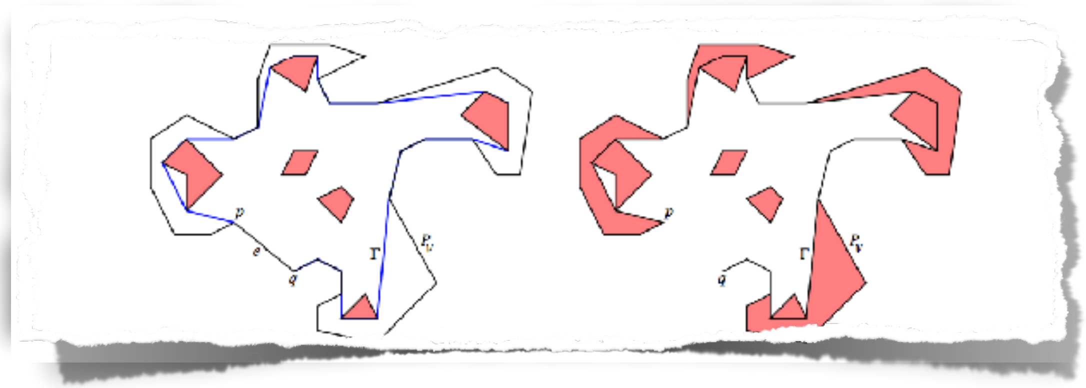
Technical Details: Merge Cycles (2)



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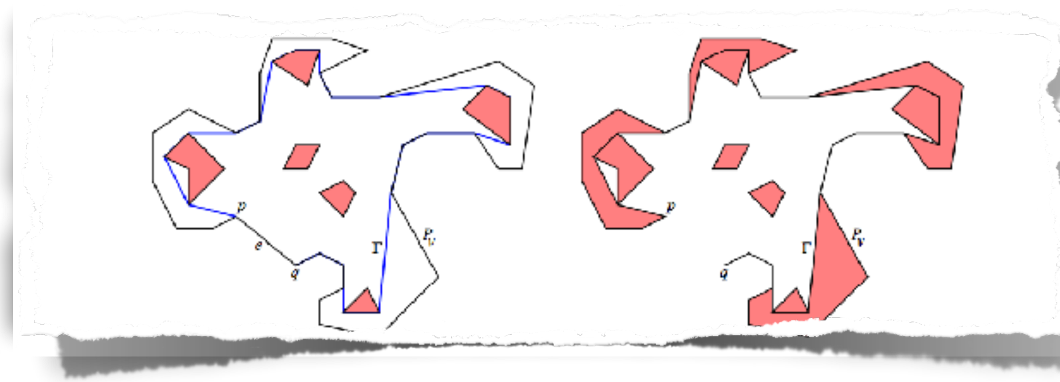
Technical Details: Merge Cycles (2)

Consider a weakly simple polygon Q, and let p be a vertex of Q that is repeated in the cycle specifying the boundary ∂Q. This implies that there are four edges of the (counterclockwise) cycle, p_0p, pp_1, p_2p, and pp_3, incident on p, all of which lie within a halfplane through p (by local optimality). There are then two subcases: (i) p_0, p, p_1 is a left turn (Figure 9, left); and (ii) p_0pp_1 is a right turn (Figure 9, right). In subcase (i), p_0p, pp_1 define a left turn at p (making p locally convex for Q), and p_2p, pp_3 define a right turn at p (making p locally reflex for Q). In this case, we replace the pair of edges p_0p, pp_1 with a shorter polygonal chain, namely the "taut" version of this path, from p_0 to p_1, along a shortest path, $\beta_{0,1}$, among the polygons Q_i, including Q, treating them as obstacles. The taut path $\beta_{0,1}$ consists of left turns only, at (locally convex) vertices of polygons Q_i ($Q_i \neq Q$) or (locally reflex) vertices of Q, where new pinch points of Q are created. Refer to Figure 9, left. Case (ii) is treated similarly; see Figure 9, right. Thus, resolving one repeated vertex, p, of Q can result in the creation of other repeated vertices of Q, or repeated vertices where two cycles come together (discussed below). The process is finite, though, since the total length of all cycles strictly decreases with each operation; in fact, there can be only a polynomial number of such adjustments, since each triple (p_0, p, p_1), is resolved at most once.

Now consider a vertex p that appears once as a reflex vertex in Q_1 (with incident ccw edges p_0p and pp_1) and once as a convex vertex in Q_2 (with incident ccw edges p_2p and pp_3). (Because cycles resulting after phase 1 are locally shortest, p must be reflex in one cycle and convex in the other.) Our local operation in this case results in a merging of the two cycles Q_1 and Q_2 into a single cycle, replacing edges p_0p (of Q_1) and pp_3 (of Q_2) with the taut shortest path, $\beta_{0,3}$. As in the process described above, this replacement can result in new repeated vertices, as the merged cycle may come into contact with other cycles, or with itself.



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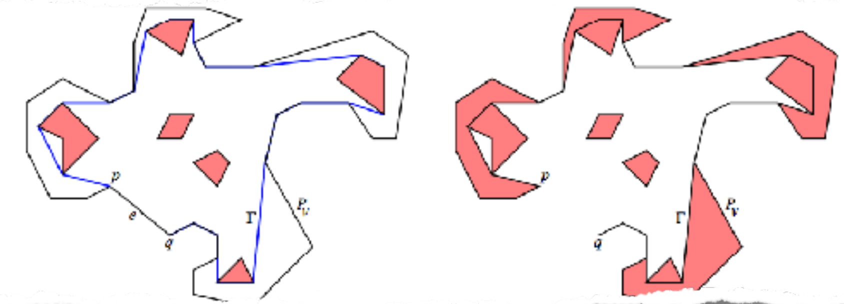

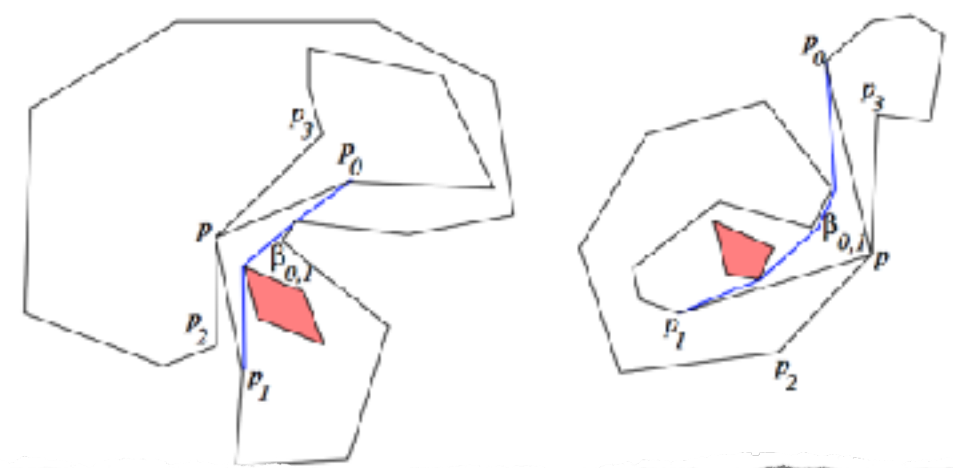

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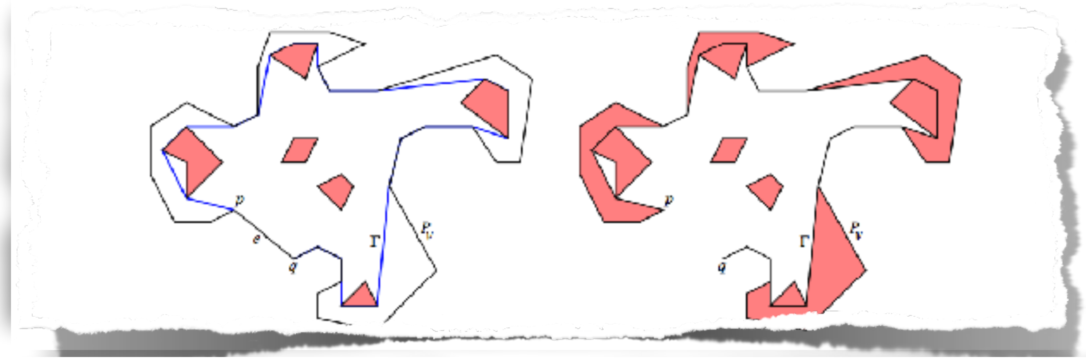

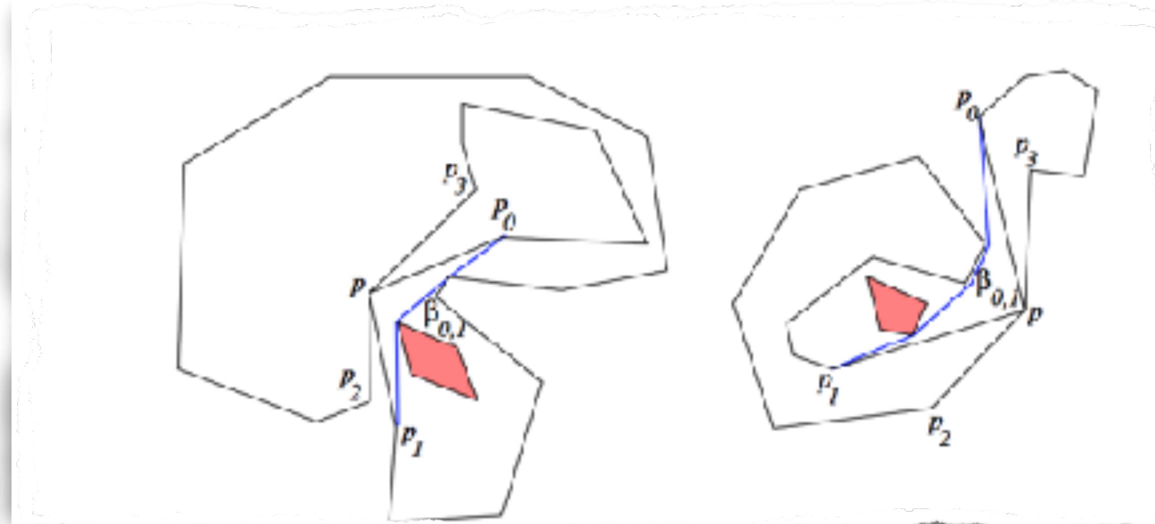

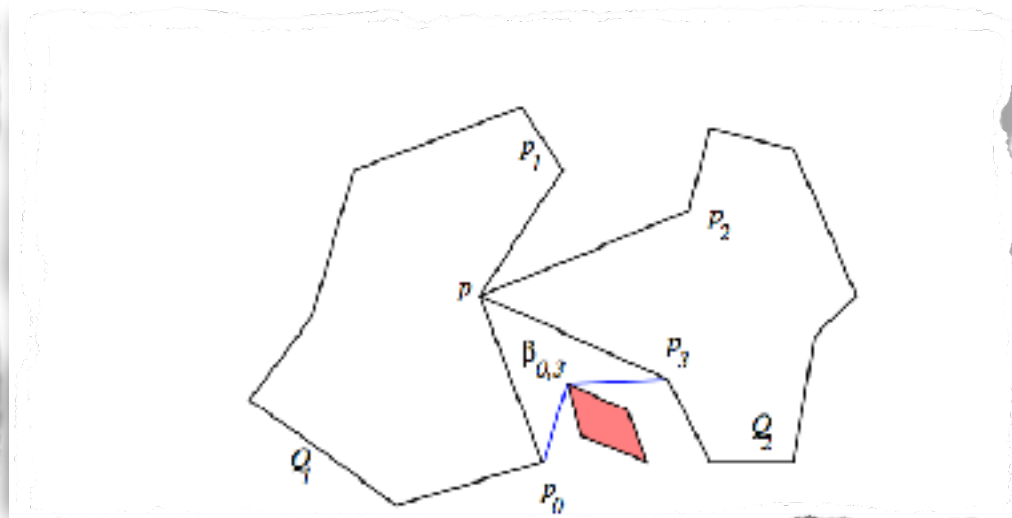

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Overall Goals

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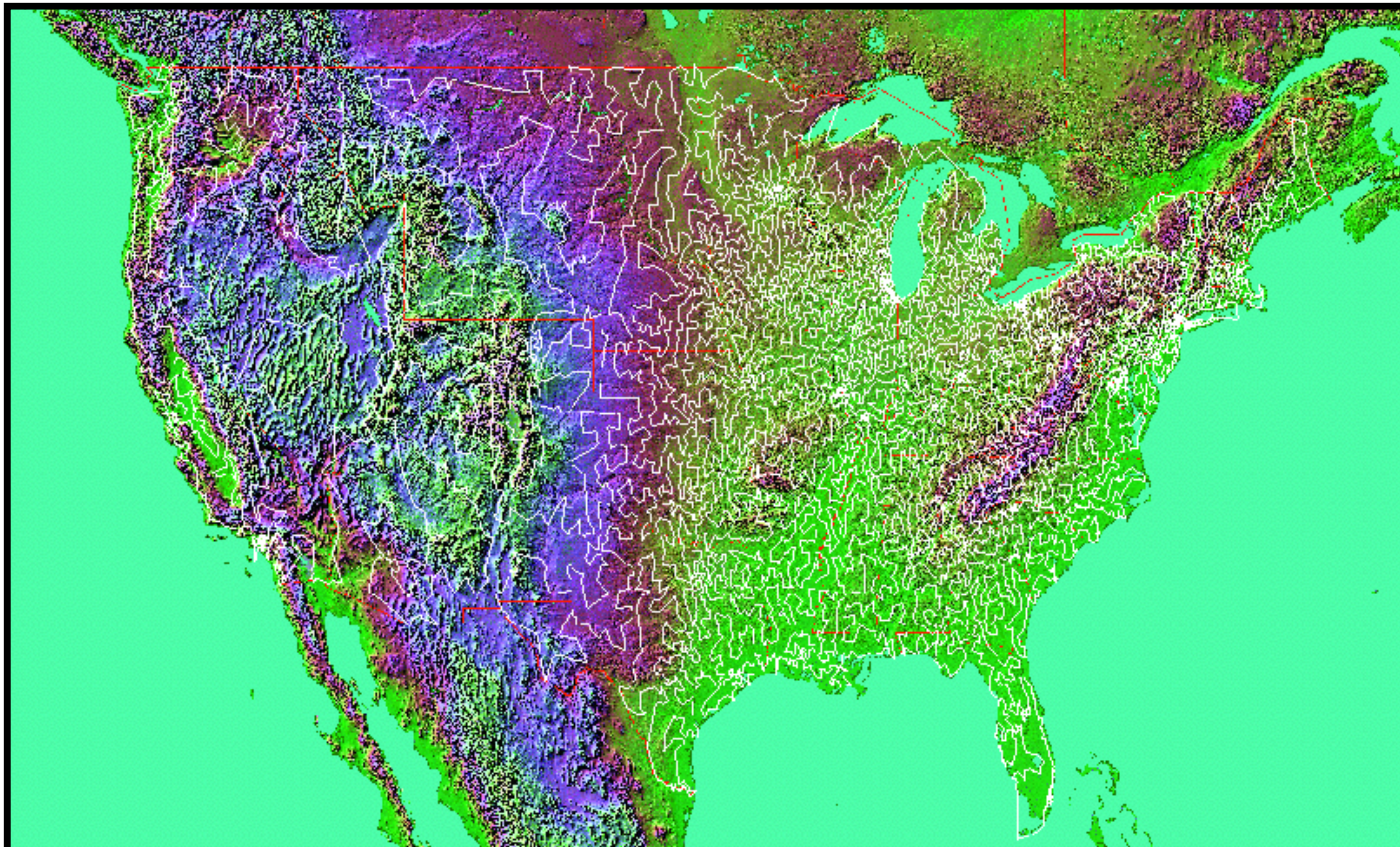

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Overall Goals

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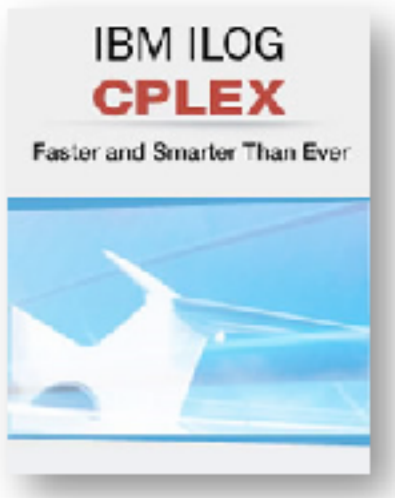
Overall Goals

CGAL

meets



Subtour Elimination (TSP)

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$$x_e \in \{0,1\}\,. \qquad (4)$$

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Subtour Elimination (TSP)

$$\min \sum_{e \in E} x_e c_e \,. \tag{1}$$

$$x_e \in \{0, 1\} \,. \tag{4}$$

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Subtour Elimination (TSP)

$$\min \sum_{e \in E} x_e c_e \,. \tag{1}$$

$$\forall v \in V : \sum_{e \in \delta(v)} x_e = 2 \,, \tag{2}$$

$$x_e \in \{0, 1\} \,. \tag{4}$$

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Subtour Elimination (TSP)

$$\min \sum_{e \in E} x_e c_e \,. \tag{1}$$

$$\forall v \in V : \sum_{e \in \delta(v)} x_e = 2 \,, \tag{2}$$

$$\forall C \in \mathcal{C} : \sum_{e \in C} x_e \leq |C| - 1 \,, \tag{3}$$

$$x_e \in \{0, 1\} \,. \tag{4}$$

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Subtour Elimination (TSP)

$$\min \sum_{e \in E} x_e c_e \,. \tag{1}$$

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Problem: Which cycles are illegal for MPP?

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Illegal Cycles

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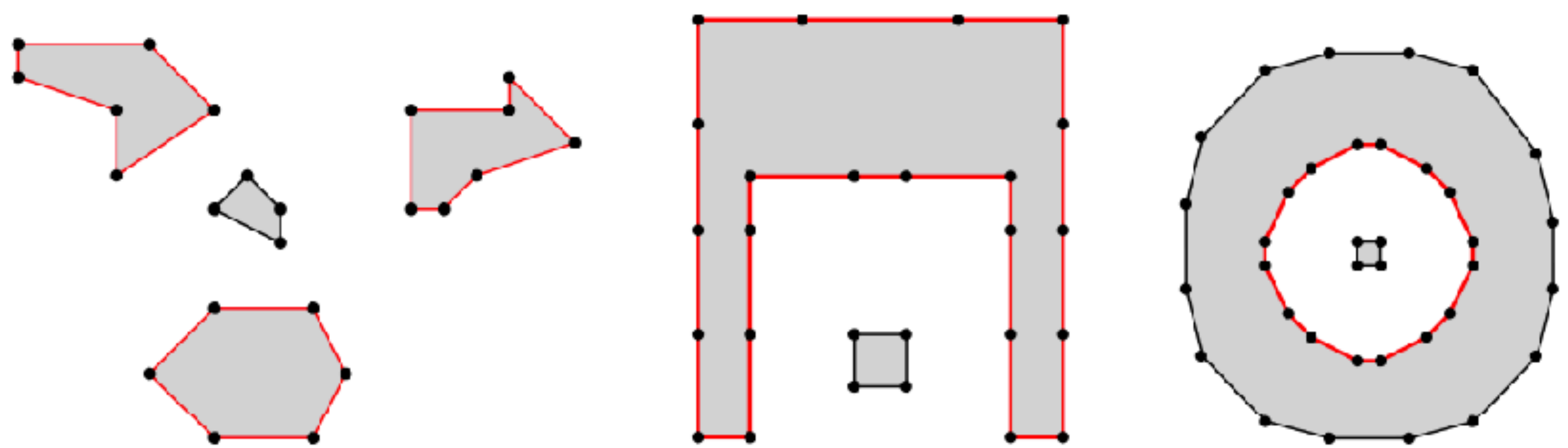
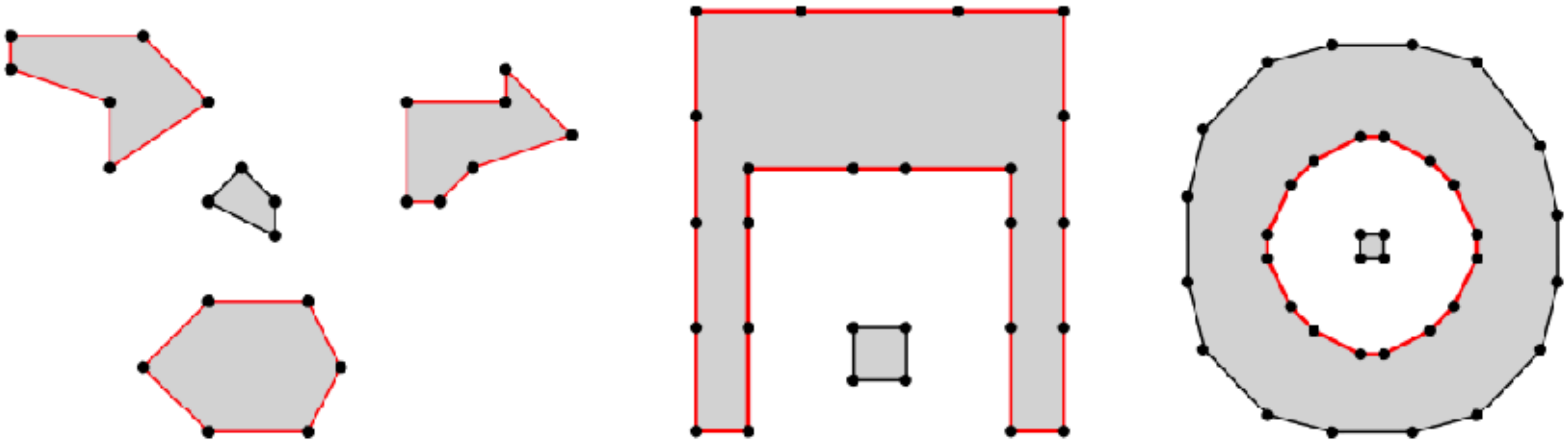
Illegal Cycles



(a) Invalid cycle of type 1 (b) Invalid cycle of type 2 (c) Invalid cycle of type 3

Illegal Cycles



(a) Invalid cycle of type 1 (b) Invalid cycle of type 2 (c) Invalid cycle of type 3

1. at least one and at most $|CH| - 1$ convex hull points. (See Figure 11(a))

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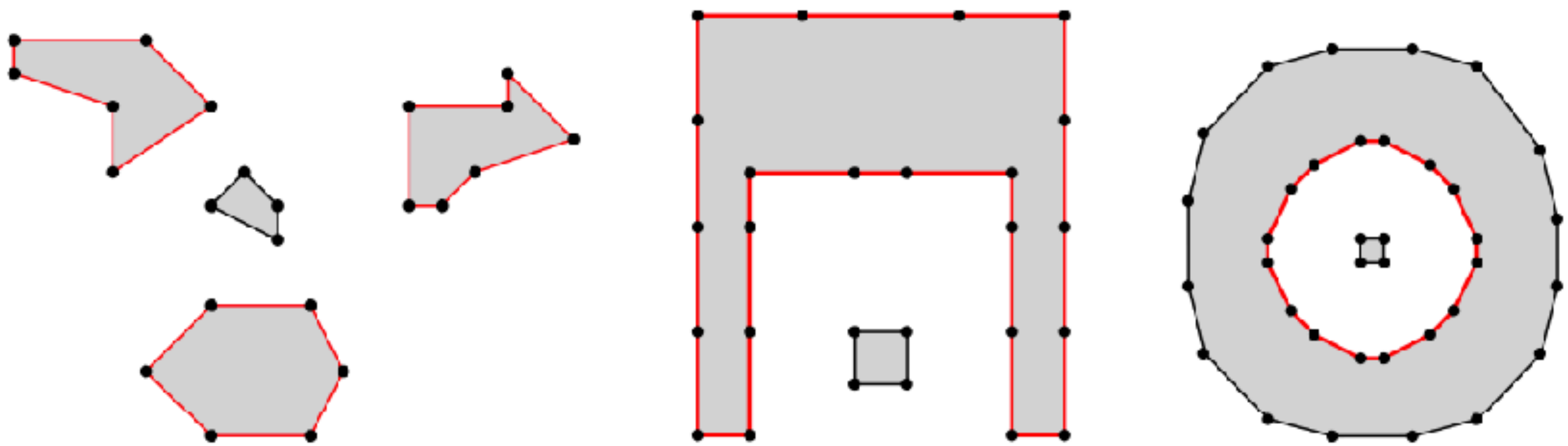
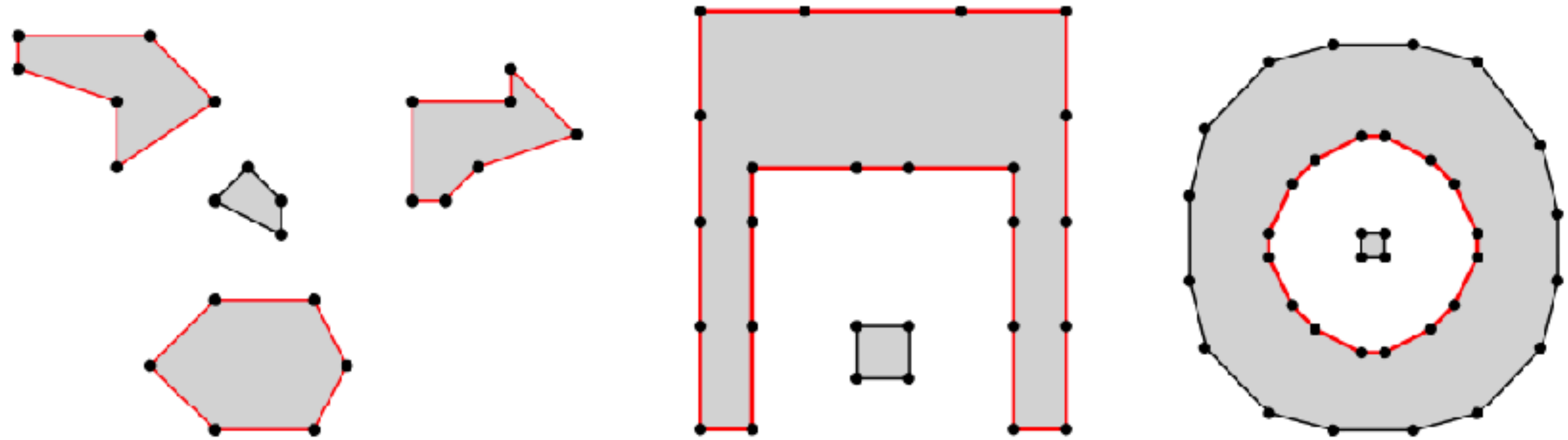
Illegal Cycles



(a) Invalid cycle of type 1 (b) Invalid cycle of type 2 (c) Invalid cycle of type 3

1. at least one and at most $|CH| - 1$ convex hull points. (See Figure 11(a))
2. all convex hull points but does not enclose all other points. (See Figure 11(b))

Illegal Cycles



(a) Invalid cycle of type 1 (b) Invalid cycle of type 2 (c) Invalid cycle of type 3

1. at least one and at most $|CH| - 1$ convex hull points. (See Figure 11(a))
2. all convex hull points but does not enclose all other points. (See Figure 11(b))
3. no convex hull point but encloses other points. (See Figure 11(c))

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Eliminating Cycles May Be Inefficient (1)

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For an invalid cycle with property 1, we use the equivalent cut constraint

$$\forall C \in \mathcal{C}_1 : \sum_{e \in \delta(C)} x_e \geq 2 \,. \tag{5}$$

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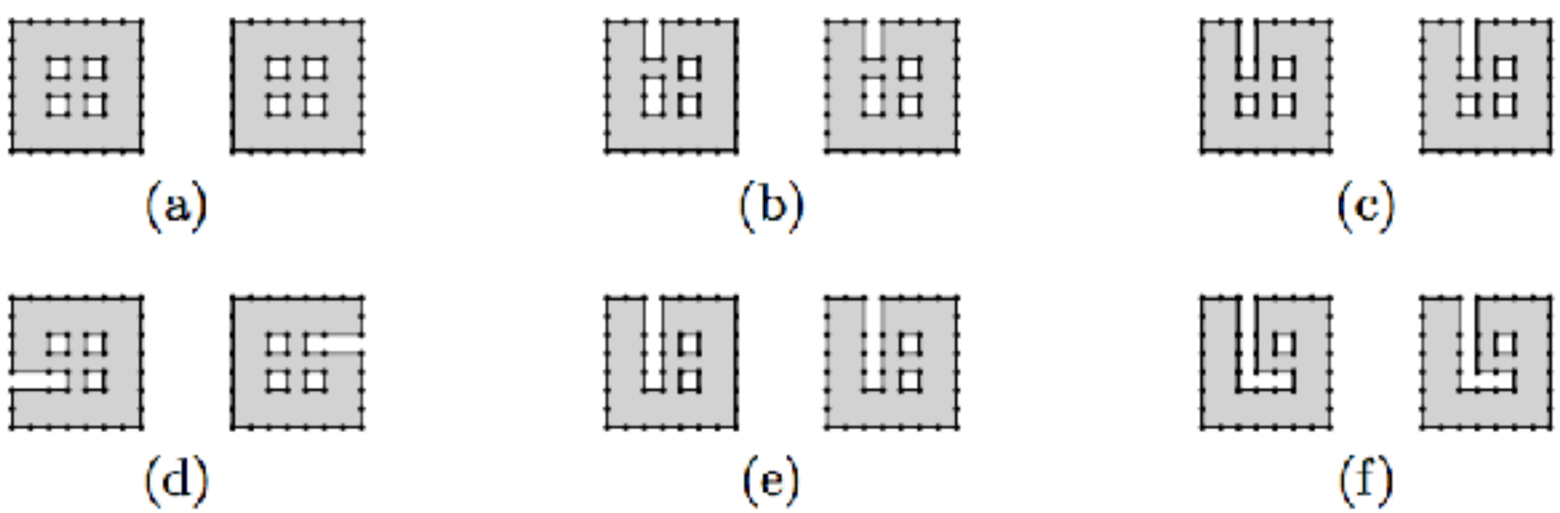

Figure 12. (a) - (f) show consecutive iterations when trying to solve an instance using only constraint (5).

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More Efficient: Glue Cuts

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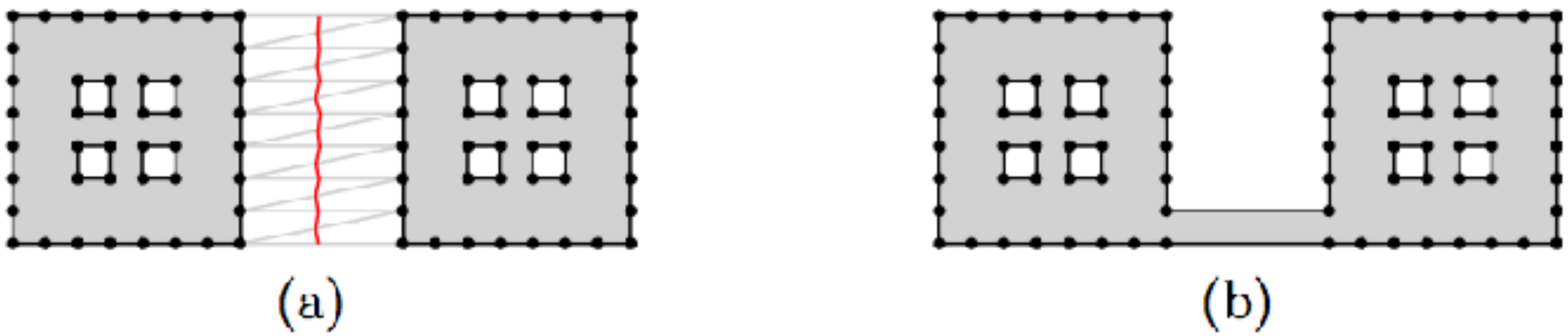

Figure 15. Solving instance from Figure 12 with a glue cut (red). (a) The red curve needs to be crossed at least twice; it is found using the Delaunay Triangulation (grey). (b) The first iteration after using the glue cut.

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TSPLIB Instances (Results)

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	BasicIP	+JS+DC +TC+HIHC	+JS+TC +HIHC	+JS+DC +HIHC	+JS+DC +TC	+DC+TC +HIHC
burma14	20	22	17	19	26	19
ulysses16	48	42	35	43	32	42
ulysses22	50	34	55	31	32	61
att48	180	58	72	62	57	129
eil51	74	82	72	78	81	99
berlin52	43	38	37	37	38	51
st70	-	329	324	-	348	414
eil76	714	144	105	530	148	239
pr76	-	711	711	-	731	1238
gr96	376	388	349	10982	384	367
rat99	922	480	485	464	513	1190
kroA100	-	961	689	-	950	1294
kroB100	-	1470	2623	-	1489	2285
kroC100	-	470	431	-	465	577
kroD100	4673	509	451	4334	514	835
kroE100	-	273	273	-	272	574
rd100	-	894	756	-	890	2861
eil101	-	575	445	-	527	1090
lin105	-	390	359	-	412	931
pr107	550	401	272	346	513	923
pr124	495	348	264	322	355	940
bier127	439	288	270	267	276	476
ch130	-	1758	1802	-	1594	2853
pr136	1505	964	1029	992	950	3001
gr137	-	1262	1361	-	1252	1724
pr144	6276	1028	2926	985	1030	2012
ch150	-	4938	5167	-	5867	7997
kroA150	-	3427	5615	-	3327	7474
kroB150	-	2993	2396	-	2943	5265
pr152	13285	2161	1619	10978	2151	19479
u159	13285	1424	1262	5339	1410	2513
rat195	106030	16188	19780	77216	16117	27580

	BasicIP	+JS+DC +TC+HIHC	+JS+TC +HIHC	+JS+DC +HIHC	+JS+DC +TC	+DC+TC +HIHC
d198	-	19329	155550	-	19398	41118
kroA200	-	26360	13093	-	26389	11844
kroB200	-	5492	6239	-	5525	15238
gr202	-	4975	7512	-	4304	9670
ts225	18902	7746	9750	7595	7603	60167
tsp225	91423	11600	9741	28756	11531	44297
pr226	-	8498	2800	-	7204	18848
gr229	-	5462	26478	-	10153	25674
gil262	-	23000	22146	-	-	72772
pr264	24690	6537	-	6719	6549	23641
a280	22023	3601	3857	3980	3619	12983
pr299	-	16251	355323	-	16173	85789
lin318	-	23863	1511219	-	24035	75312
linhp318	-	23107	1313680	-	23064	79352
rd400	-	111128	92995	-		302363
fl417	-	198013	-	-	215210	825808
gr431	-	56716	173609	-	78133	265416
pr439	-	46685	36592	-	48231	273873
pcb442	-	1356796	-	-	-	-
d493	-	359072	-	-	-	837229
att532	-	217679	256394	-	218665	817096
ali535	-	93771	427800	-	91828	323104
u574	-	371523	199114	-	-	1010276
rat575	-	417494	191198	-	580320	934988
p654	-	864066	-	-	-	-
d657	-	455378	253374	-	646148	1352747
gr666	-	366157	-	-	670818	-

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Other Instances

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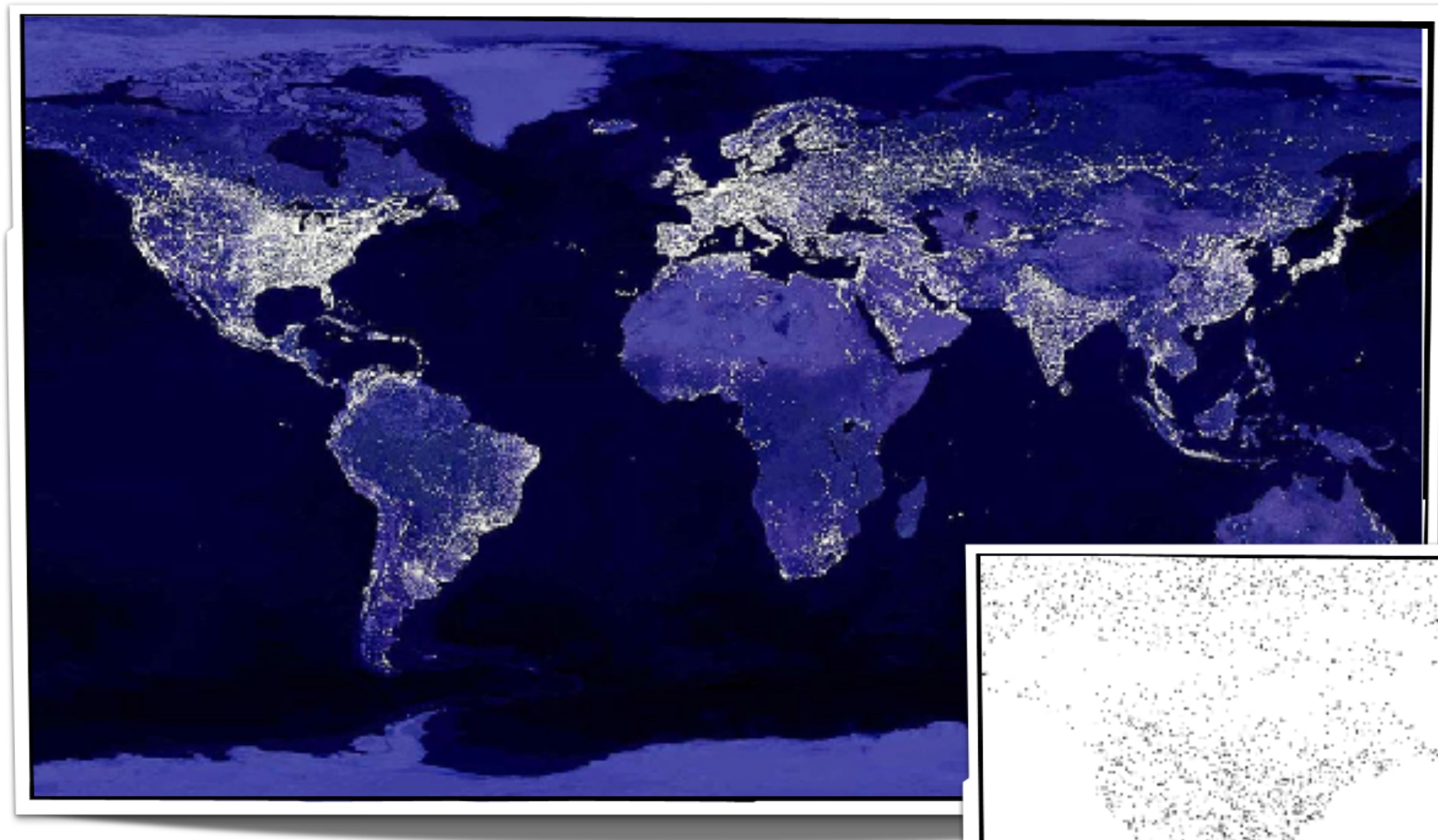
Other Instances



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A Sparser Version: Delaunay Edges

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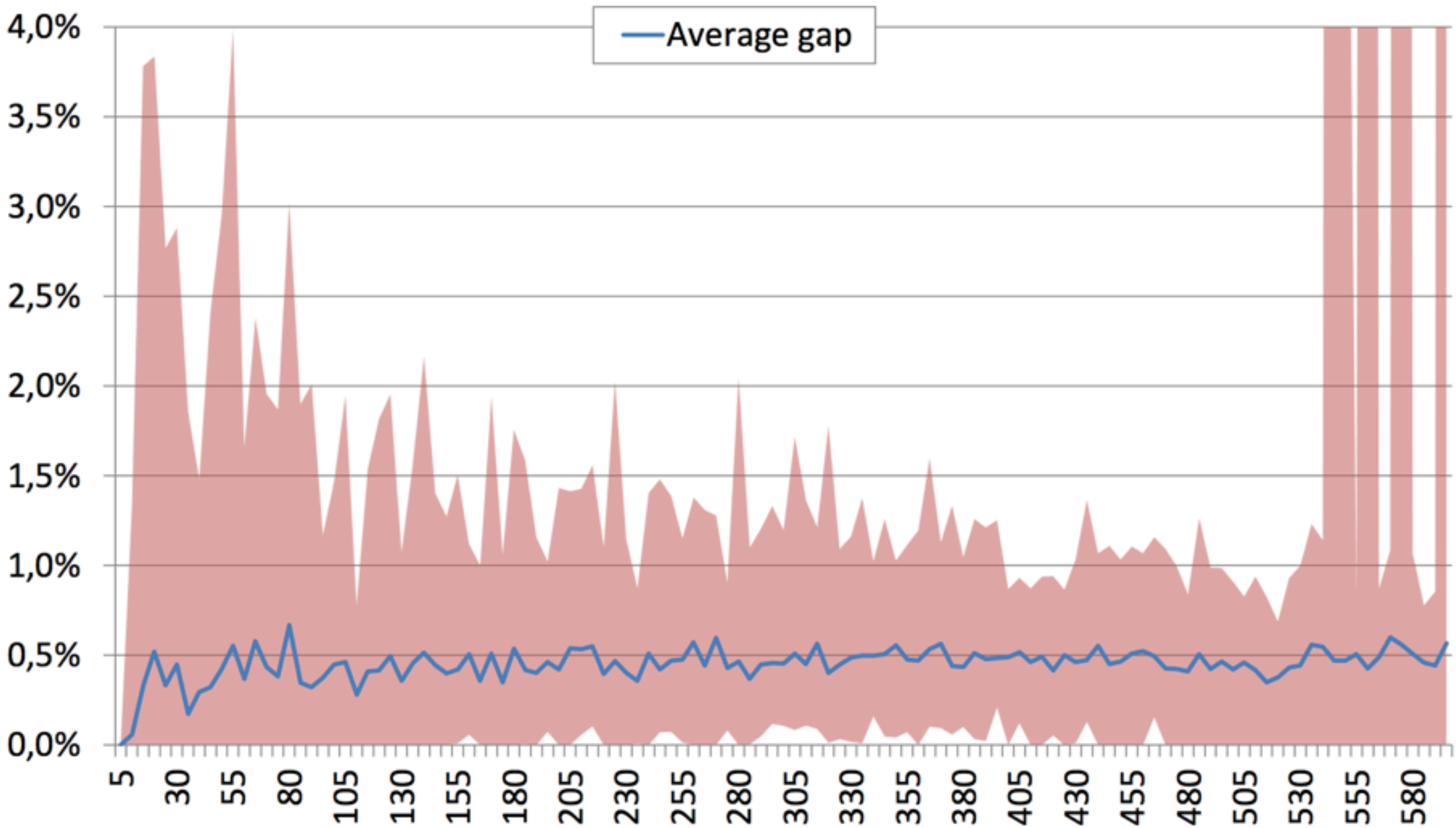

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Optimal Polygonizations

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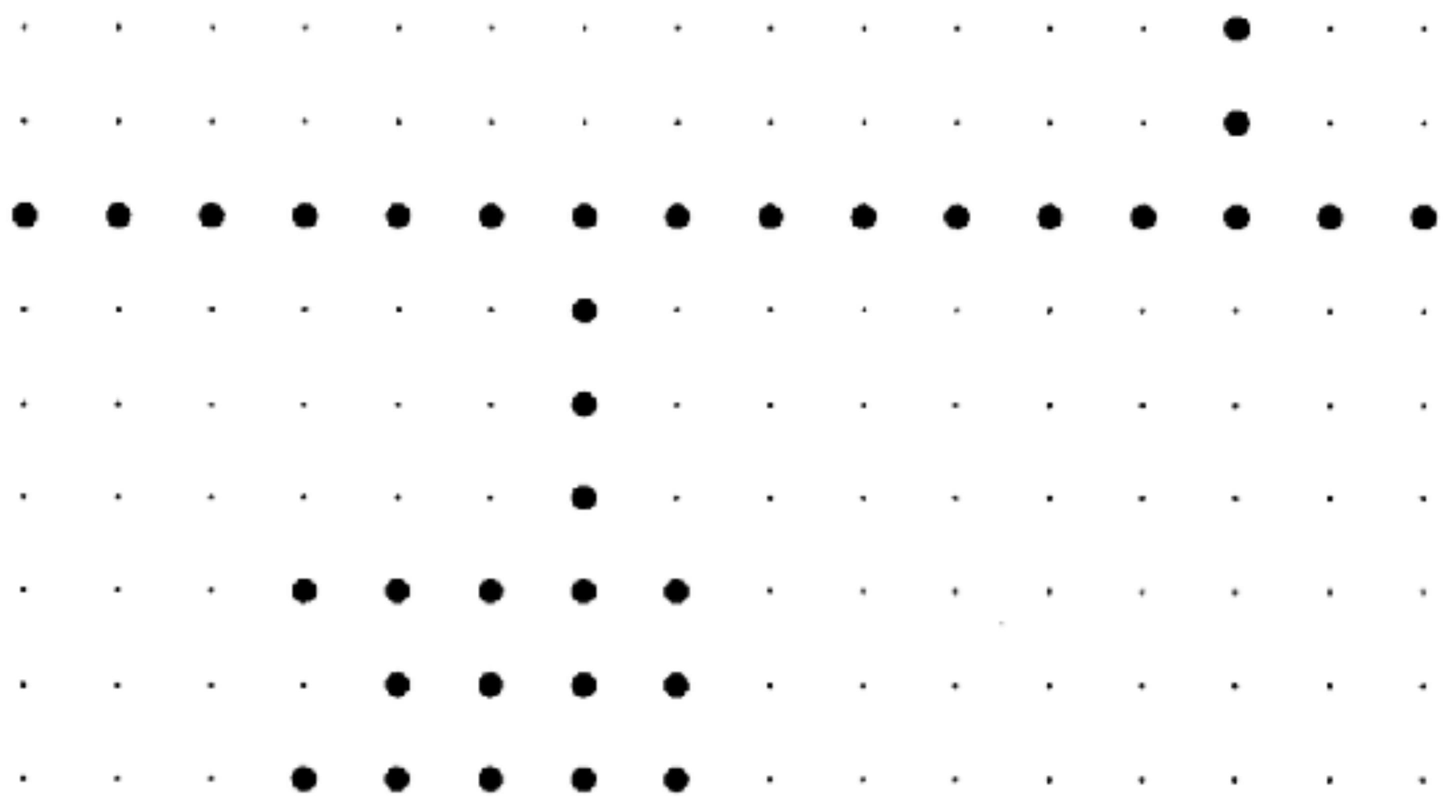
Optimal Polygonizations

Given: A planar point set P

Optimal Polygonizations

Given: A planar point set *P*



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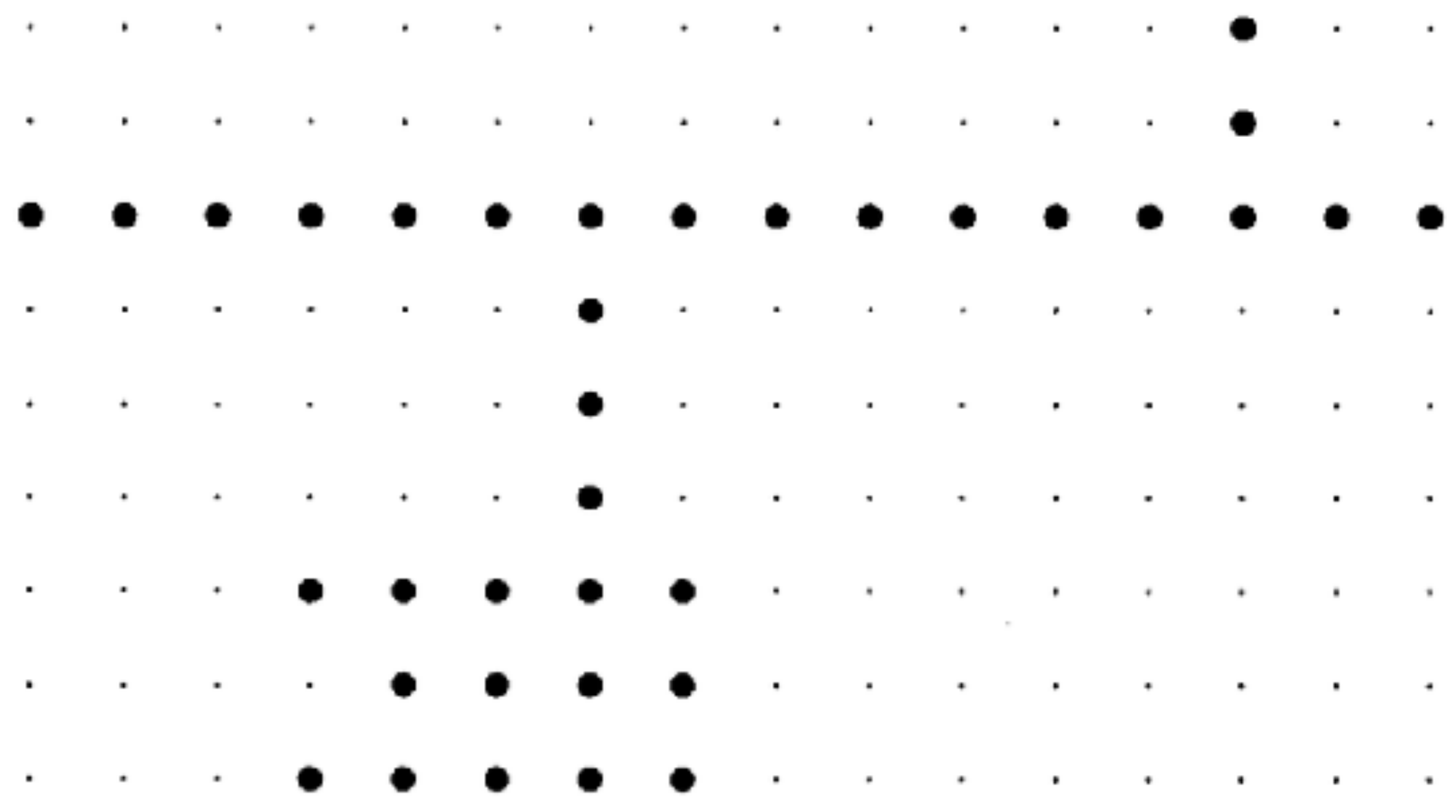
Optimal Polygonizations

Given: A planar point set *P*

Wanted: A simple polygon with vertex set *P*



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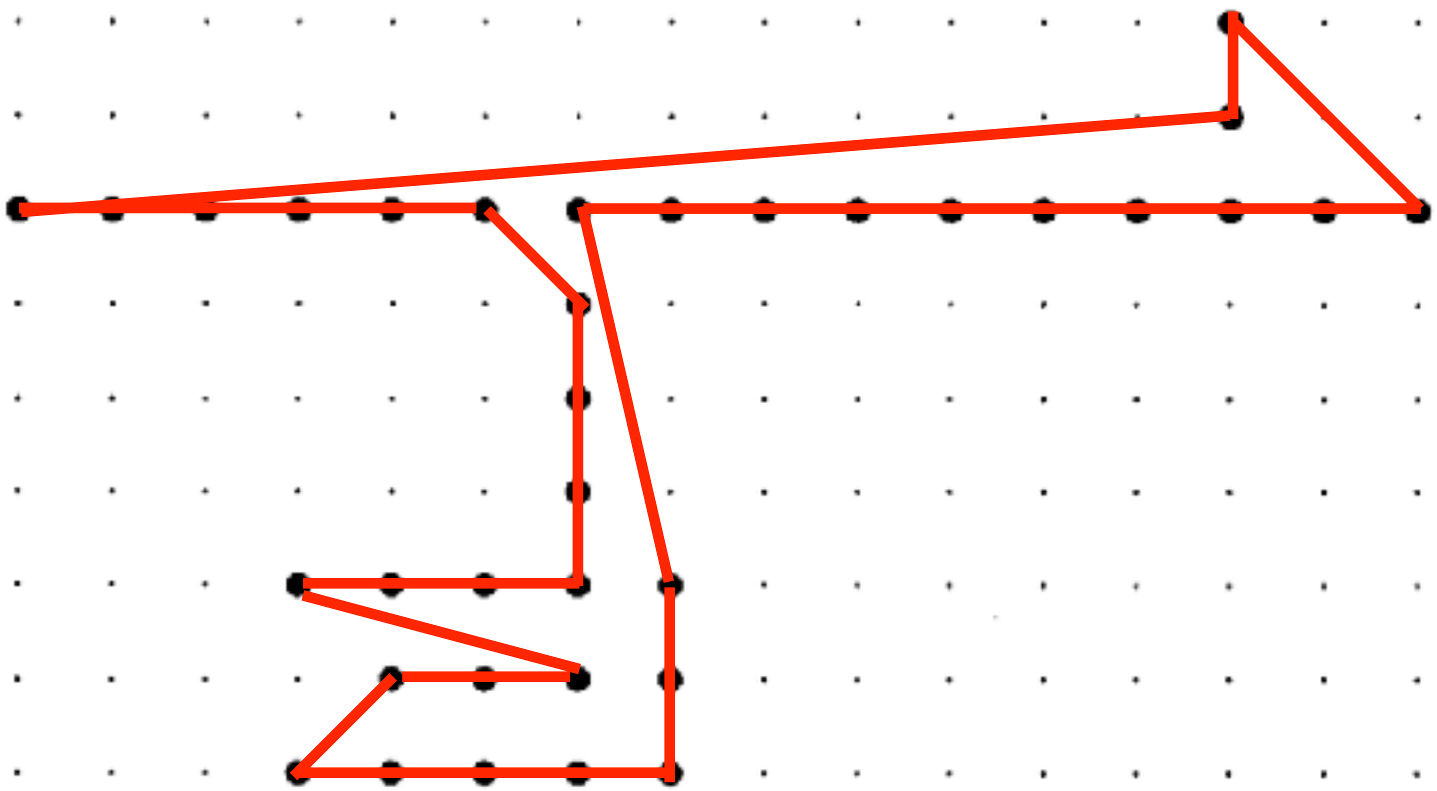

Technische Universität Braunschweig

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Given: A planar point set *P*

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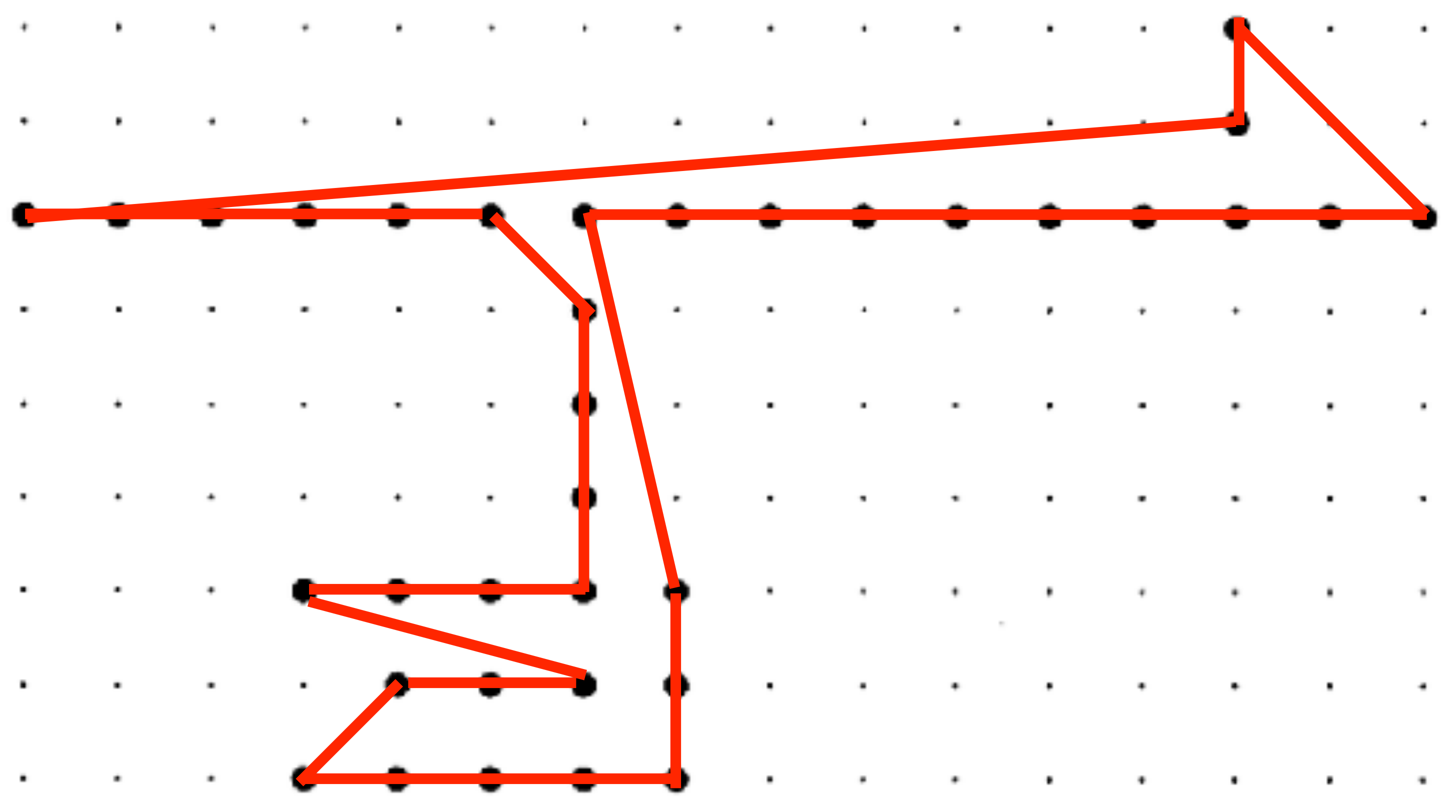
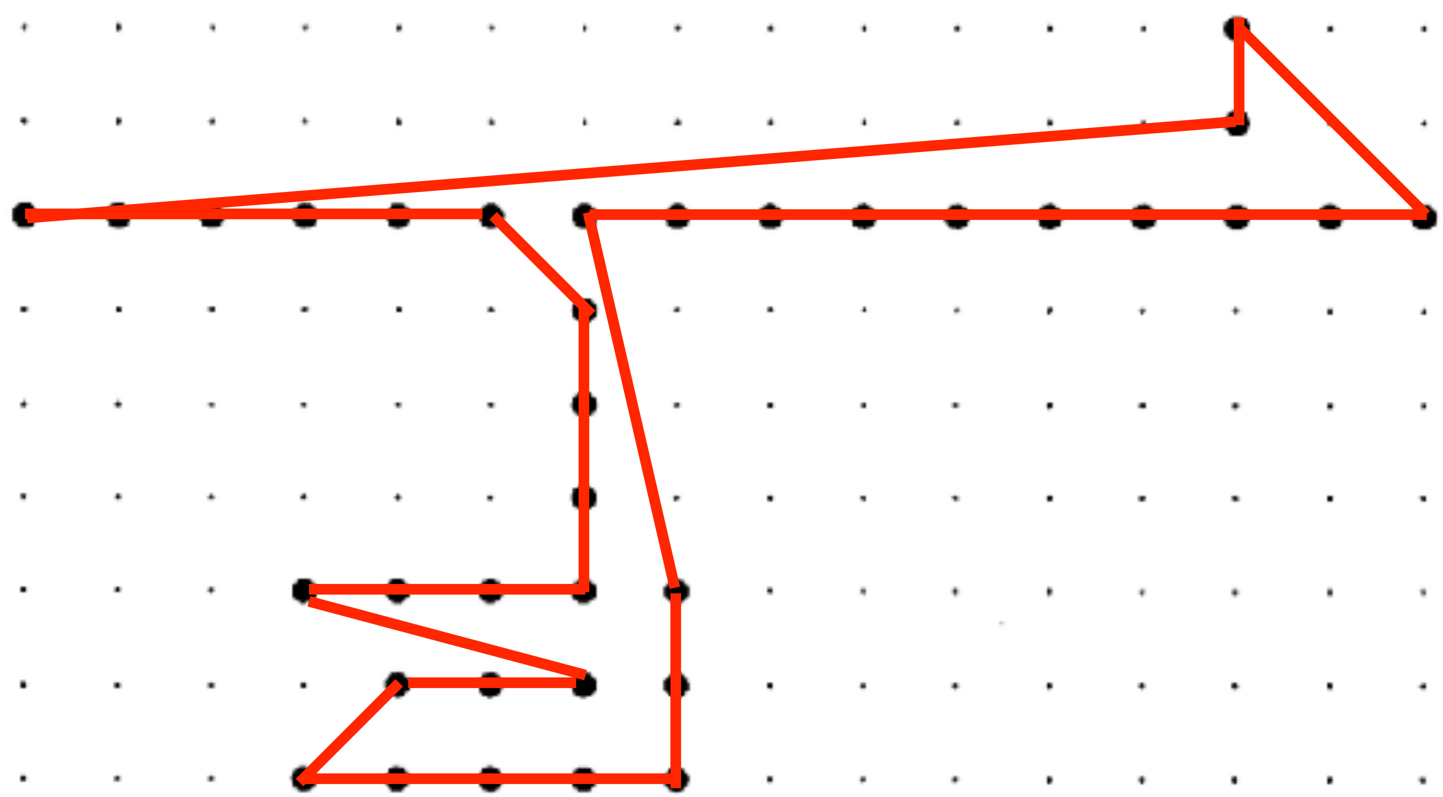


Objective:

Optimal Polygonizations

Given: A planar point set *P*

Wanted: A simple polygon with vertex set *P*



Objective: Minimize Perimeter

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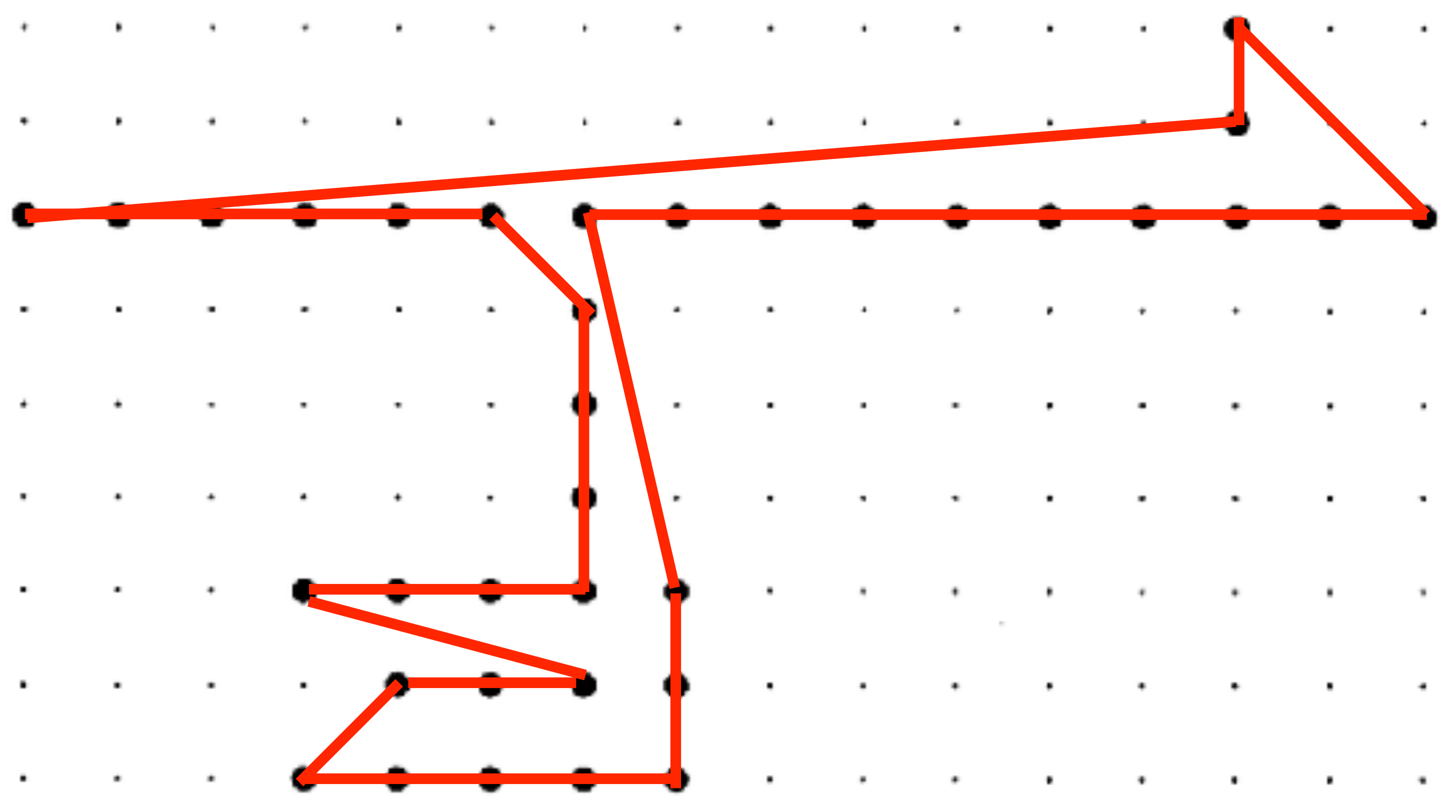
Optimal Polygonizations

Given: A planar point set *P*

Wanted: A simple polygon with vertex set *P*

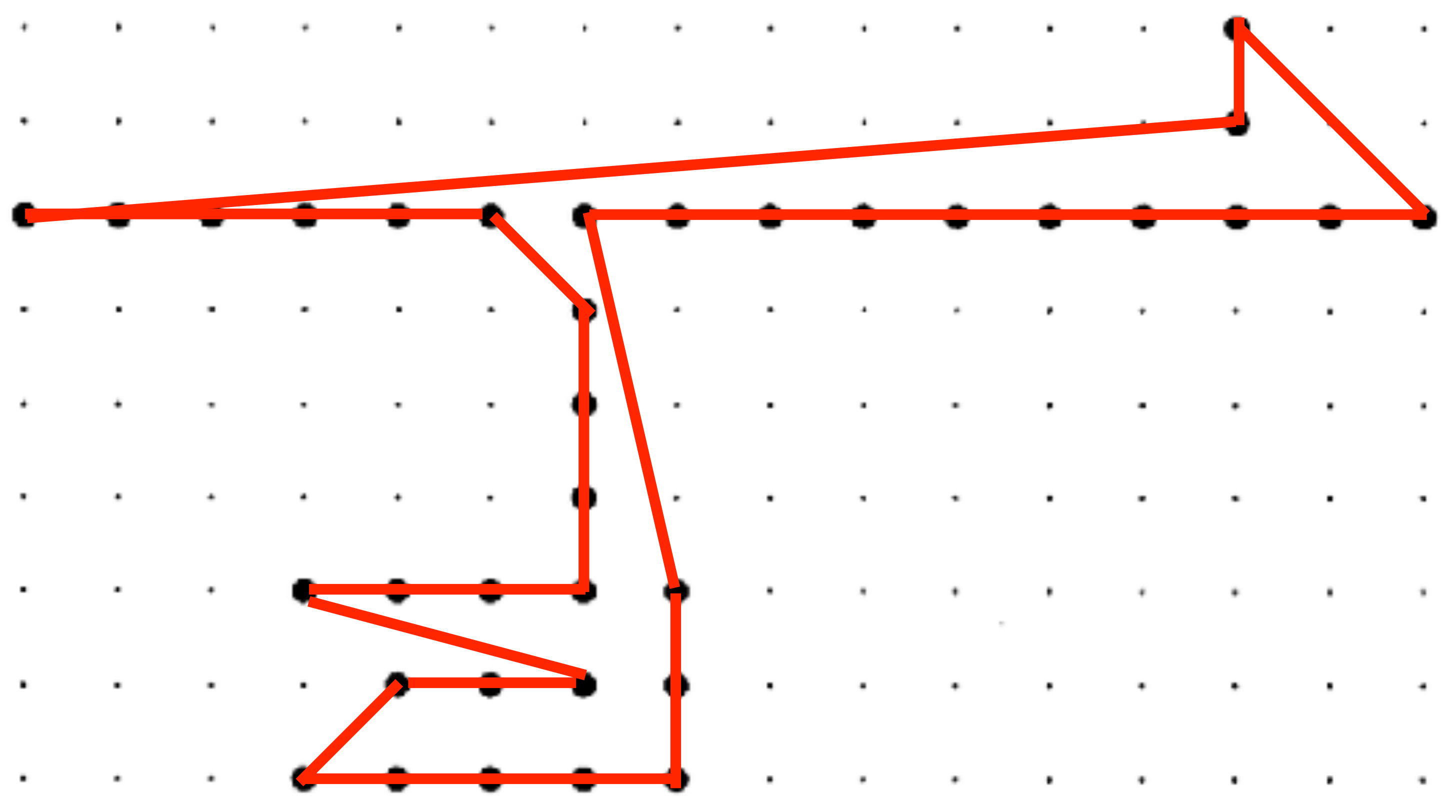


Objective: Minimize Perimeter ⟶ Traveling Salesman Problem

Optimal Polygonizations

Given: A planar point set P

Wanted: A simple polygon with vertex set P

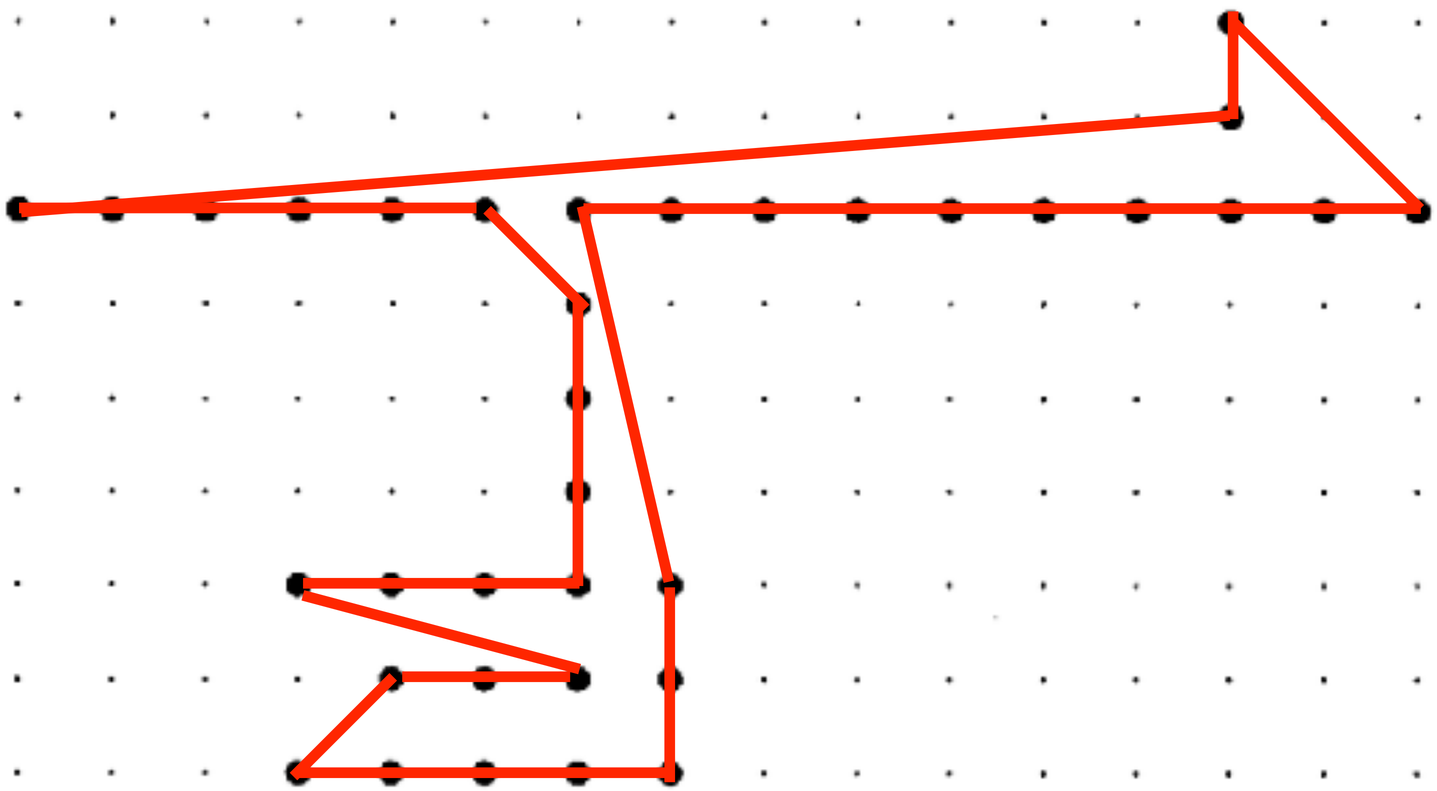


Objective:

Optimal Polygonizations

Given: A planar point set *P*

Wanted: A simple polygon with vertex set *P*



Objective: Minimize Area!

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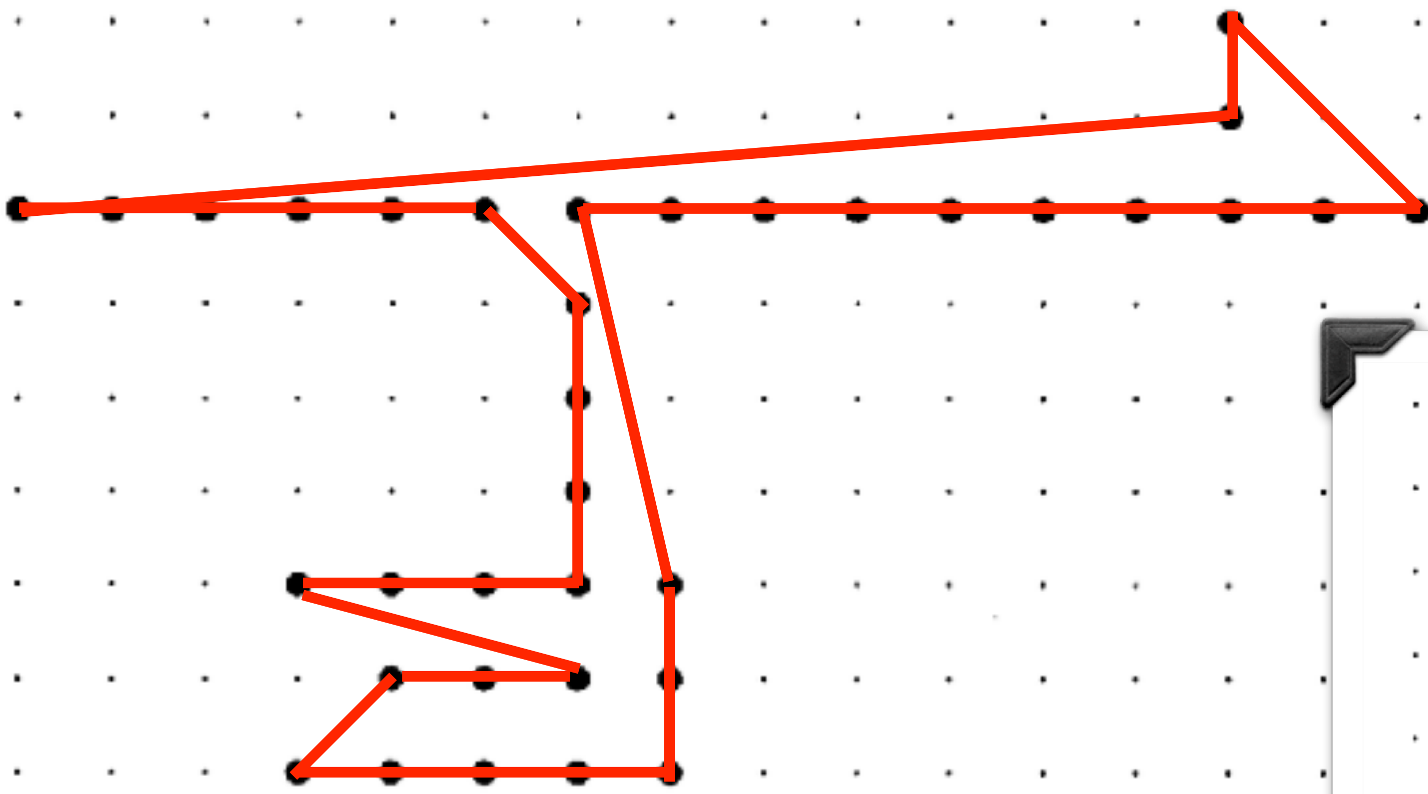
Optimal Polygonizations

Given: A planar point set *P*

Wanted: A simple polygon with vertex set *P*



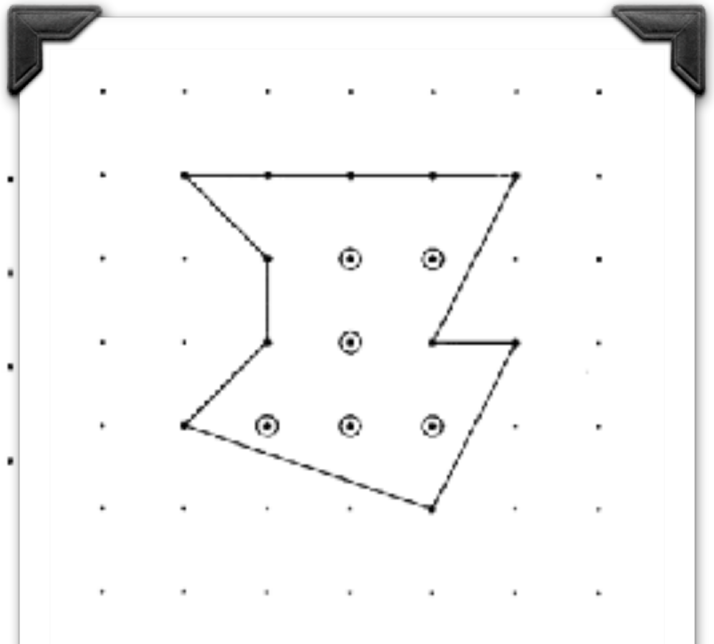


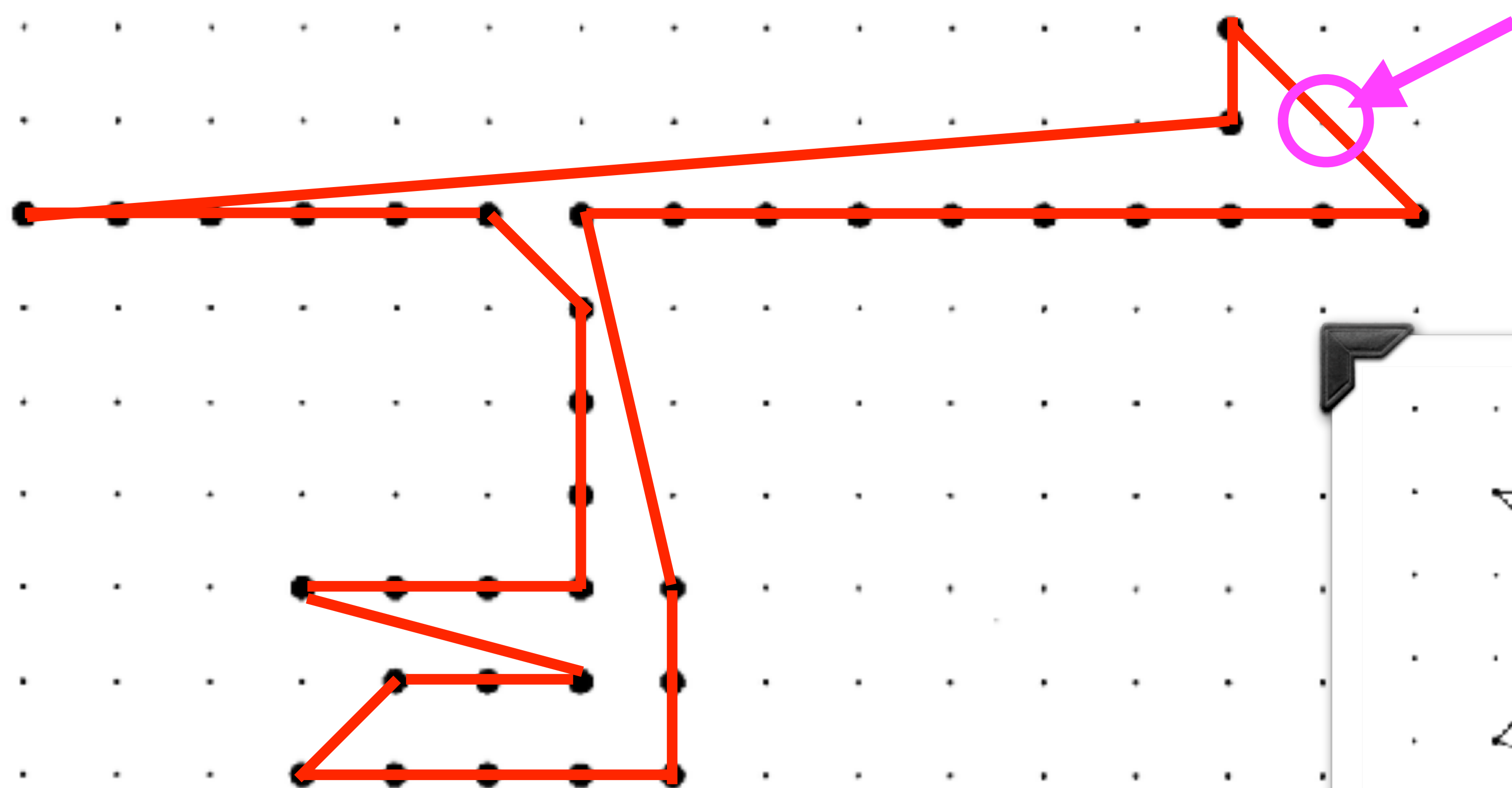
Fig. 2. Pick's theorem.

Objective: Minimize Area!

Optimal Polygonizations

Given: A planar point set *P*

Wanted: A simple polygon with vertex set *P*



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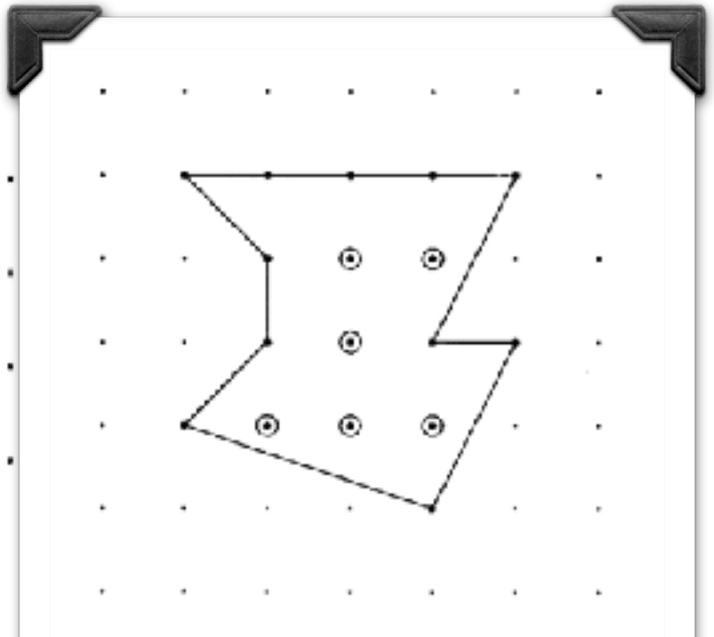

Fig. 2. Pick's theorem.

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Optimal Polygonizations

Given: A planar point set *P*

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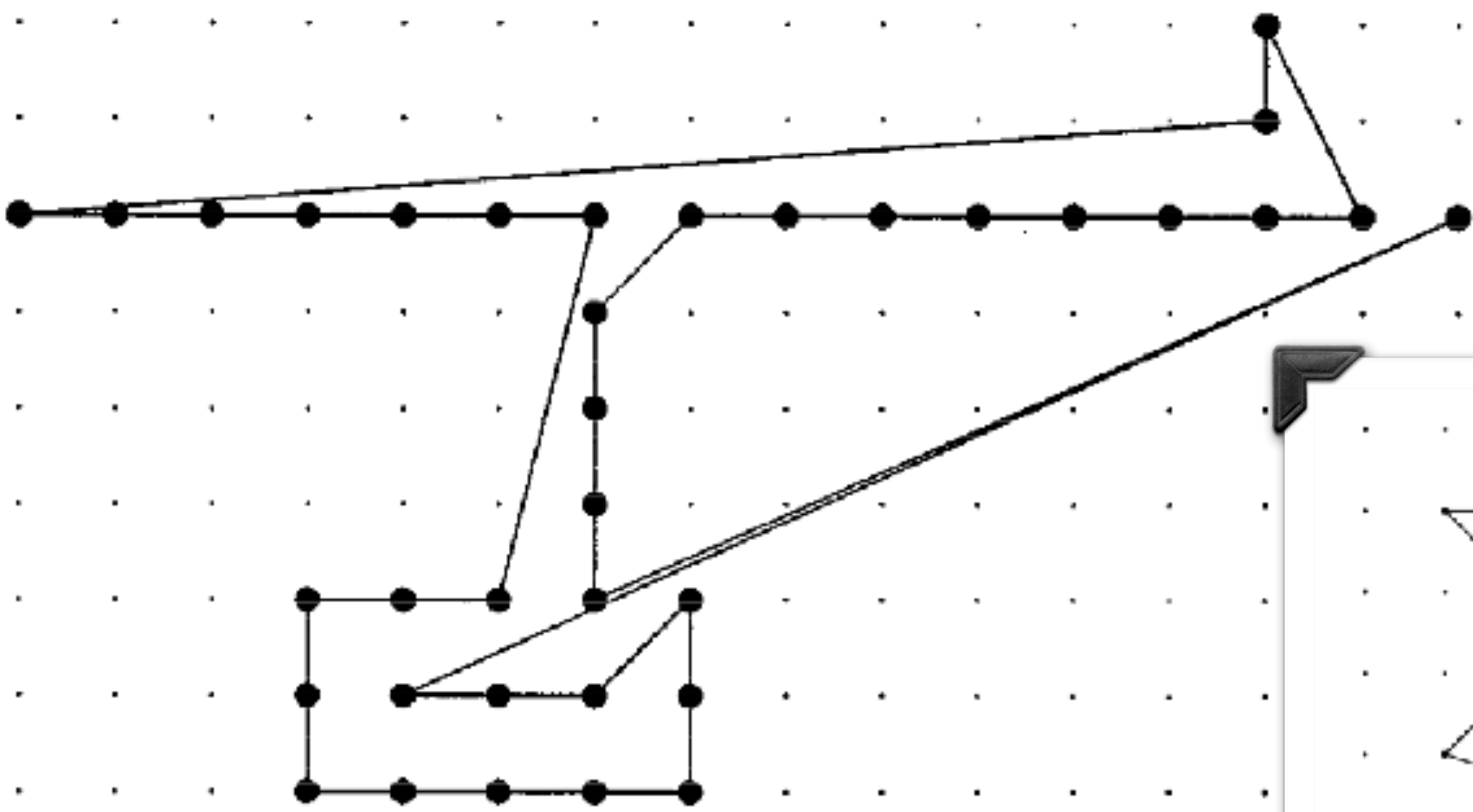
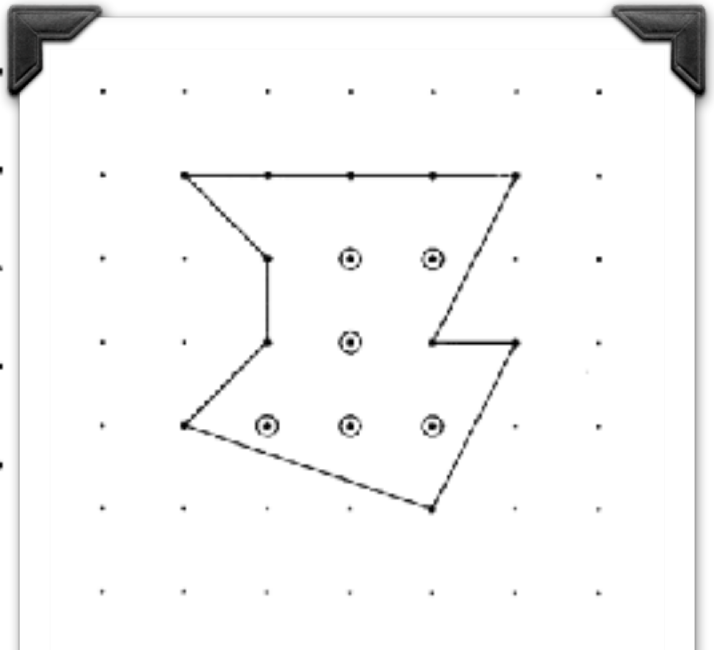


Objective: Minimize Area!



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Some History

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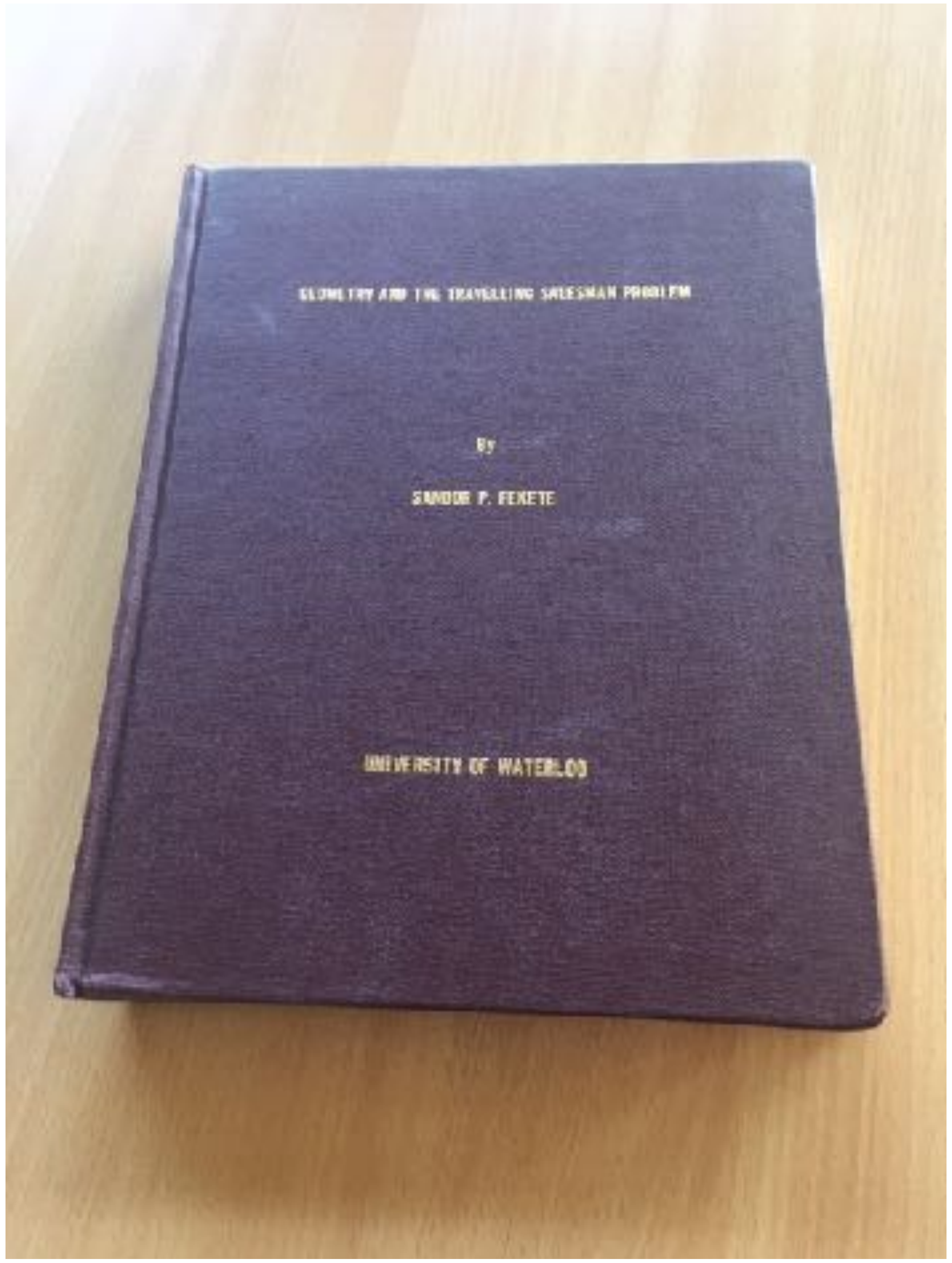

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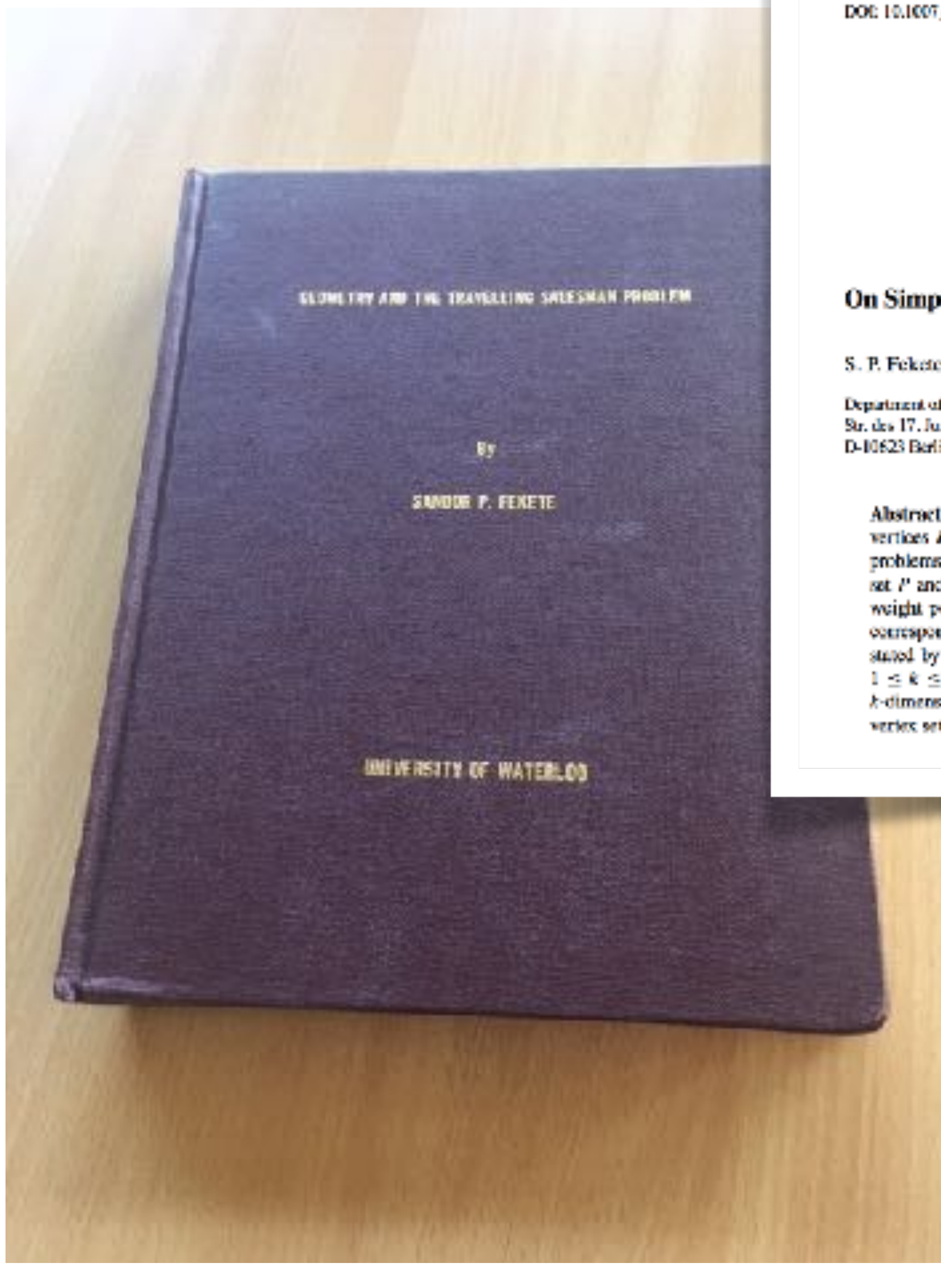

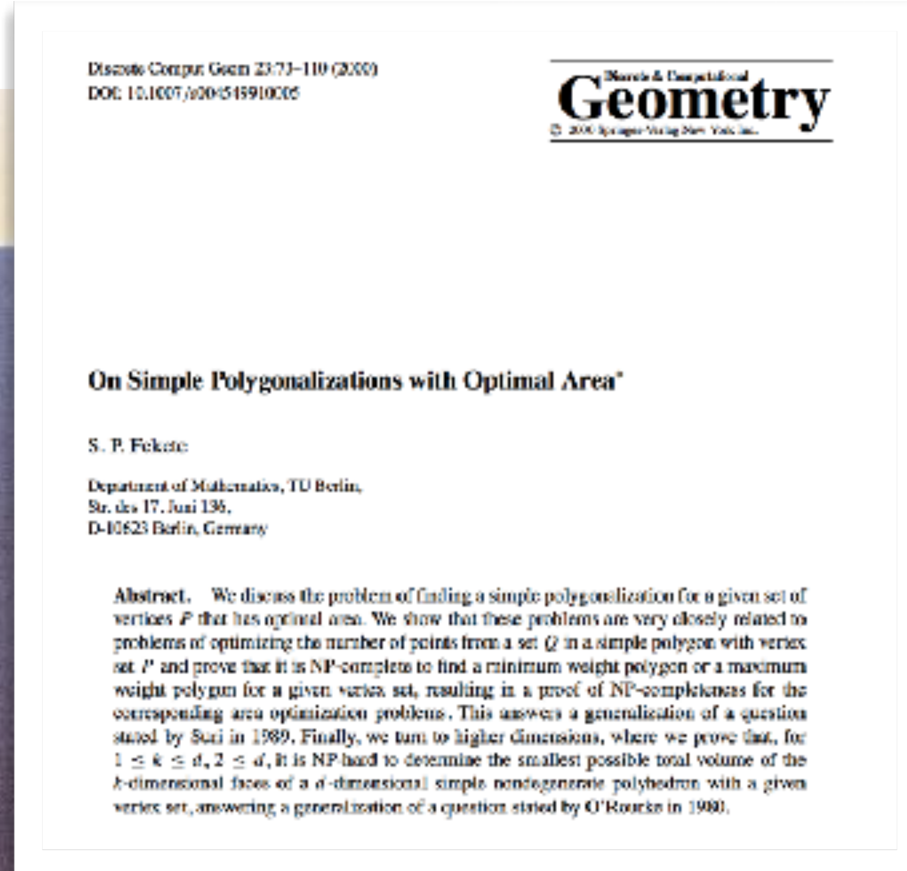
Discrete Comput Geom 23:73–110 (2000)
DOI: 10.1007/s004549910005

Discrete & Computational Geometry

On Simple Polygonalizations with Optimal Area*

S. P. Fekete

Department of Mathematics, TU Berlin,
Str. des 17. Juni 136,
D-10623 Berlin, Germany

Abstract. We discuss the problem of finding a simple polygonalization for a given set of vertices P that has optimal area. We show that these problems are very closely related to problems of optimizing the number of points from a set Q in a simple polygon with vertex set P and prove that it is NP-complete to find a minimum weight polygon or a maximum weight polygon for a given vertex set, resulting in a proof of NP-completeness for the corresponding area optimization problems. This answers a generalization of a question stated by Suri in 1989. Finally, we turn to higher dimensions, where we prove that, for $1 \le k \le d$, $2 \le d$, it is NP-hard to determine the smallest possible total volume of the k-dimensional faces of a d-dimensional simple nondegenerate polyhedron with a given vertex set, answering a generalization of a question stated by O'Rourke in 1980.

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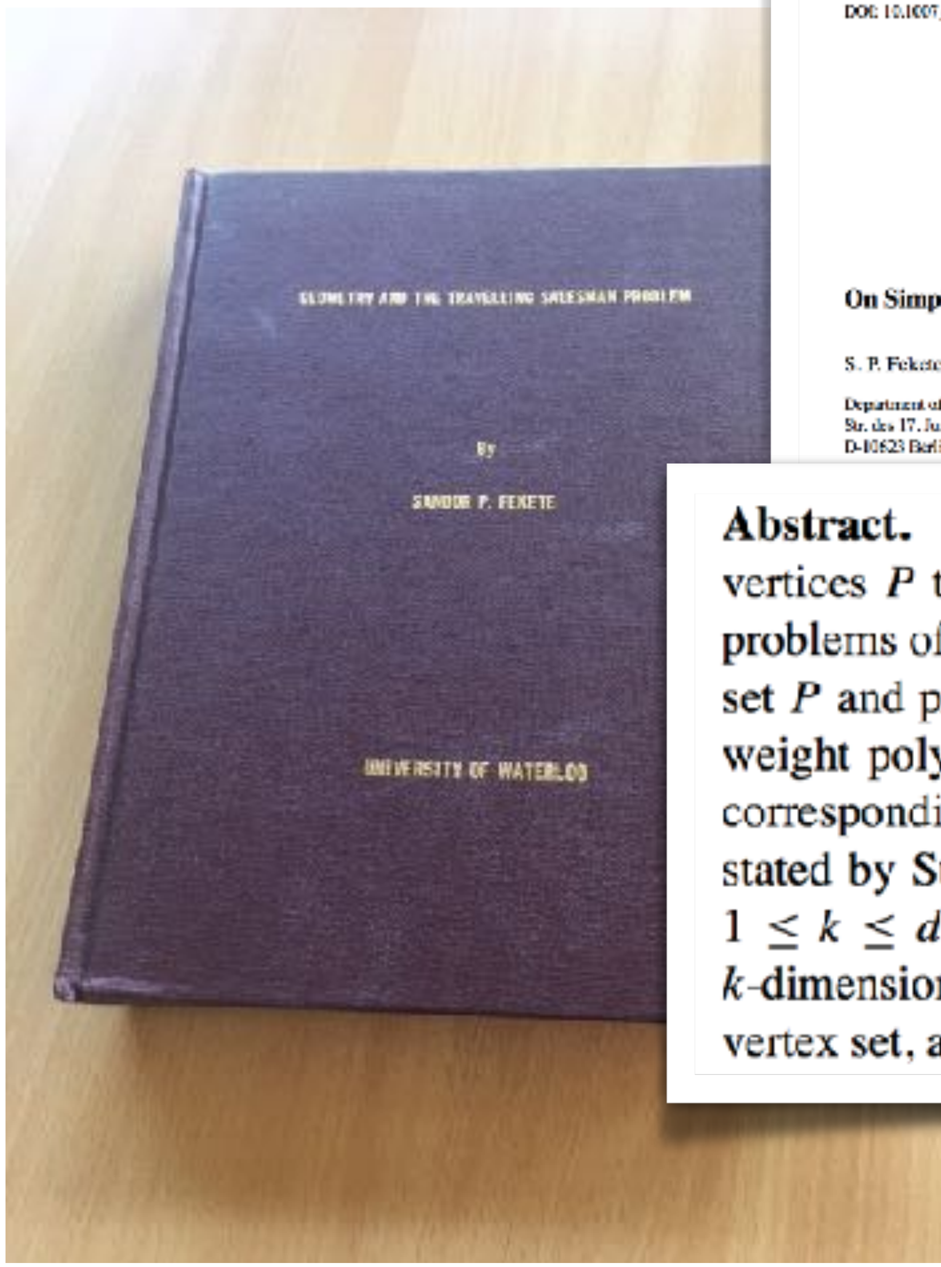

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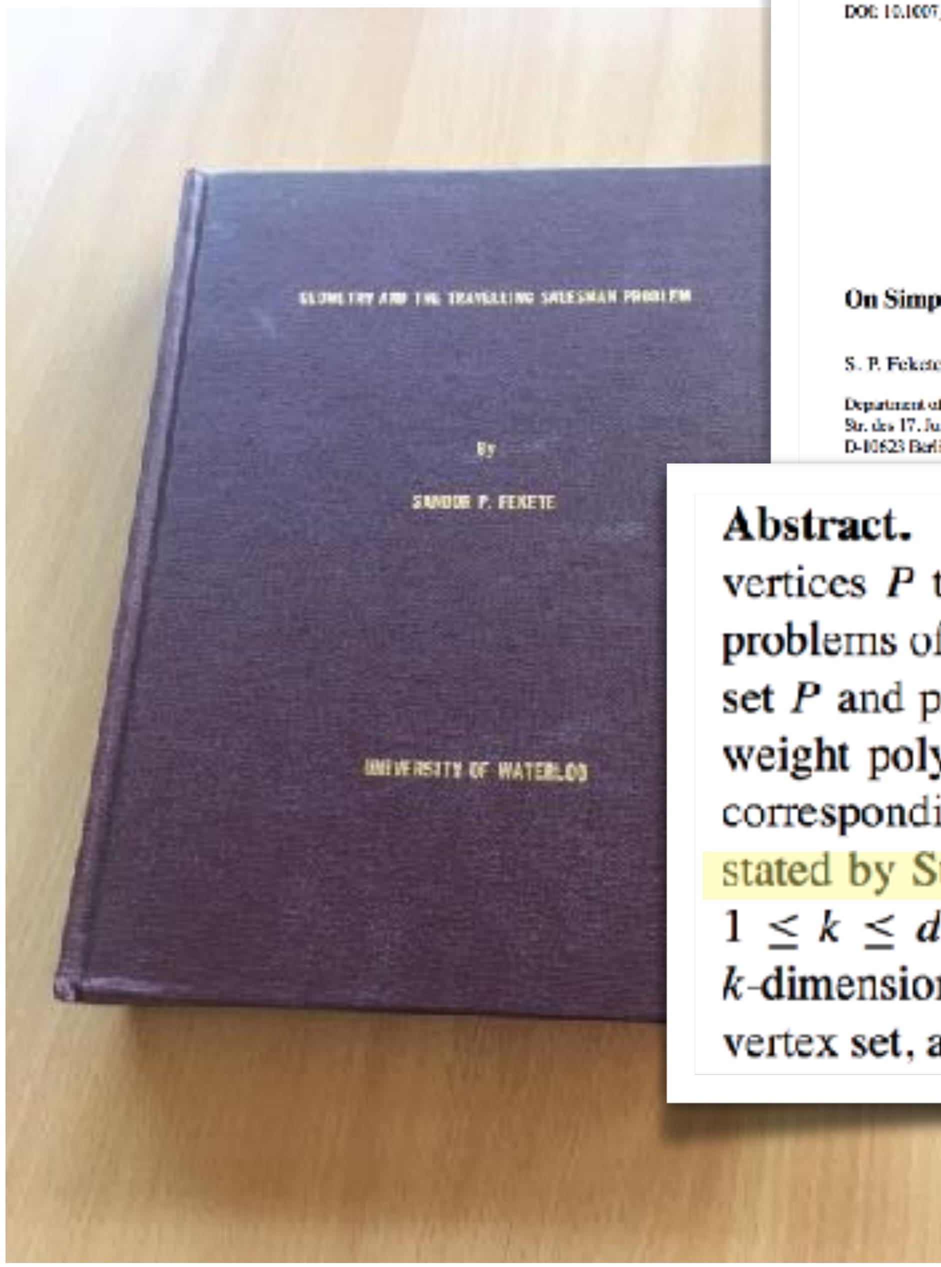

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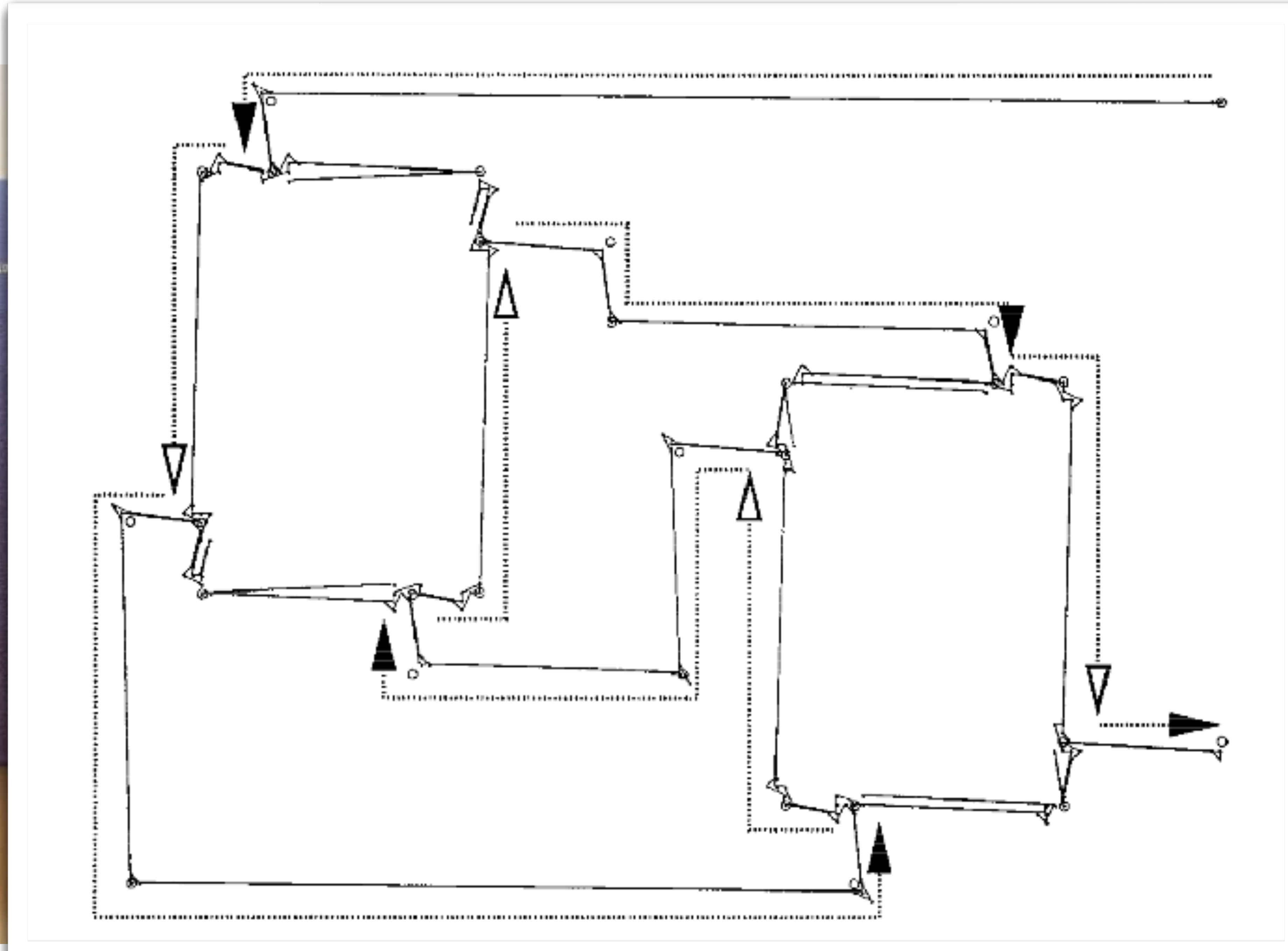

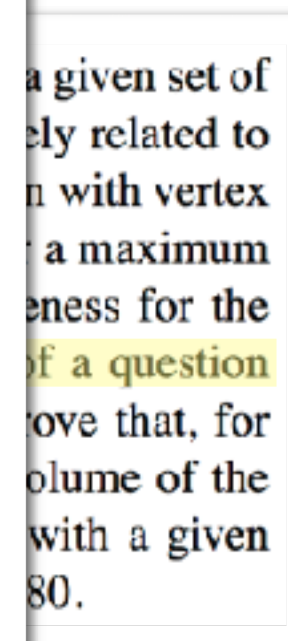

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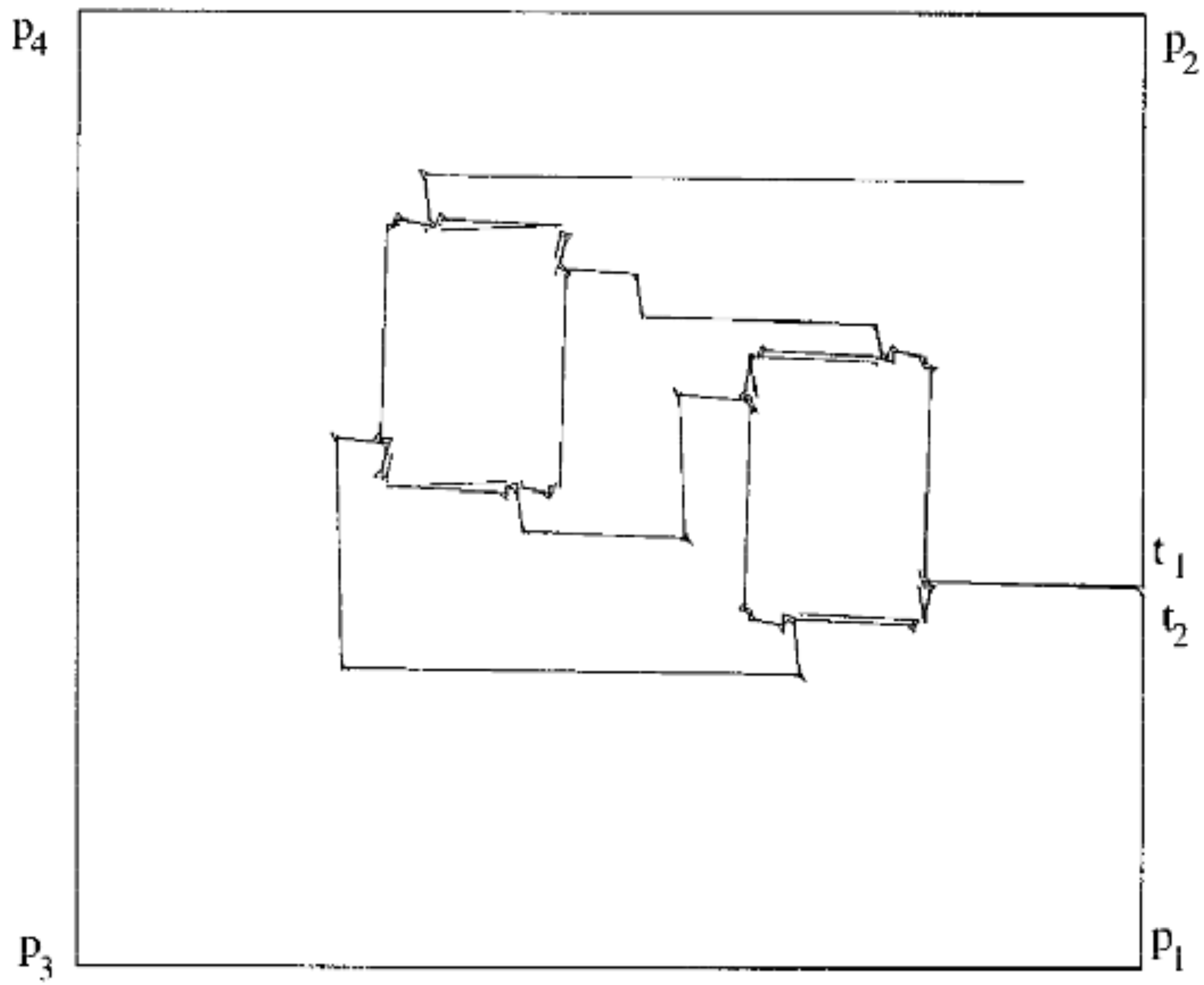
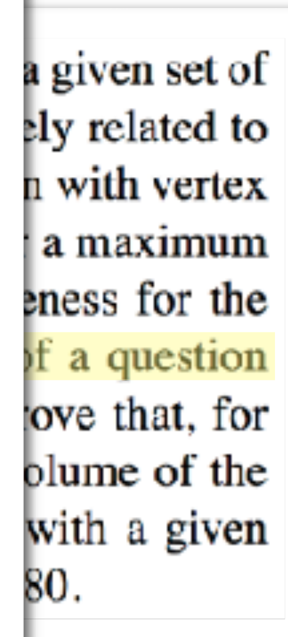


Fig. 24. GRID-FULL solves GRID-EMPTY: turning a small polygon inside out.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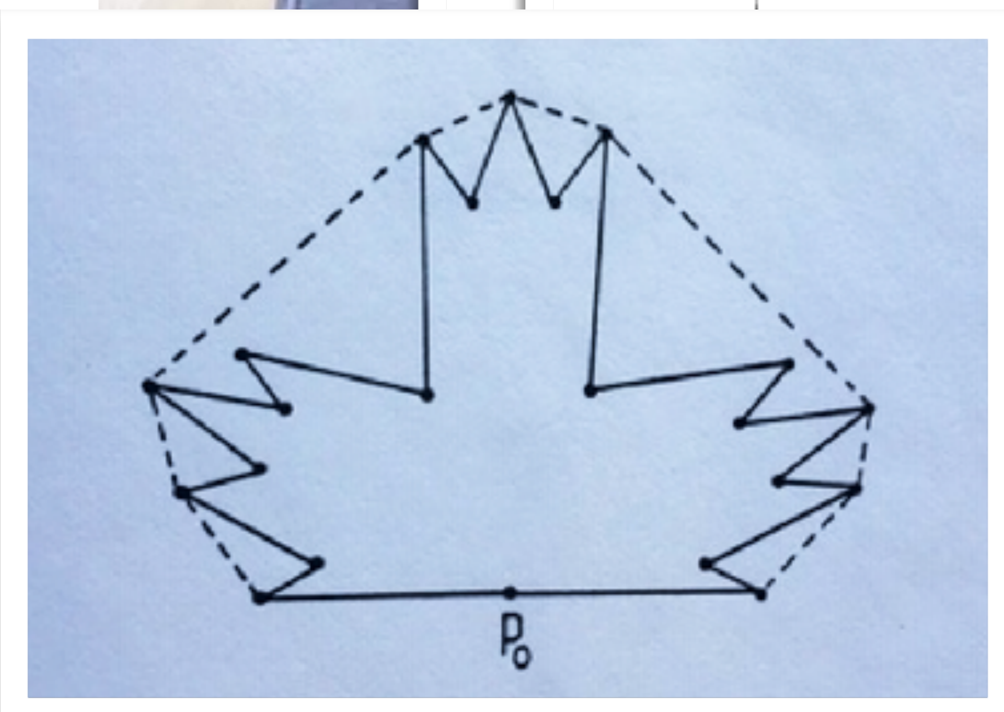

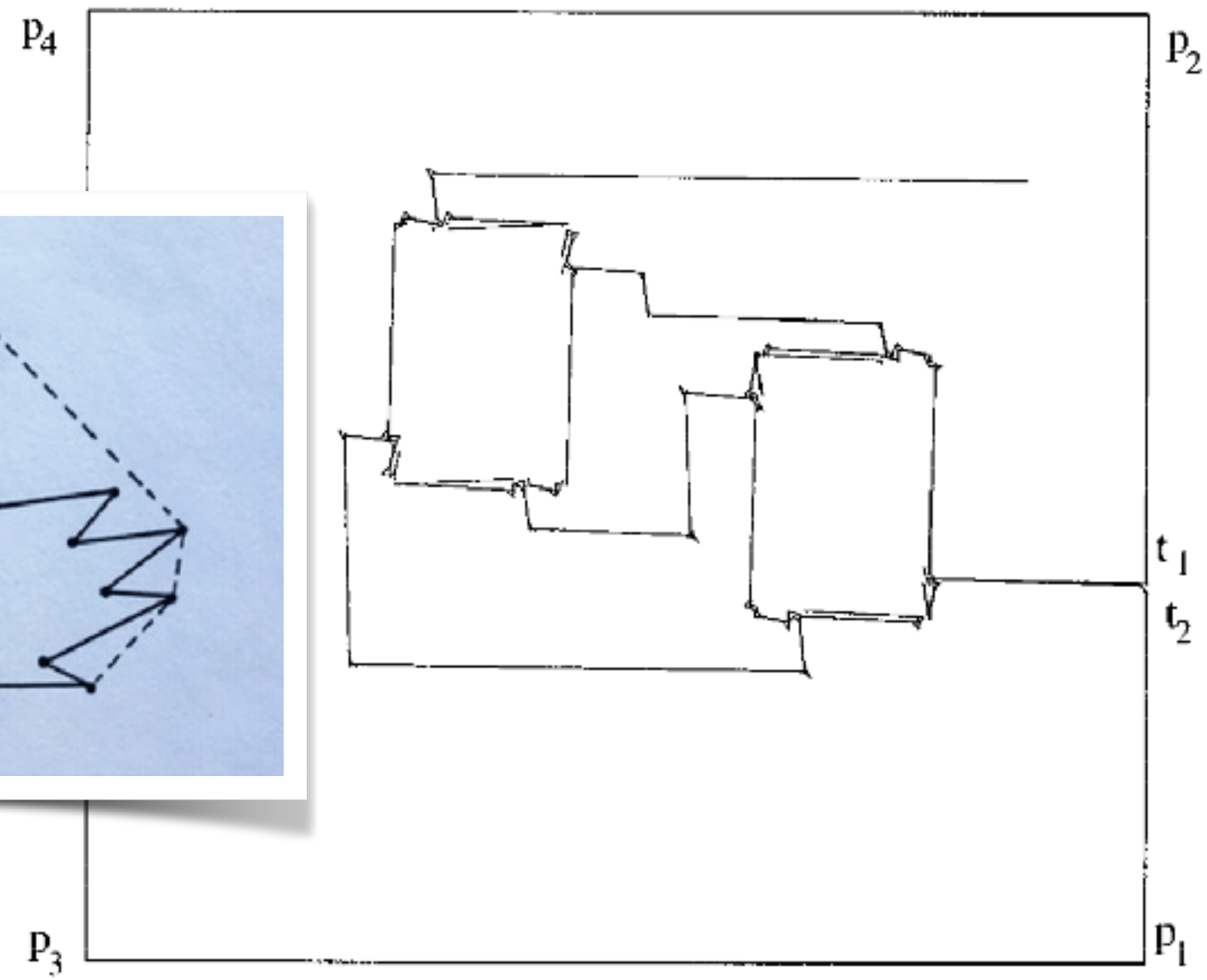

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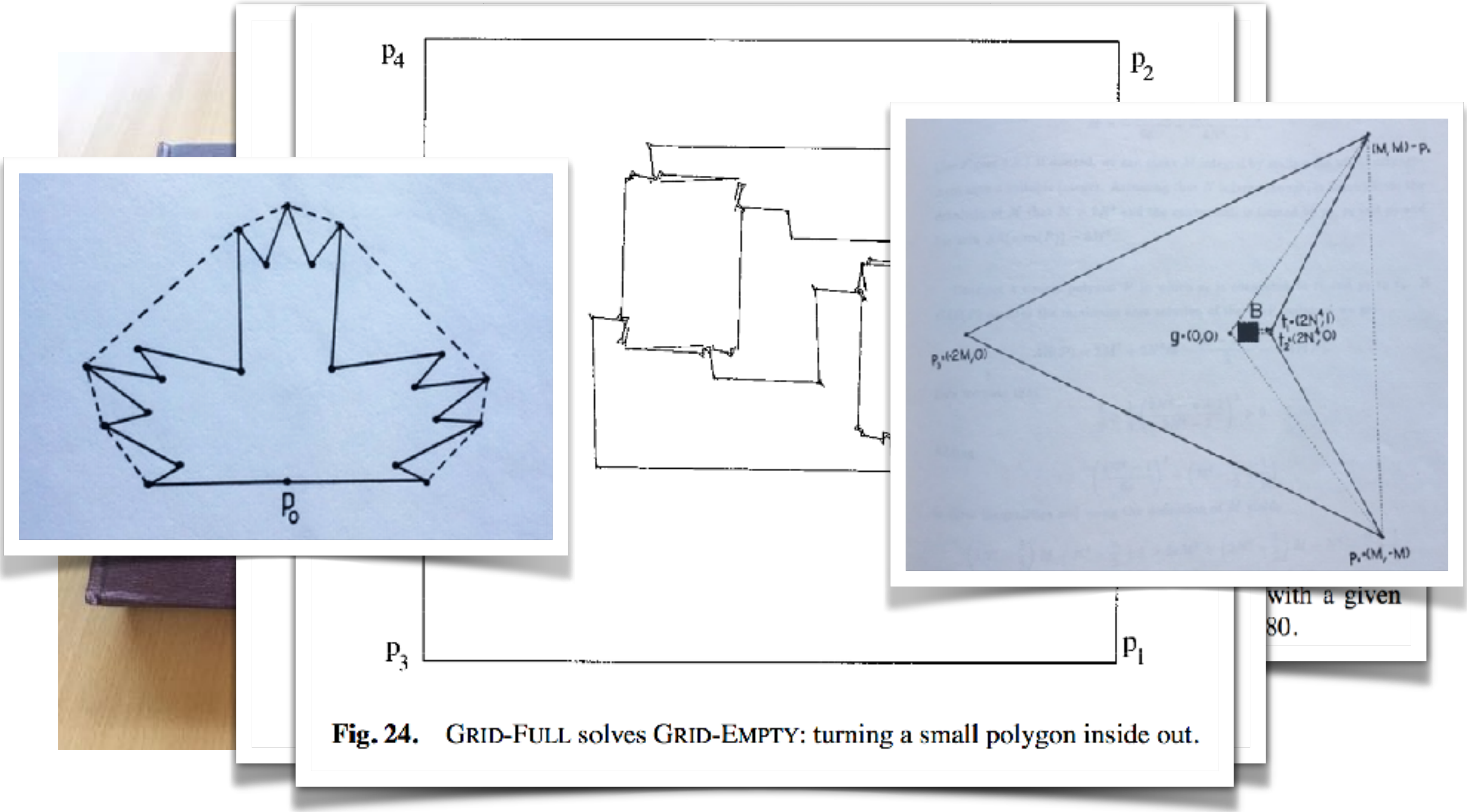

Some Recent Progress

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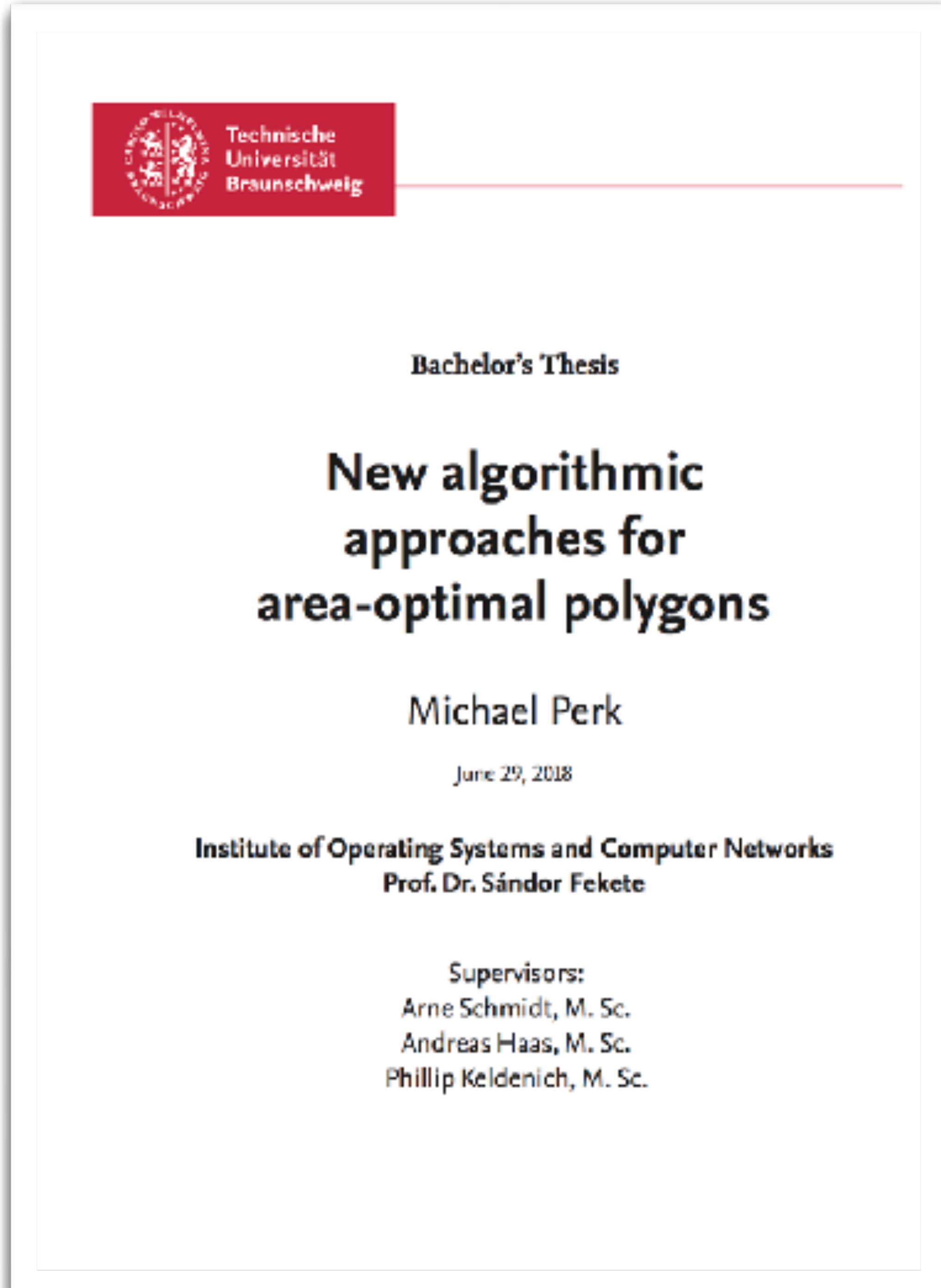
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Bachelor's Thesis

New algorithmic approaches for area-optimal polygons

Michael Perk

June 29, 2018

Institute of Operating Systems and Computer Networks
Prof. Dr. Sándor Fekete

Supervisors:
Arne Schmidt, M. Sc.
Andreas Haas, M. Sc.
Phillip Keldenich, M. Sc.

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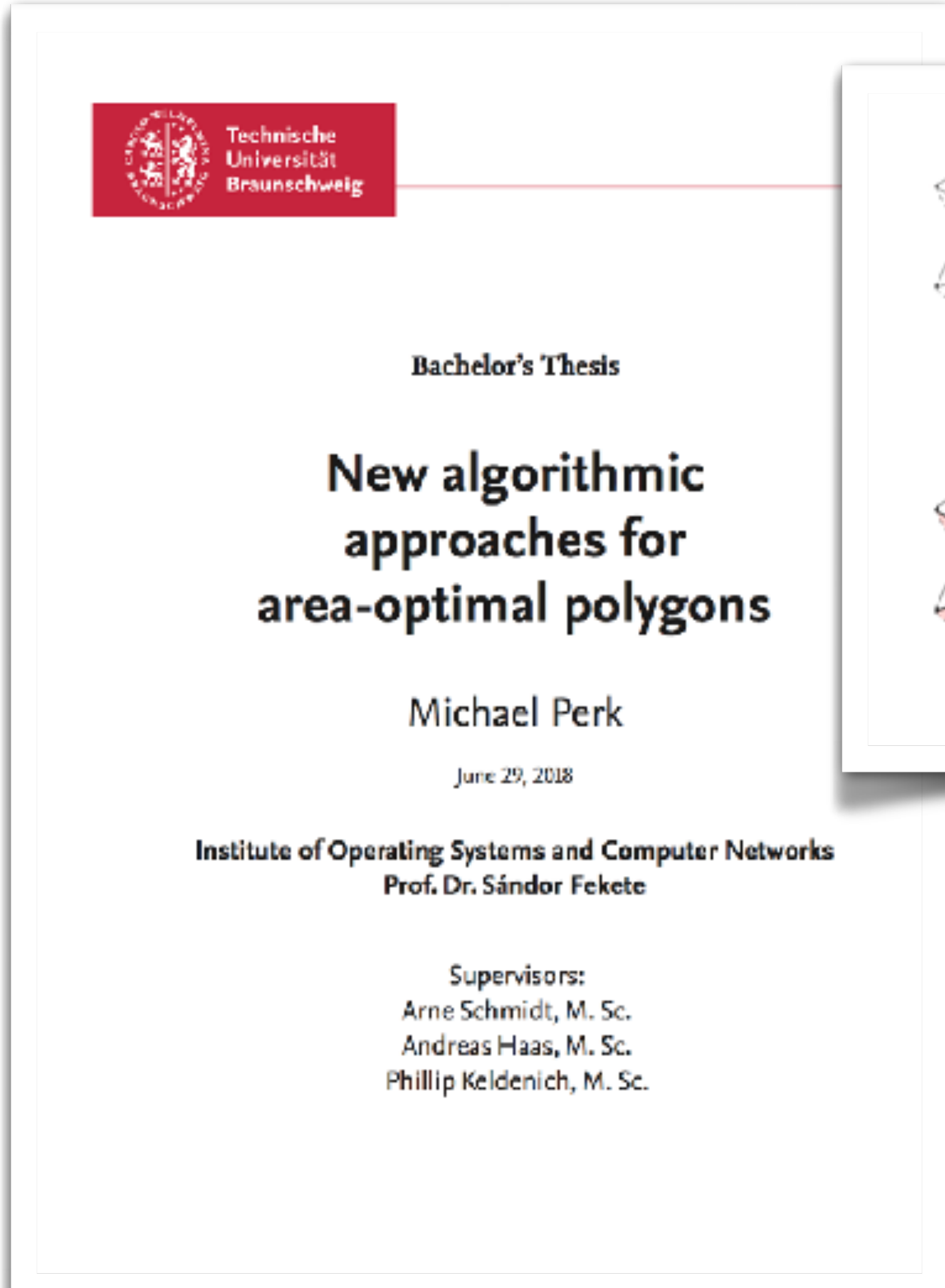
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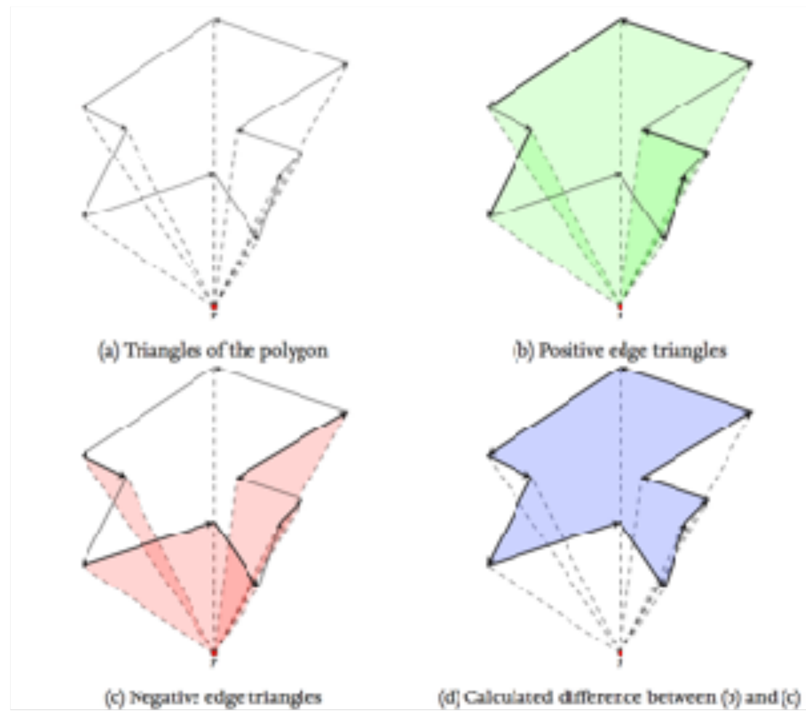
(a) Triangles of the polygon

(b) Positive edge triangles

(c) Negative edge triangles

(d) Calculated difference between (b) and (c)

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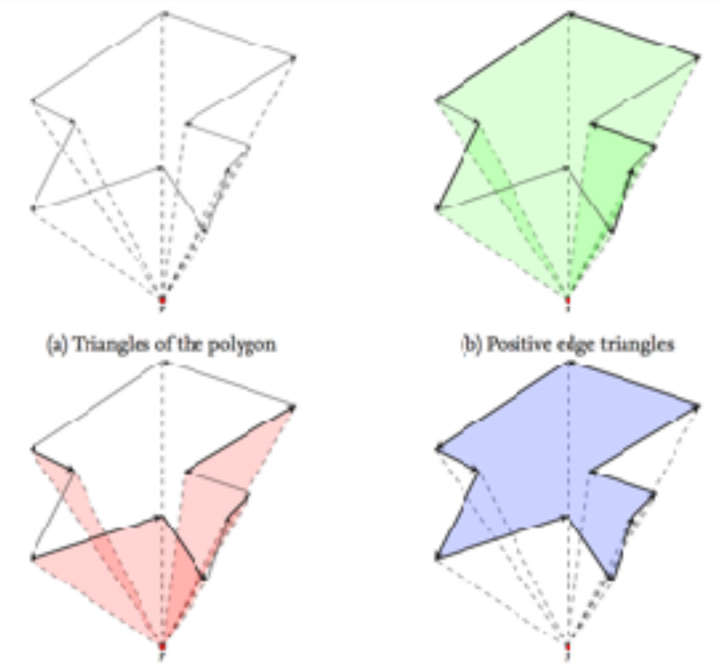

(a) Triangles of the polygon
(b) Positive edge triangles
(c) Negative edge triangles
(d) Calculated difference between (b) and (c)

$$\{\min, \max\} \sum_{e^+ \in E'} z_{e^+} \cdot f_e - \sum_{e^- \in E'} z_{e^-} \cdot f_e$$

$$\forall s_i \in S: \quad \sum_{(j,i) \in \delta^+(s_i)} z_{ji} = 1$$

$$\forall s_i \in S: \quad \sum_{(i,j) \in \delta^-(s_i)} z_{ij} = 1$$

$$\forall e = \{i,j\} \in E: \quad z_{ij} + z_{ji} \leq 1$$

$$\forall \text{intersecting} \quad \{i,j\},\{k,l\} \in E: \quad z_{ij} + z_{ji} + z_{kl} + z_{lk} \leq 1$$

$$(\forall \text{slabs} \quad D)(\forall m = 1, \ldots, |D|): \quad \sum_{i=1}^{m} z_{e^{l_r}_{i_D}} - z_{e^{l}_{i_D}}$$

$$\forall \emptyset \neq D \subsetneq S: \quad \begin{array}{l} \sum_{(k,l) \in \delta^-(D)} z_{kl} \geq 1 \\ \sum_{(k,l) \in \delta^+(D)} z_{kl} \geq 1 \end{array}$$

$$\forall e \in E^+ \cup E^-: \quad e \in \{0,1\}$$

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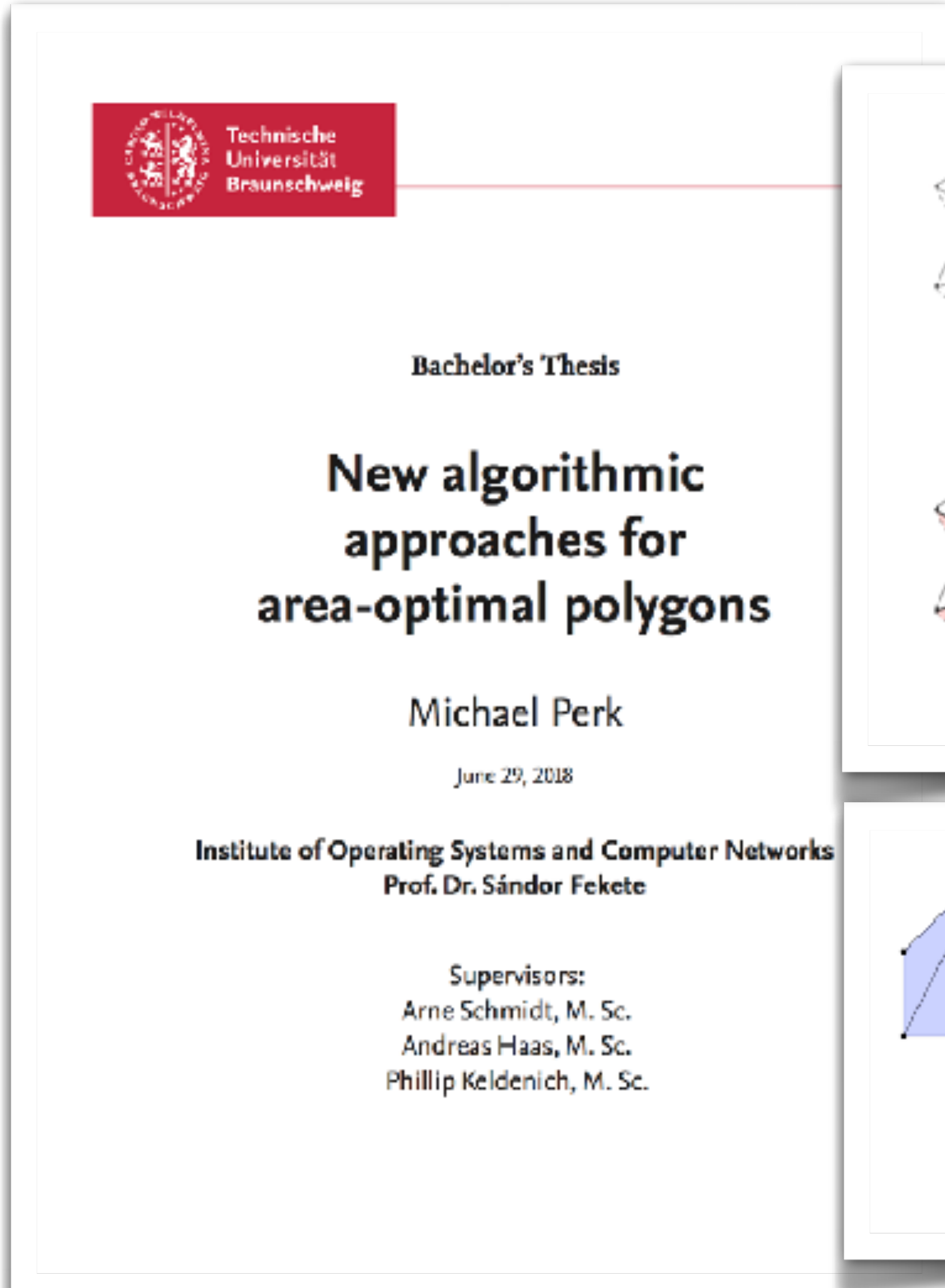
Some Recent Progress



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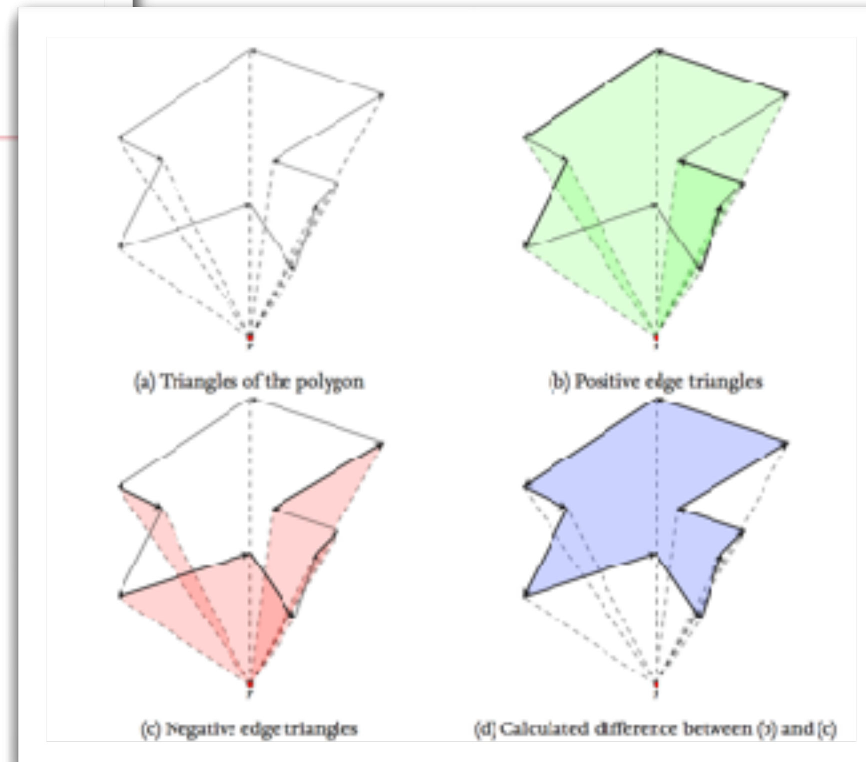

(a) Triangles of the polygon

(b) Positive edge triangles

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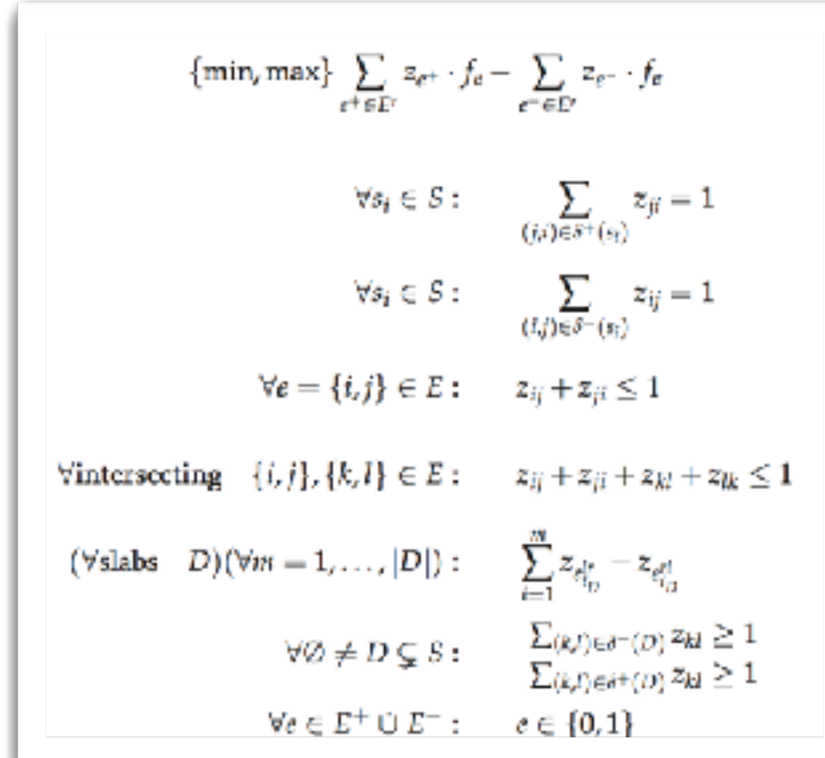

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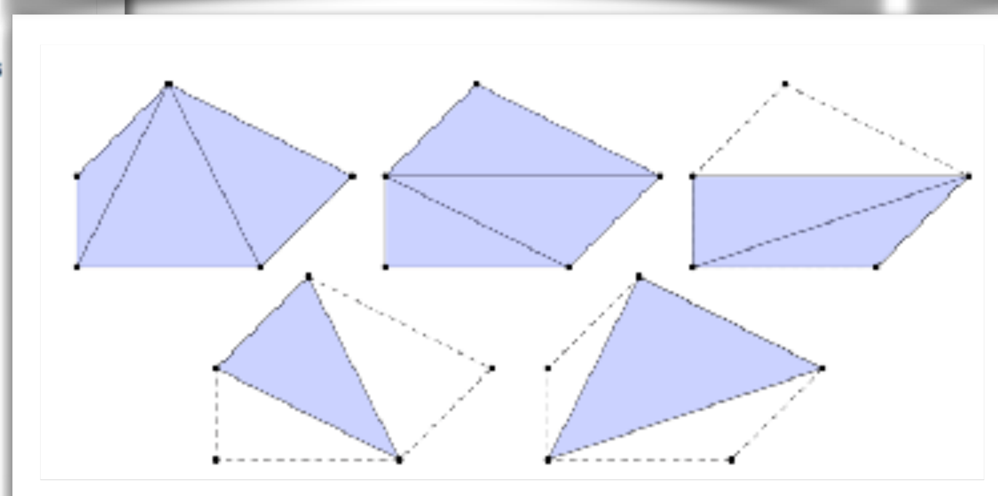
$$\forall e \in E^+ \cup E^-: \quad e \in \{0,1\}$$



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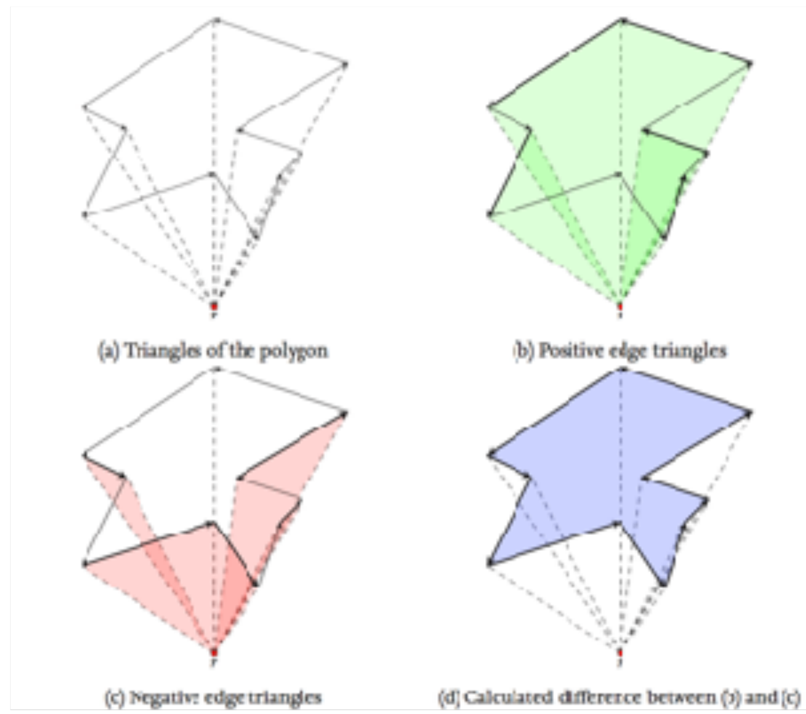
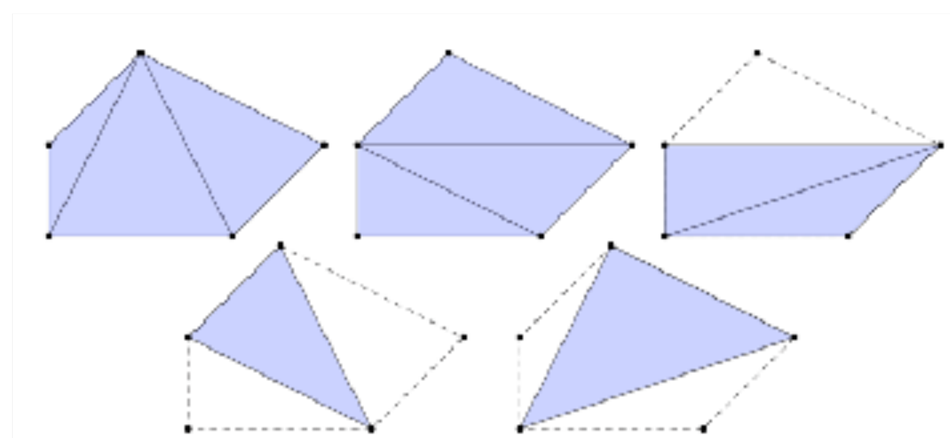


$$\{\min,\max\} \sum_{e^+\in E'} z_{e^+}\cdot f_e - \sum_{e^-\in E'} z_{e^-}\cdot f_e$$

$$\begin{aligned}
\forall s_i \in S:&\quad \sum_{(j,i)\in\delta^+(s_i)} z_{ji} = 1\\
\forall s_i \in S:&\quad \sum_{(i,j)\in\delta^-(s_i)} z_{ij} = 1\\
\forall e=\{i,j\}\in E:&\quad z_{ij}+z_{ji}\le 1\\
\forall \text{intersecting}\quad \{i,j\},\{k,l\}\in E:&\quad z_{ij}+z_{ji}+z_{kl}+z_{lk}\le 1\\
(\forall \text{slabs}\quad D)(\forall m=1,\dots,|D|):&\quad \sum_{i=1}^{m} z_{e^{l_r}_{i_D}} - z_{e^{rl}_{i_D}}\\
\forall\emptyset\neq D\subsetneq S:&\quad \textstyle\sum_{(k,l)\in\delta^-(D)} z_{kl}\ge 1\\
&\quad \textstyle\sum_{(k,l)\in\delta^+(D)} z_{kl}\ge 1\\
\forall e\in E^+\cup E^-:&\quad e\in\{0,1\}
\end{aligned}$$



$$\{\min,\max\} \sum_{\Delta\in T} f_\Delta\cdot x_\Delta$$

$$\begin{aligned}
&\quad \sum_{\Delta\in T} x_\Delta = n-2\\
\forall s_i\in S:&\quad \sum_{\Delta\in\delta(s_i)} x_\Delta \ge 1\\
\forall\text{intersecting}\quad \Delta_i,\Delta_j\in T:&\quad x_{\Delta_i}+x_{\Delta_j}\le 1\\
\forall\emptyset\neq D\subsetneq T, |D|\le n-3:&\quad \sum_{\Delta\in D} x_\Delta \le \sum_{\Delta\in\delta(D)} x_\Delta + |D| - 1\\
\forall\Delta\in T:&\quad x_\Delta\in\{0,1\}
\end{aligned}$$

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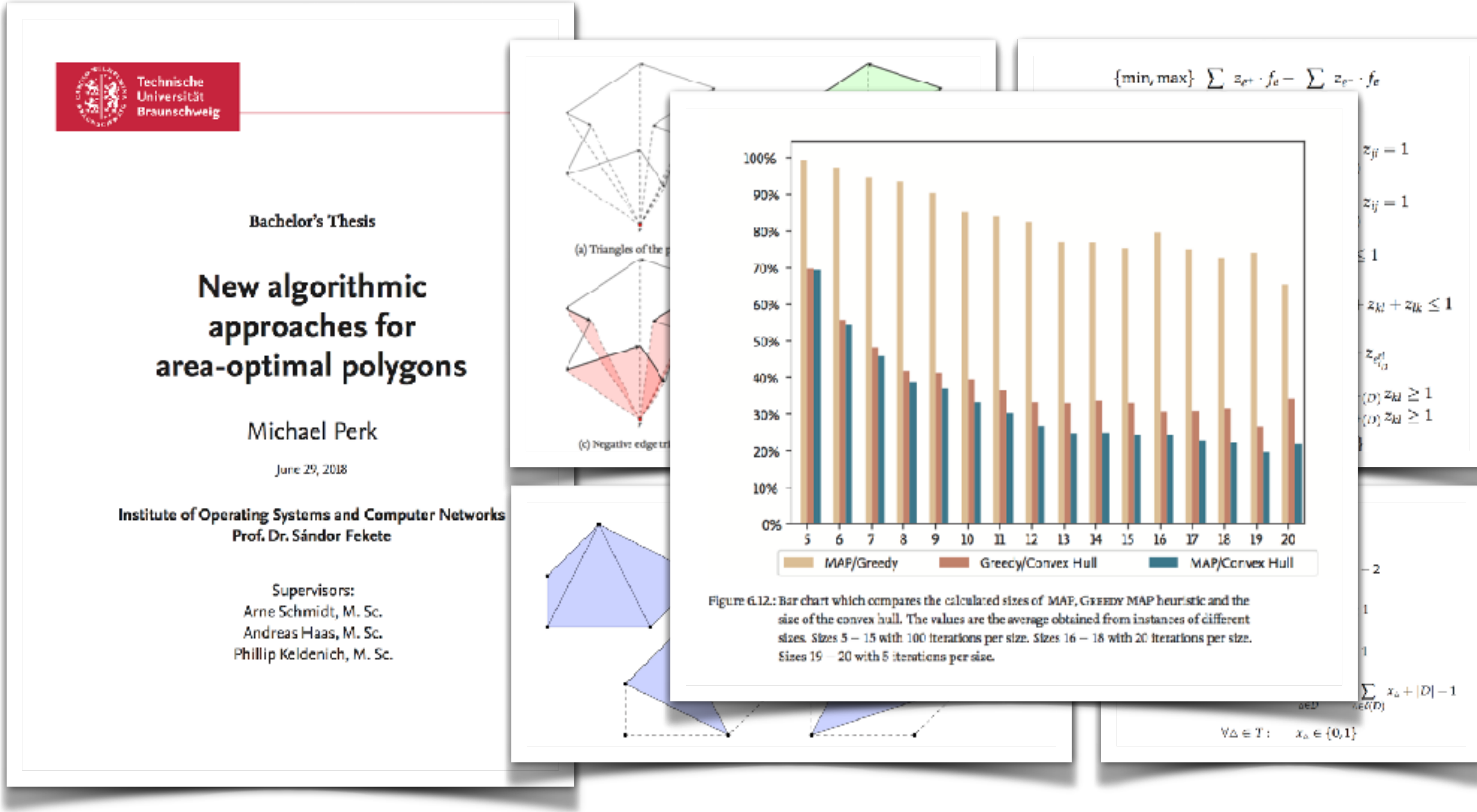
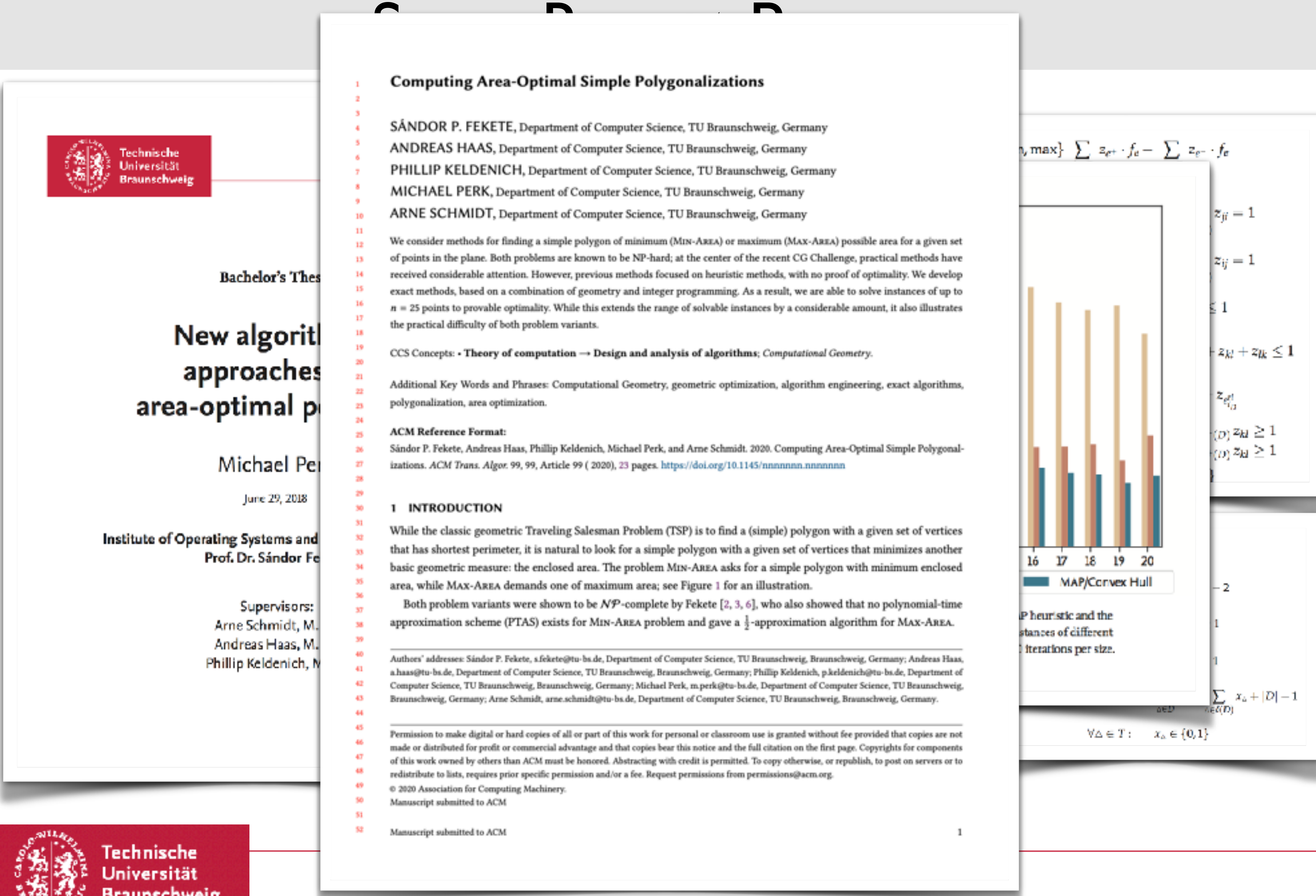
Some Recent Progress





Computing Area-Optimal Simple Polygonalizations

SÁNDOR P. FEKETE, Department of Computer Science, TU Braunschweig, Germany
ANDREAS HAAS, Department of Computer Science, TU Braunschweig, Germany
PHILLIP KELDENICH, Department of Computer Science, TU Braunschweig, Germany
MICHAEL PERK, Department of Computer Science, TU Braunschweig, Germany
ARNE SCHMIDT, Department of Computer Science, TU Braunschweig, Germany

We consider methods for finding a simple polygon of minimum (Min-Area) or maximum (Max-Area) possible area for a given set of points in the plane. Both problems are known to be NP-hard; at the center of the recent CG Challenge, practical methods have received considerable attention. However, previous methods focused on heuristic methods, with no proof of optimality. We develop exact methods, based on a combination of geometry and integer programming. As a result, we are able to solve instances of up to $n = 25$ points to provable optimality. While this extends the range of solvable instances by a considerable amount, it also illustrates the practical difficulty of both problem variants.

CCS Concepts: • **Theory of computation → Design and analysis of algorithms**; *Computational Geometry.*

Additional Key Words and Phrases: Computational Geometry, geometric optimization, algorithm engineering, exact algorithms, polygonalization, area optimization.

ACM Reference Format:
Sándor P. Fekete, Andreas Haas, Phillip Keldenich, Michael Perk, and Arne Schmidt. 2020. Computing Area-Optimal Simple Polygonalizations. *ACM Trans. Algor.* 99, 99, Article 99 (2020), 23 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

1 INTRODUCTION

While the classic geometric Traveling Salesman Problem (TSP) is to find a (simple) polygon with a given set of vertices that has shortest perimeter, it is natural to look for a simple polygon with a given set of vertices that minimizes another basic geometric measure: the enclosed area. The problem Min-Area asks for a simple polygon with minimum enclosed area, while Max-Area demands one of maximum area; see Figure 1 for an illustration.

Both problem variants were shown to be $\mathcal{NP}$-complete by Fekete [2, 3, 6], who also showed that no polynomial-time approximation scheme (PTAS) exists for Min-Area problem and gave a $\frac{1}{2}$-approximation algorithm for Max-Area.

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Manuscript submitted to ACM

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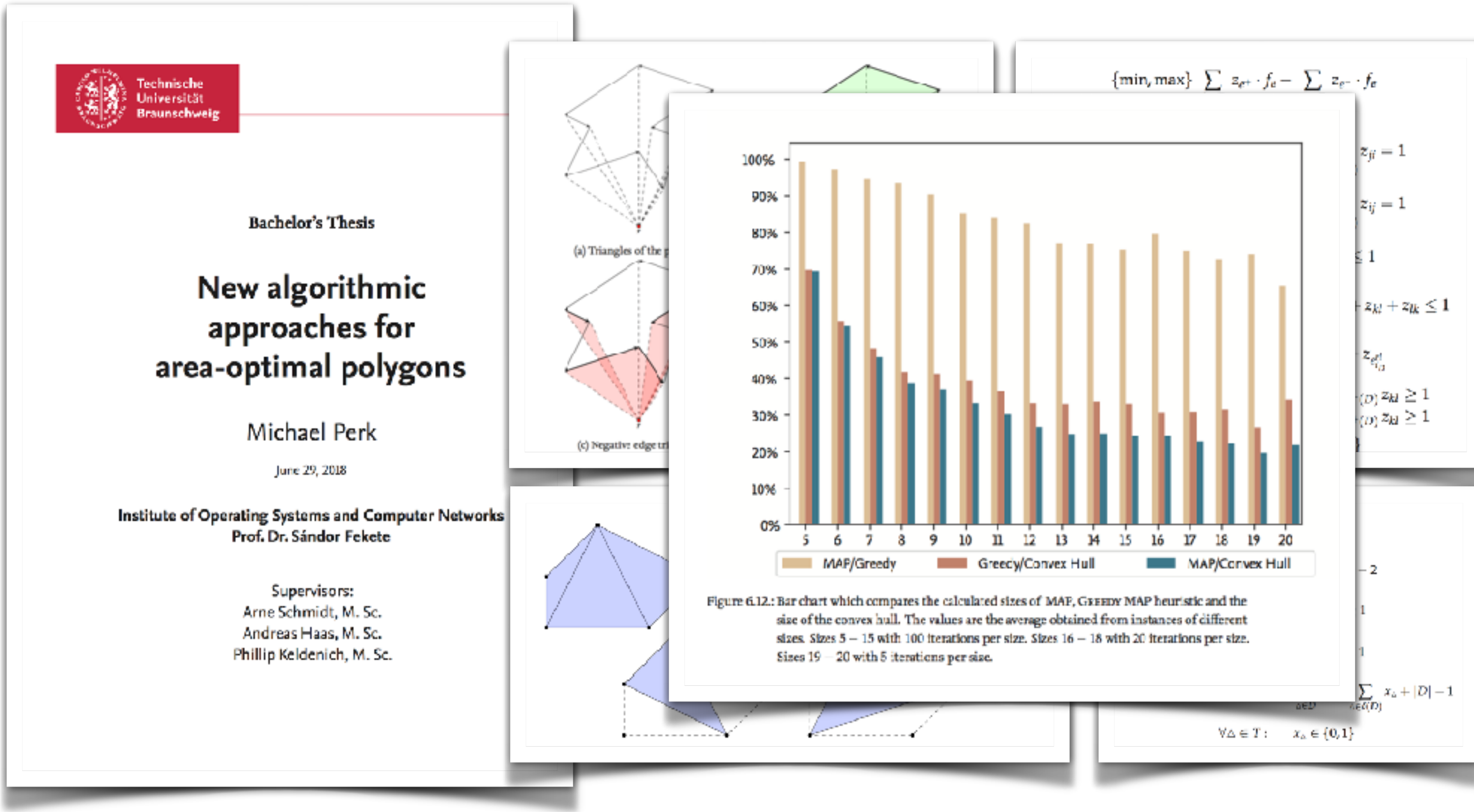
Some Recent Progress



The 2019 CG Challenge

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Area-Optimal Simple Polygonalizations: The CG Challenge 2019

ERIK D. DEMAINE, CSAIL, MIT, USA
SÁNDOR P. FEKETE, Department of Computer Science, TU Braunschweig, Germany
PHILLIP KELDENICH, Department of Computer Science, TU Braunschweig, Germany
DOMINIK KRUPKE, Department of Computer Science, TU Braunschweig, Germany
JOSEPH S. B. MITCHELL, Department of Applied Mathematics and Statistics, Stony Brook University, USA

We give an overview of theoretical and practical aspects of finding a simple polygon of minimum (MIN-AREA) or maximum (MAX-AREA) possible area for a given set of n points in the plane. Both problems are known to be $\mathcal{NP}$-hard and were the subject of the 2019 Computational Geometry Challenge, which presented the quest of finding good solutions to more than 200 instances, ranging from $n = 10$ all the way to $n = 1,000,000$.

CCS Concepts: • **Theory of computation → Design and analysis of algorithms**; *Computational Geometry*.

ACM Reference Format:
Erik D. Demaine, Sándor P. Fekete, Phillip Keldenich, Dominik Krupke, and Joseph S. B. Mitchell. 2020. Area-Optimal Simple Polygonalizations: The CG Challenge 2019. *ACM J. Exp. Algor.* 99, 99, Article 99 (2020), 12 pages. https://doi.org/10.1145/nnnnnnn.nnnnnnn

1 INTRODUCTION

1.1 The Computational Geometry Challenge

The "CG:SHOP Challenge" (Computational Geometry: Solving Hard Optimization Problems) originated as a workshop at the 2019 Computational Geometry Week (CG Week) in Portland, Oregon in June, 2019. The goal was to conduct a computational challenge competition that focused attention on a specific hard geometric optimization problem, encouraging researchers to devise and implement solution methods that could be compared scientifically based on how well they performed on a database of instances. While much of computational geometry research has targeted theoretical research, often seeking provable approximation algorithms for $\mathcal{NP}$-hard optimization problems, the goal of the CG Challenge was to set the metric of success based on computational results on a specific set of benchmark geometric instances. The 2019 CG Challenge focused on the problem of computing simple polygons whose vertices were a given set of points in the plane.

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The 2019 CG Challenge

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The 2019 CG Challenge

The contest consists of a total of **247 instances**, as follows. For n= {10,15,20,25,30,35,40,45,50,60,70,80,90,100,200,300,400,500,600,700,800,900,1000, 2000,3000,4000,5000,6000,7000,8000,9000,10000,20000,30000,40000,50000,60000, 70000,80000,90000,100000}, there are six instances each. In addition, there will be one instance of size n=1000000.

These instances are of three different types:

- uniform: uniformly at random from a square
- edge: randomly generated according to the distribution of the rate of change (the "edges") of an image
- illumination: randomly generated according to the distribution of brightness of an image (such as an illumination map)

Each instance consists of n points in the plane with even integer coordinates. (This ensures that the area of any simple polygon will be an integer.)

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For each instance, the **score** is the ratio between the achieved area, divided by the area of the convex hull, i.e., a number between 0 and 1. For instances without a feasible solution, the default score is 1 (for minimization) or 0 (for maximization). The total score is the sum of all 247 individual scores.

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The contest will be run in several different categories. These categories include:

(Score_min) The best total score for minimum area polygons
(Score_max) The best total score for maximum area polygons
(Opt_min) The largest number of instances solved to optimality for a minimum area polygon
(Opt_max) The largest number of instances solved to optimality for a maximum area polygon
(Bound_min) The best bounds for minimization
(Bound_max) The best bounds for maximization

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The 2019 CG Challenge

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(Opt_max) The largest number of instances solved to optimality for a maximum area polygon
(Bound_min) The best bounds for minimization
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Contest opens 24:00 (midnight, CET), February 28, 2019.
Contest closes 24:00 (midnight, CET), May 31, 2019.

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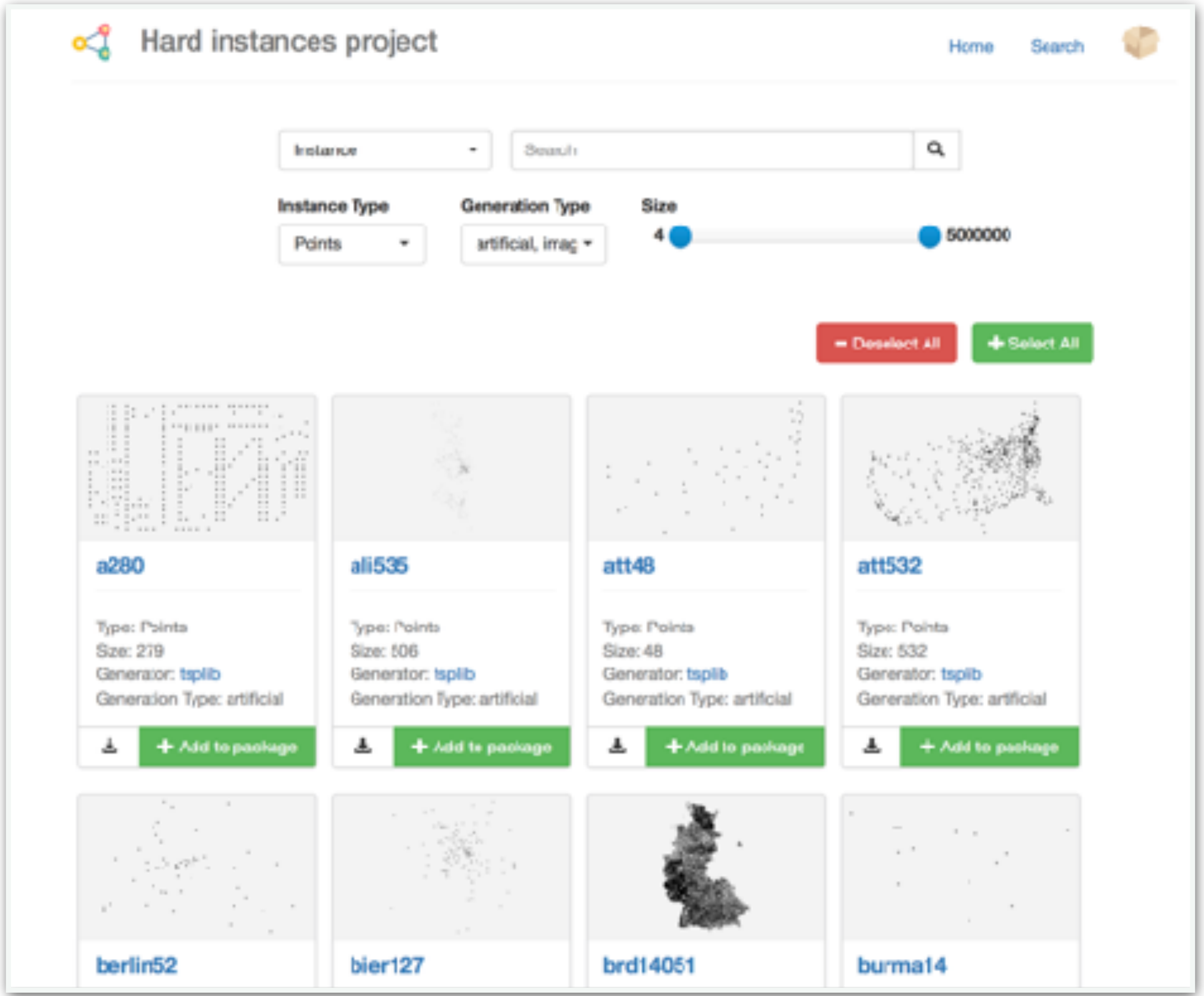
Instances

Instances

Hard instances project

Instances

Hard instances project



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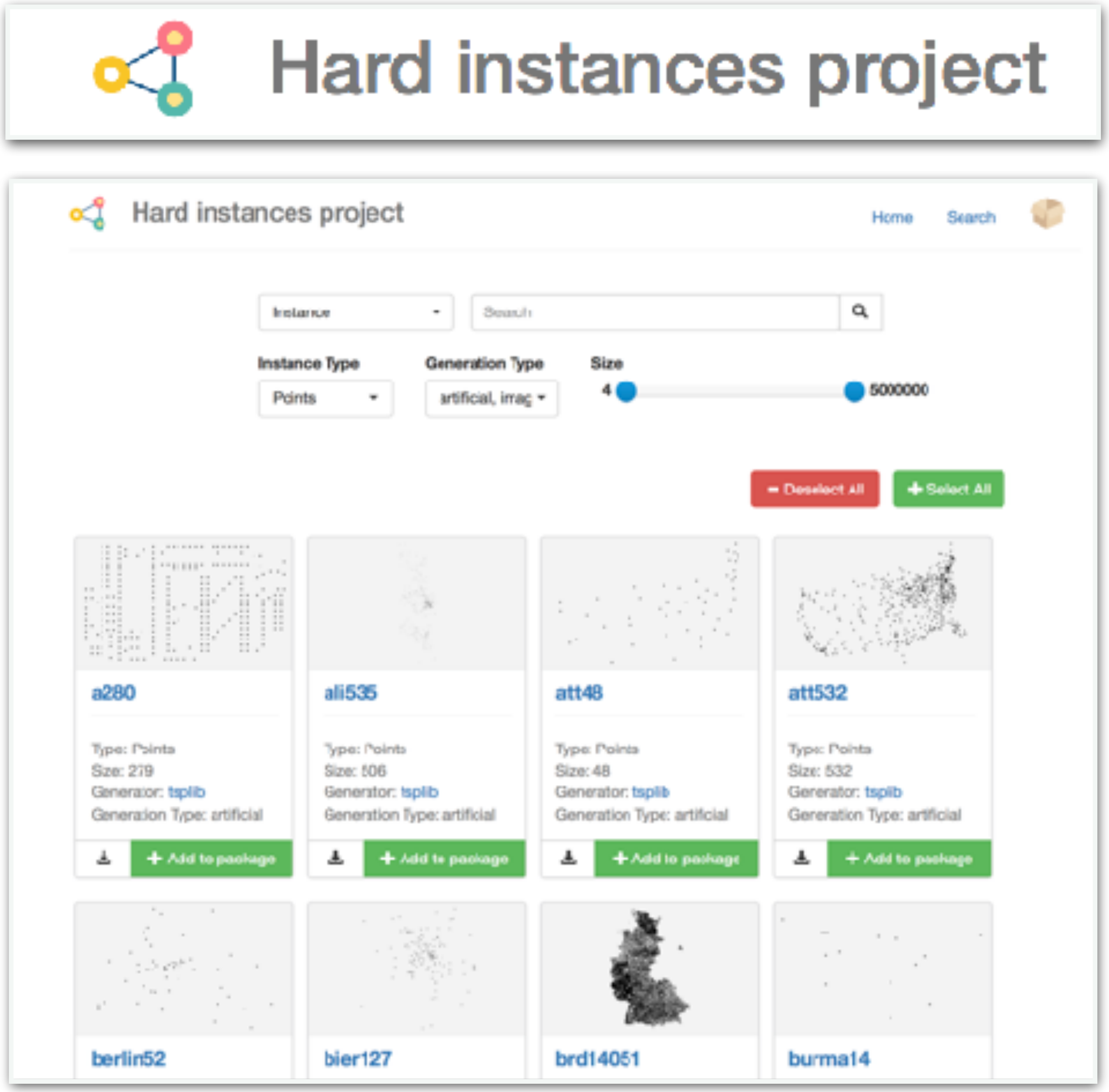

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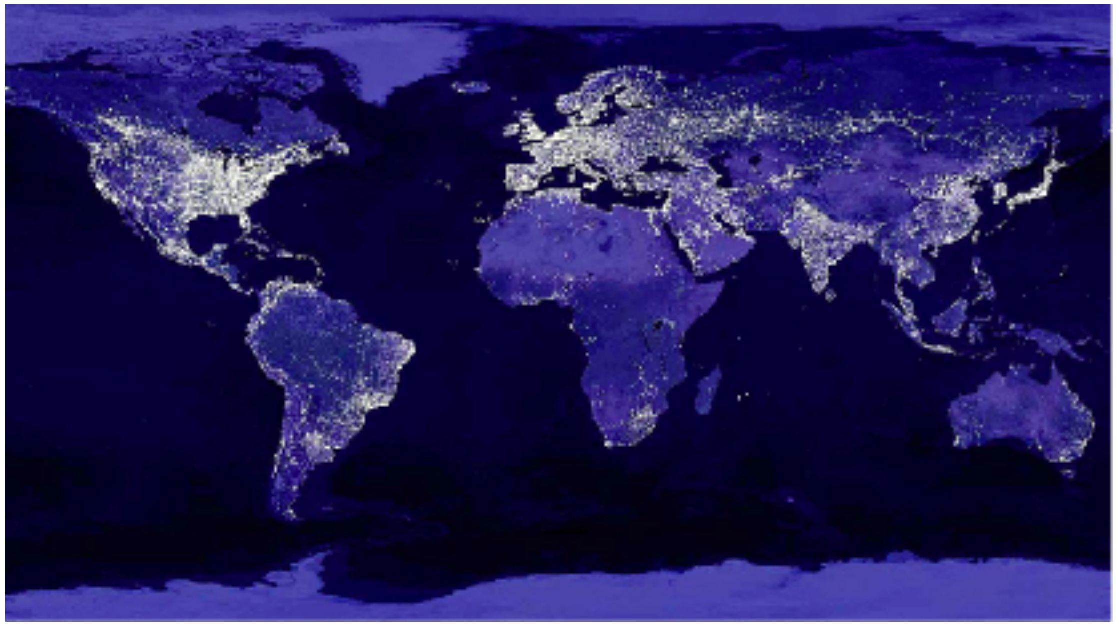

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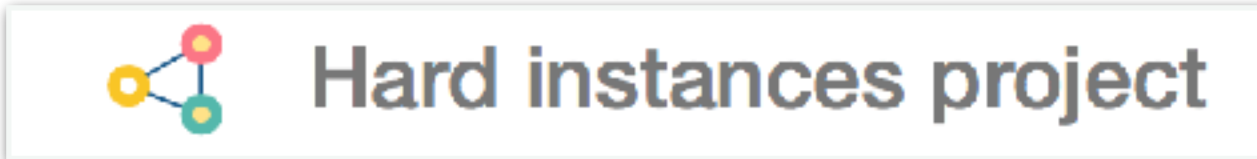

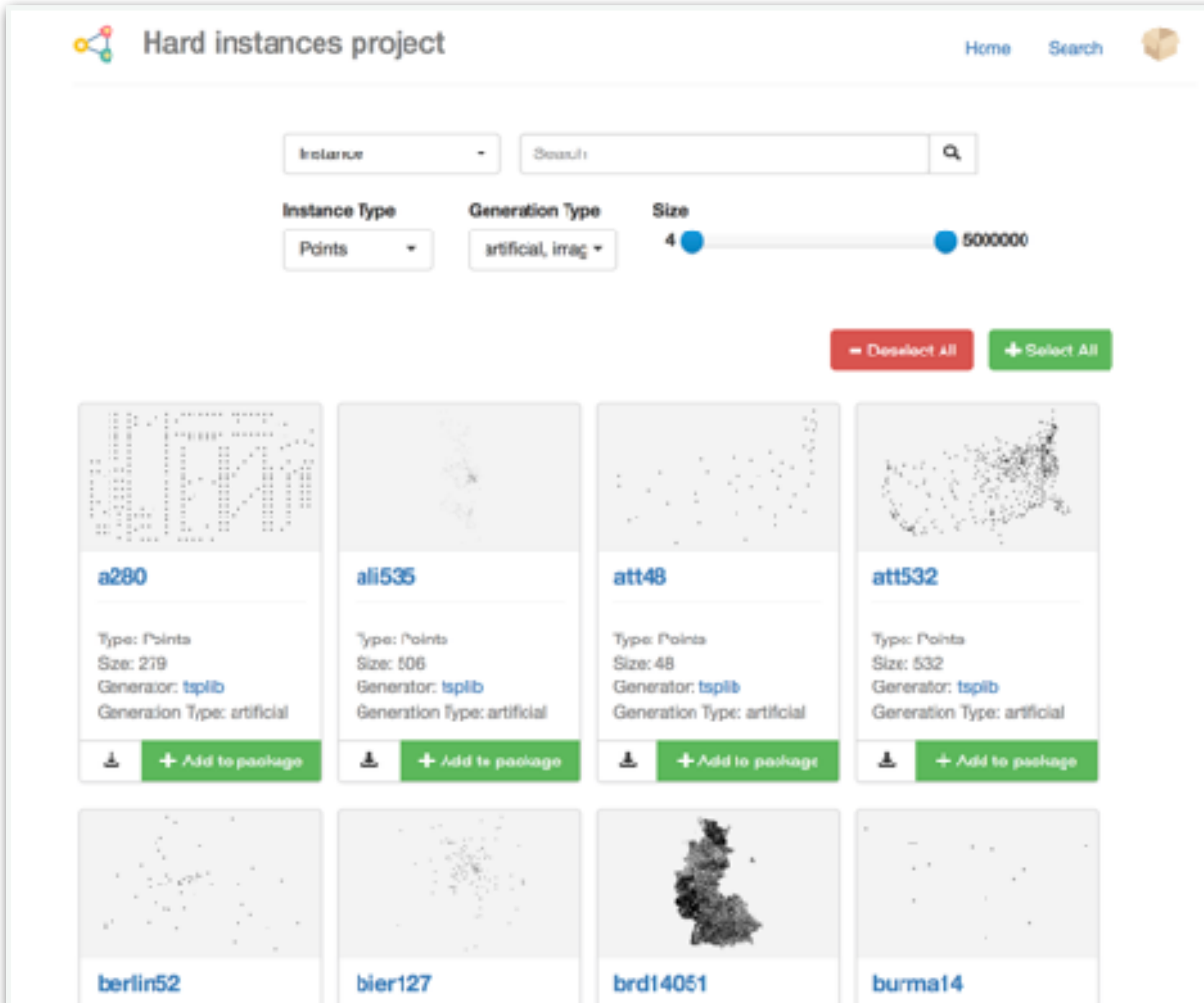

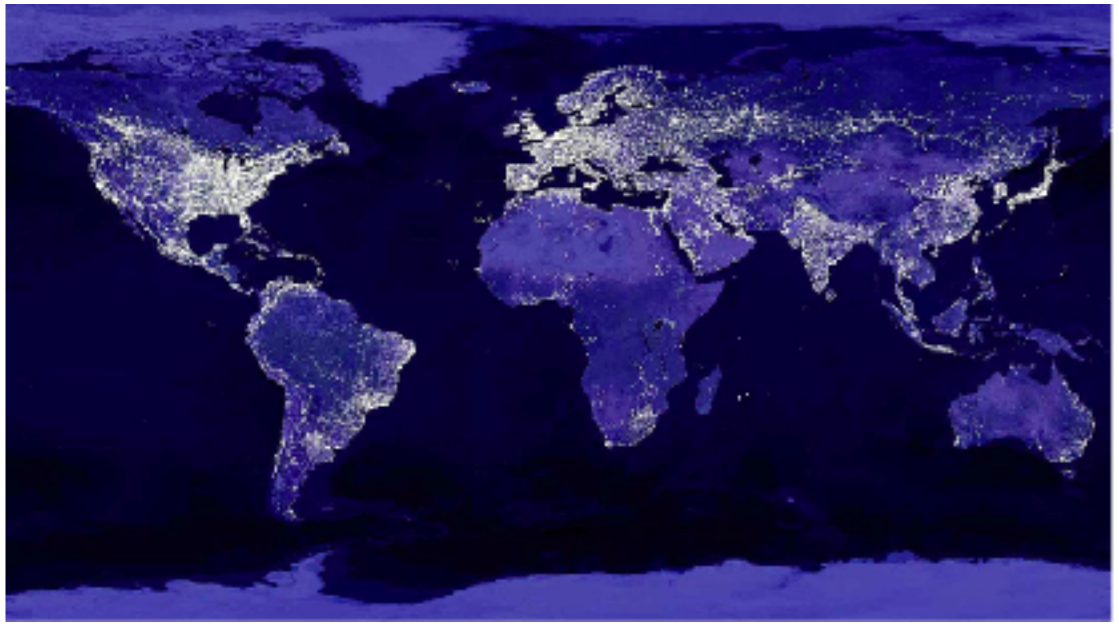

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Progress

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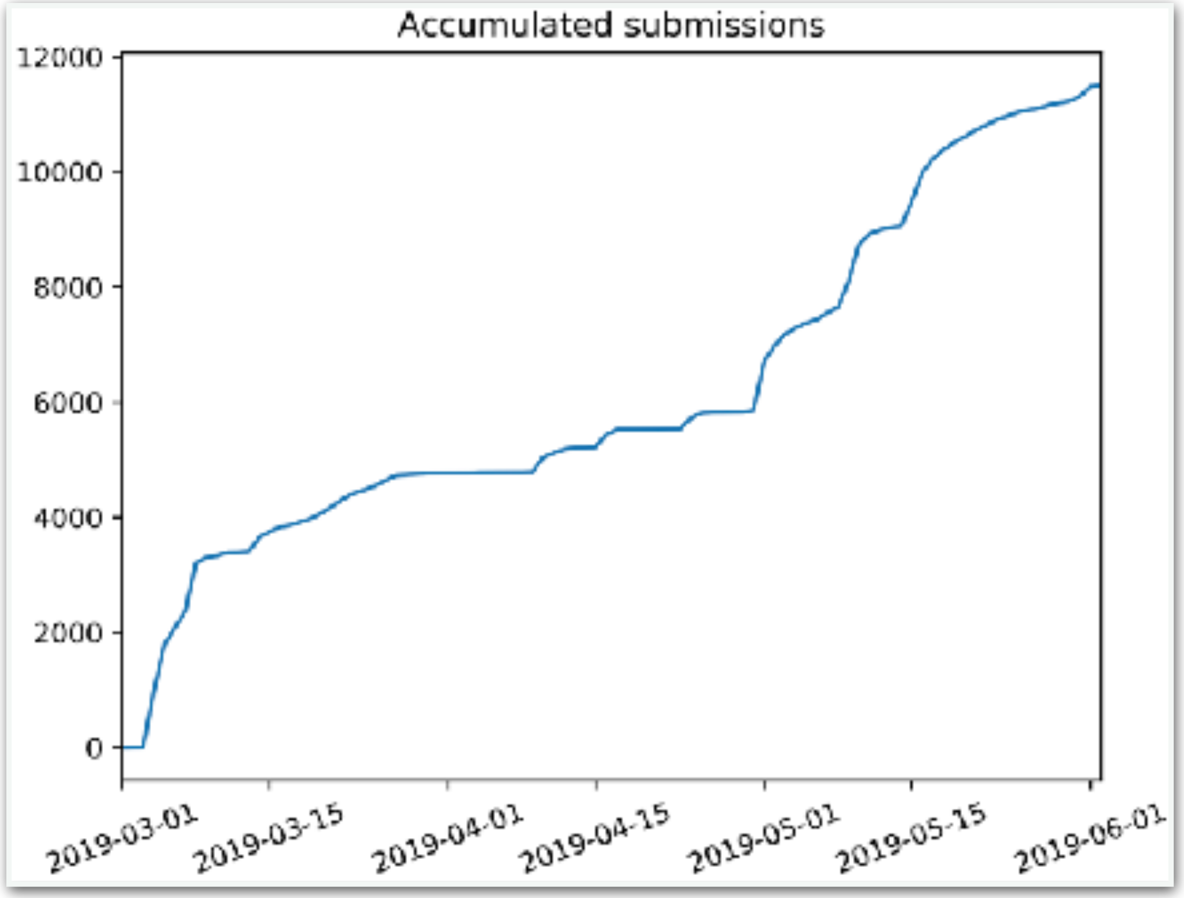
Progress



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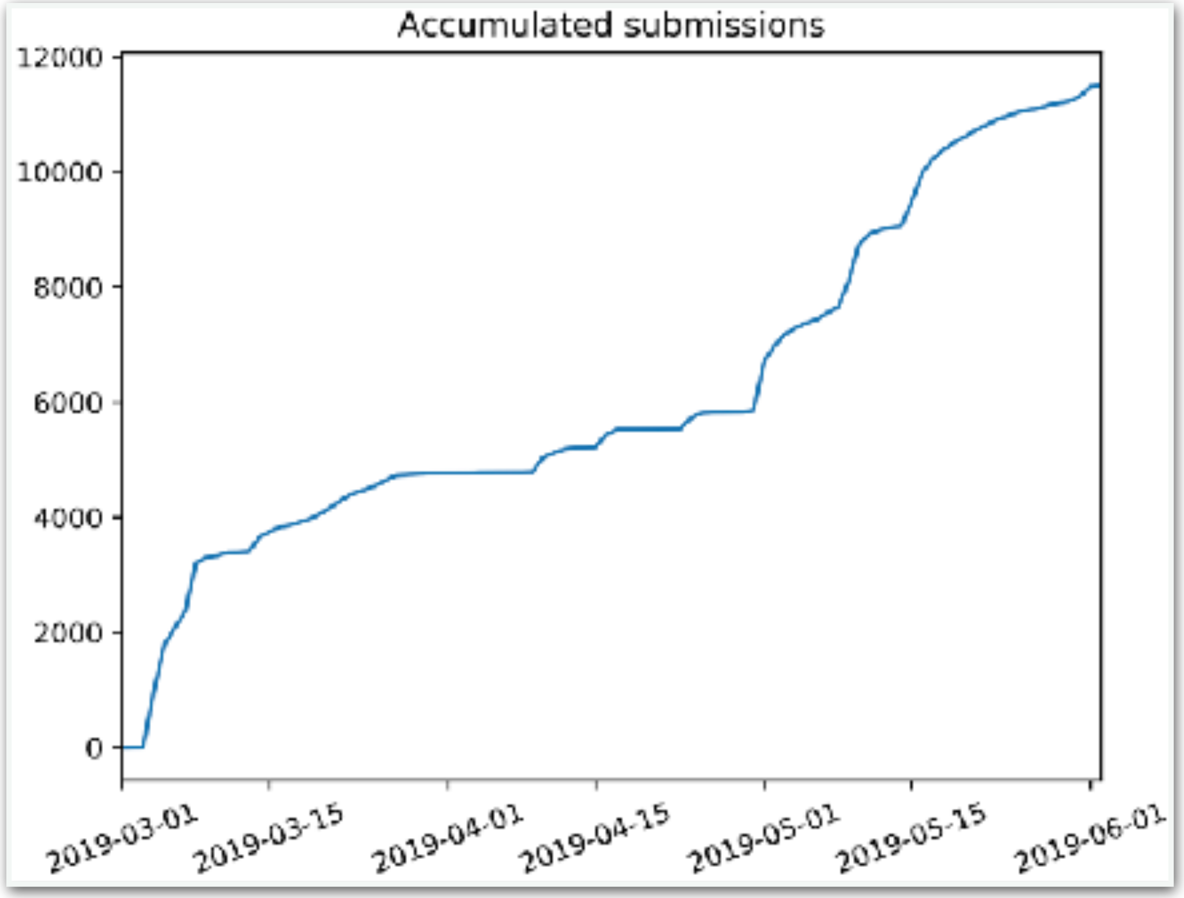
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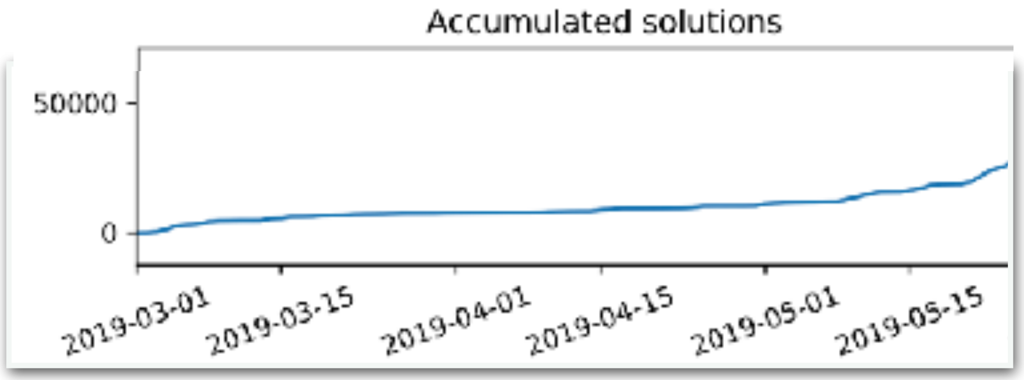
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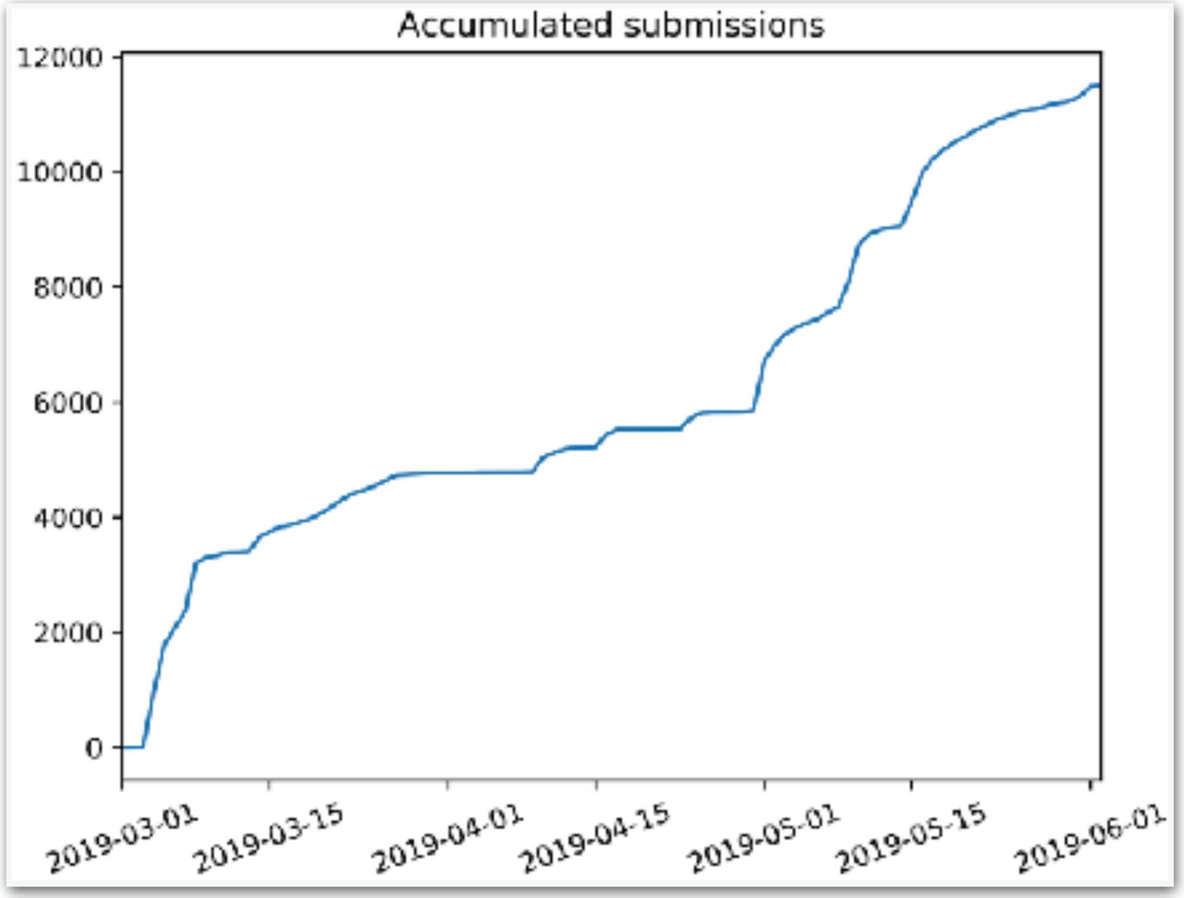
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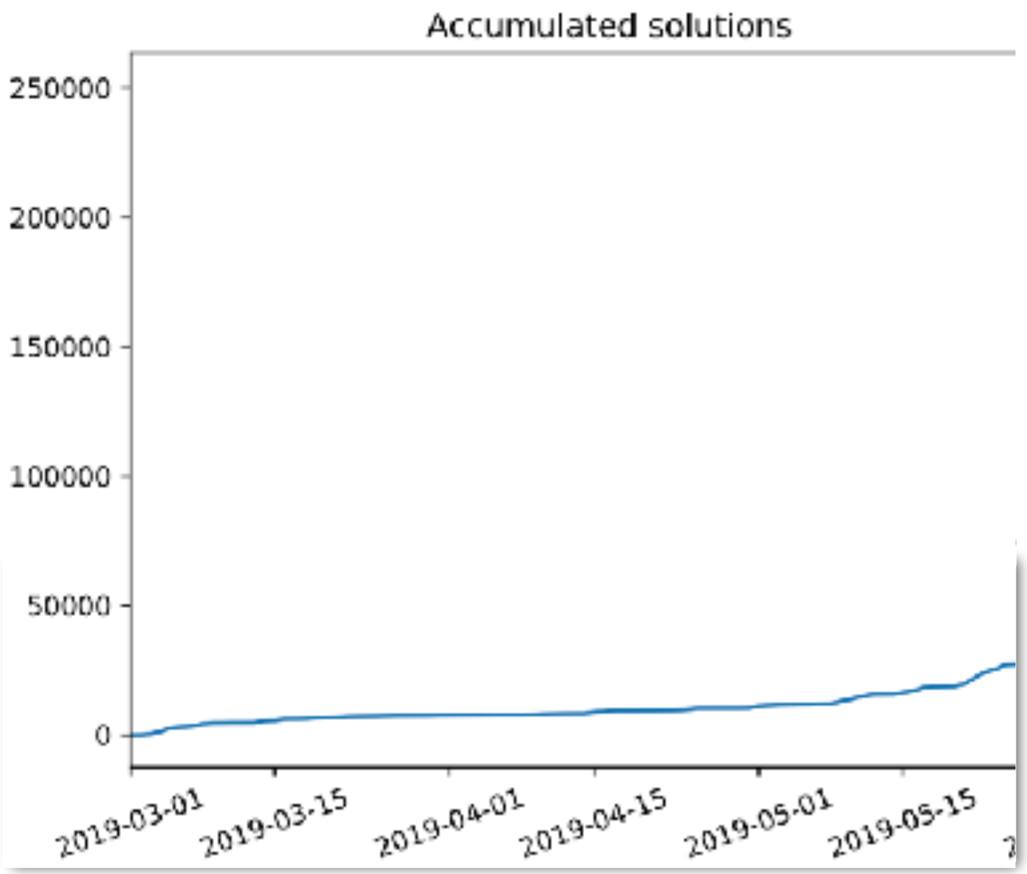
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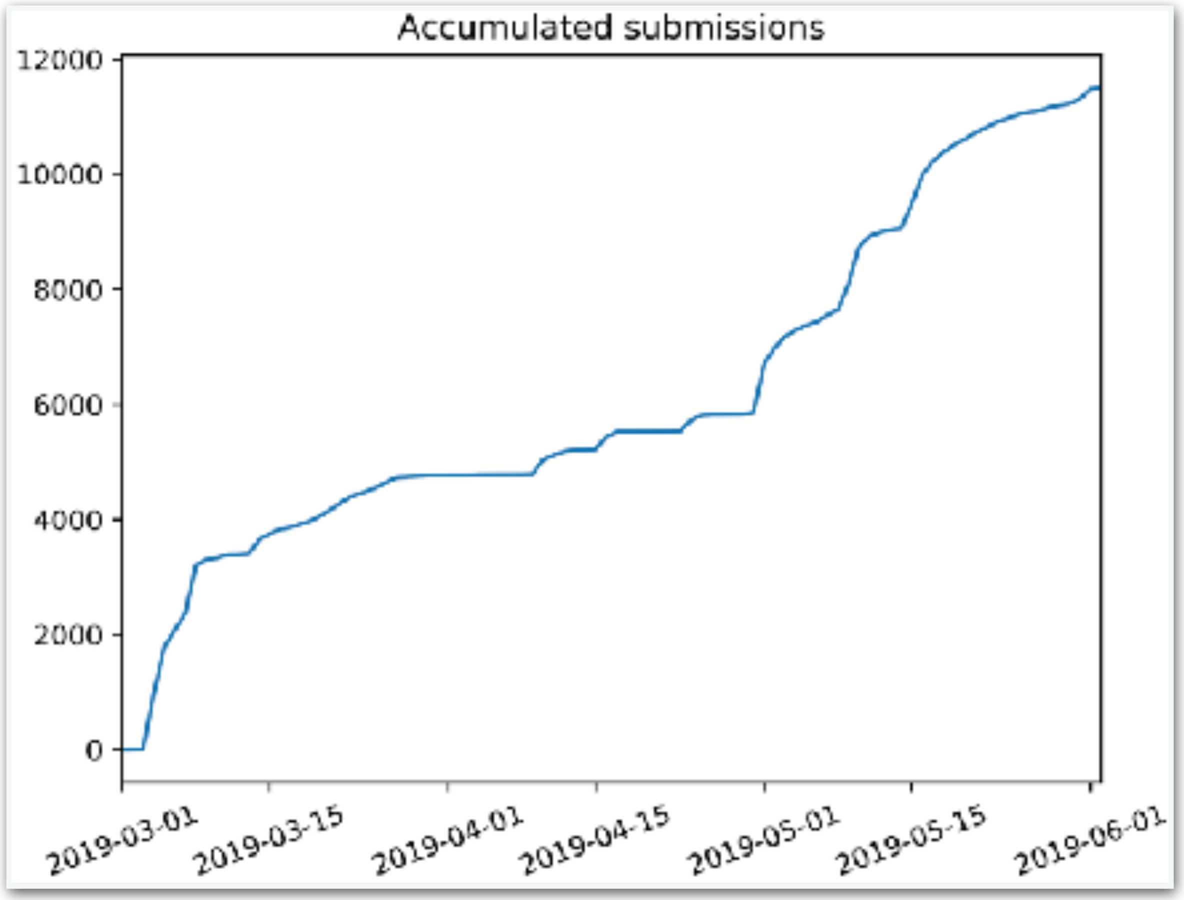

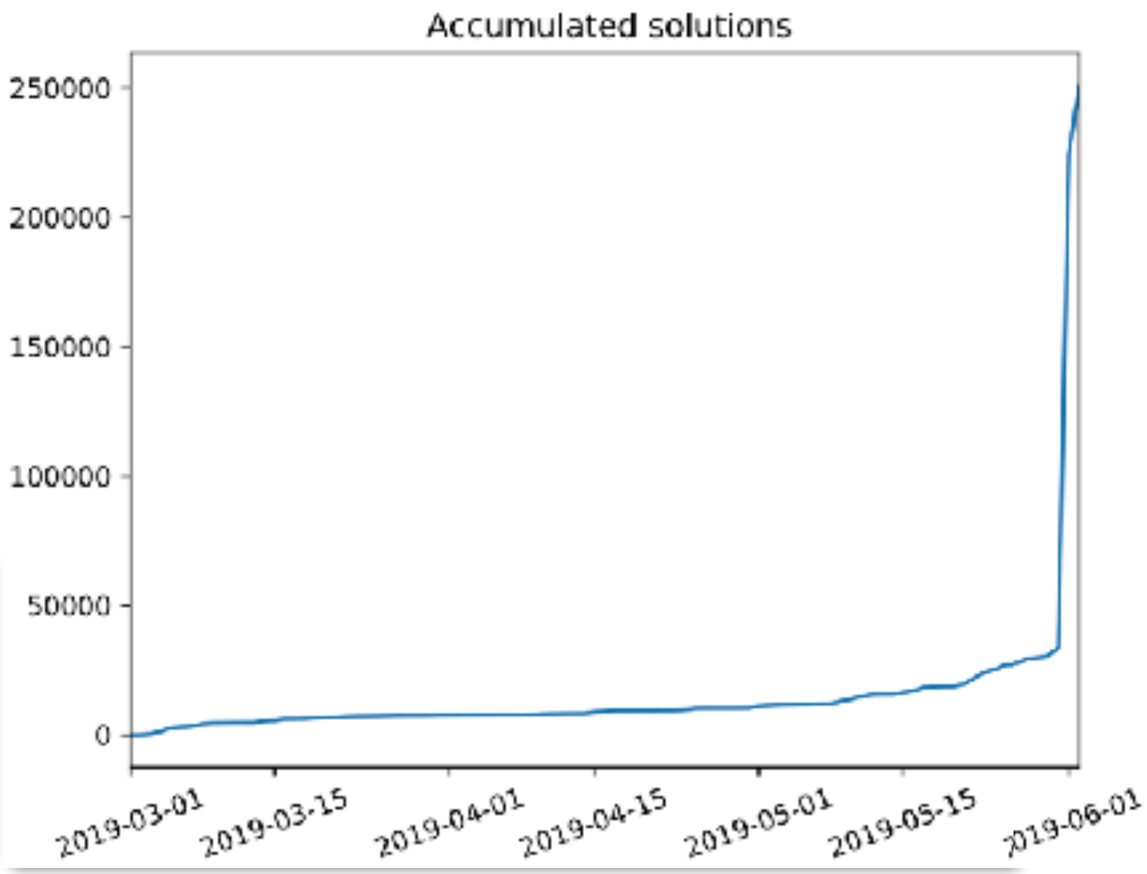

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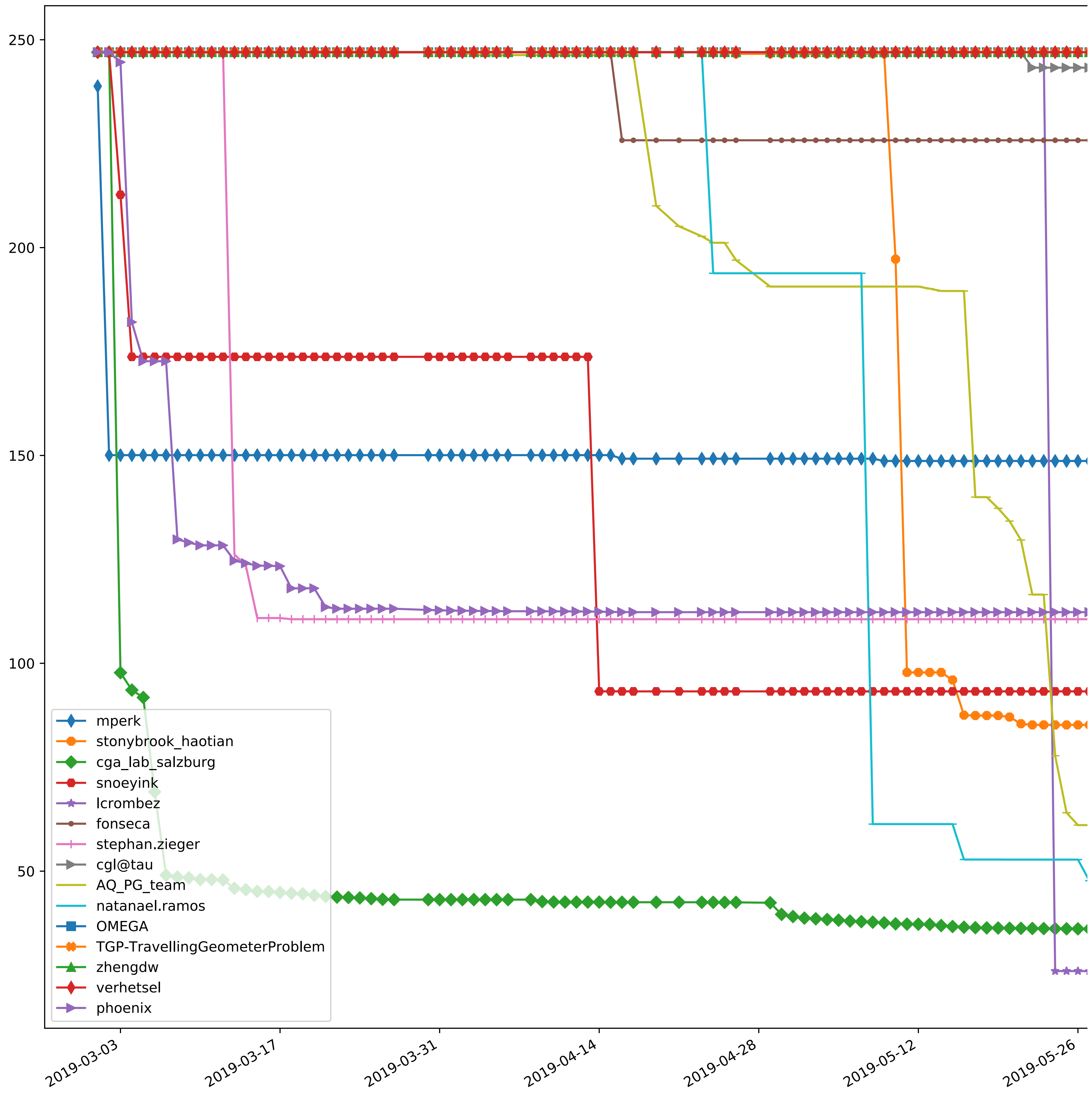
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TGP-TravellingGeometerProblem
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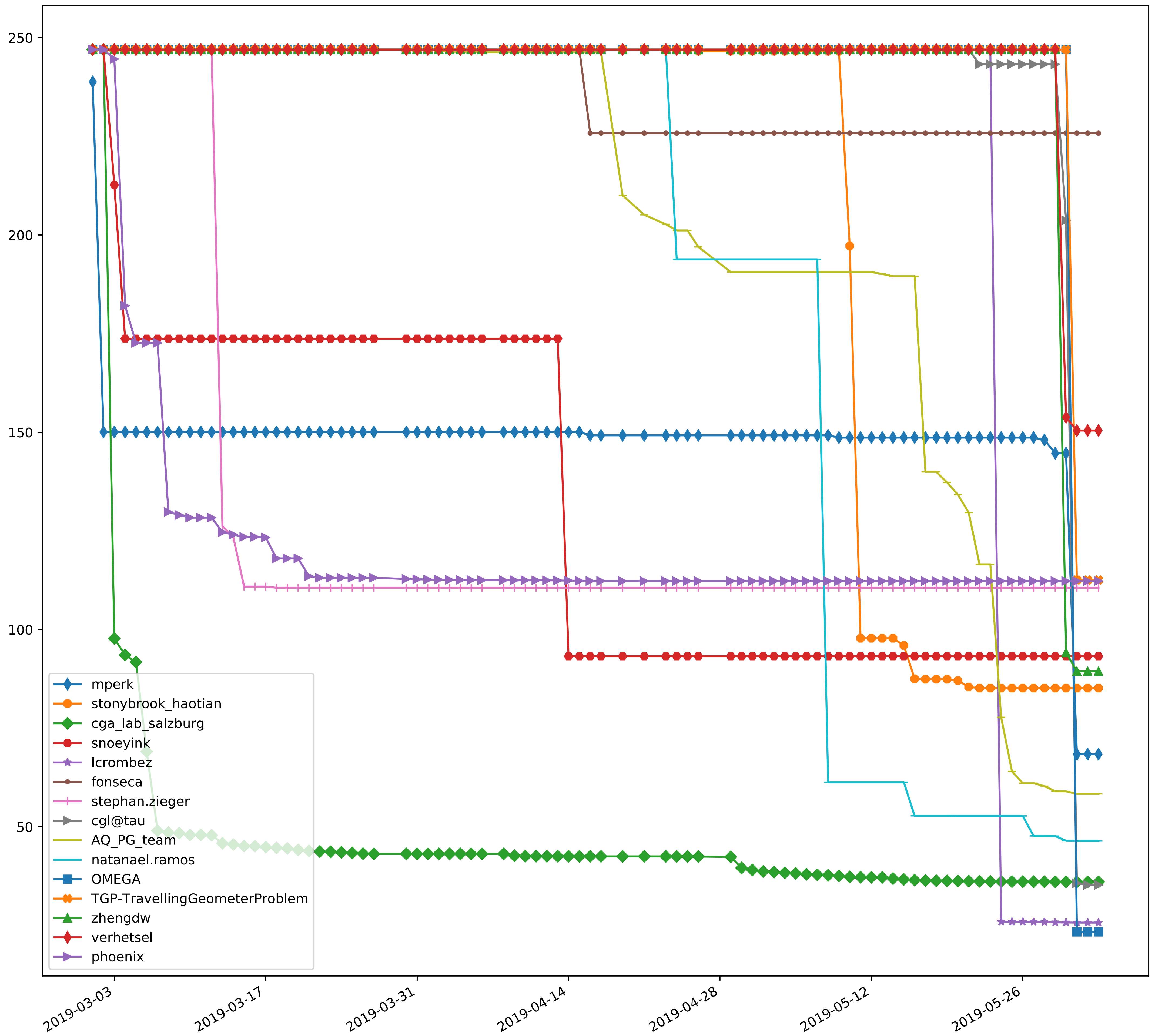

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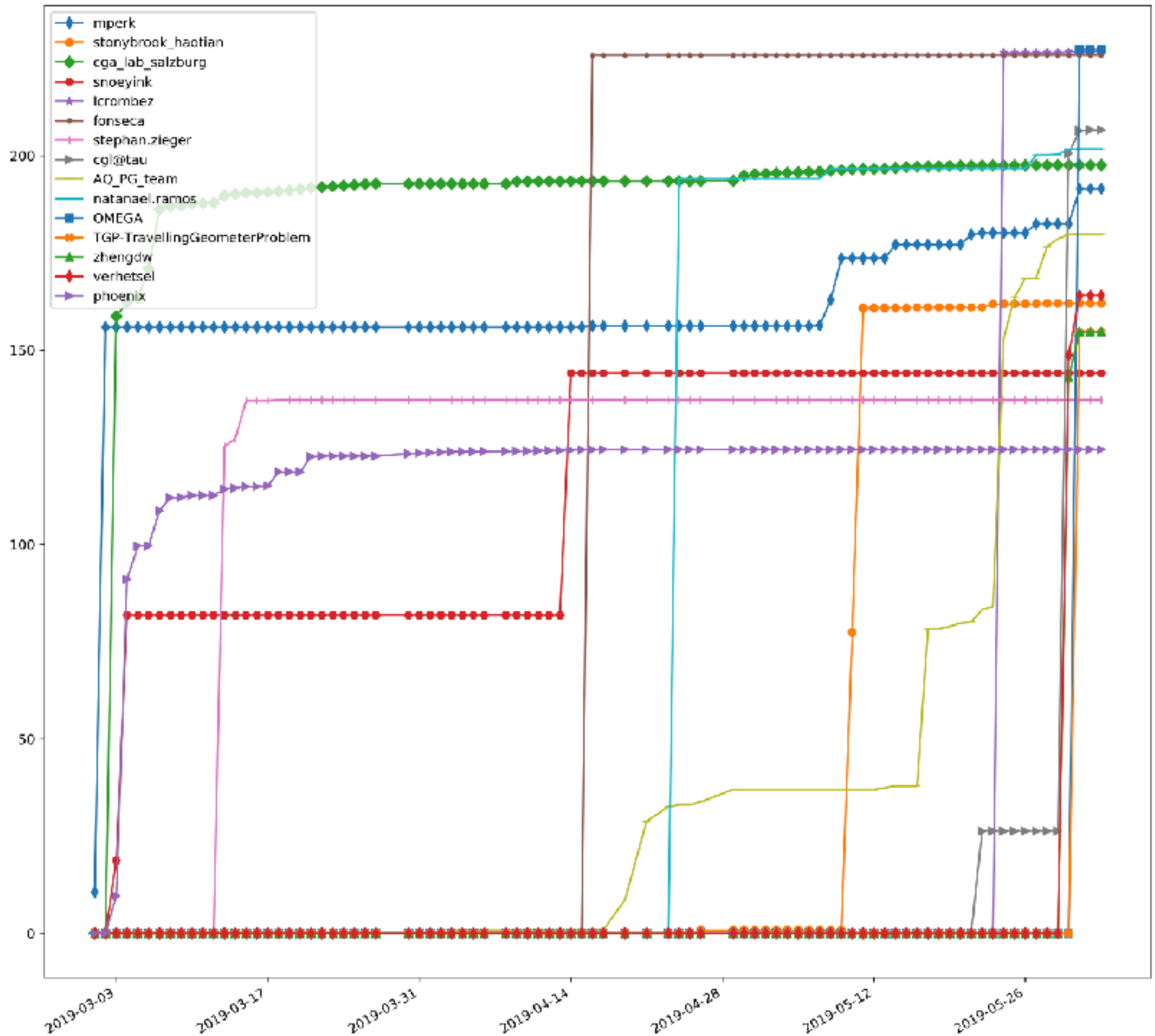

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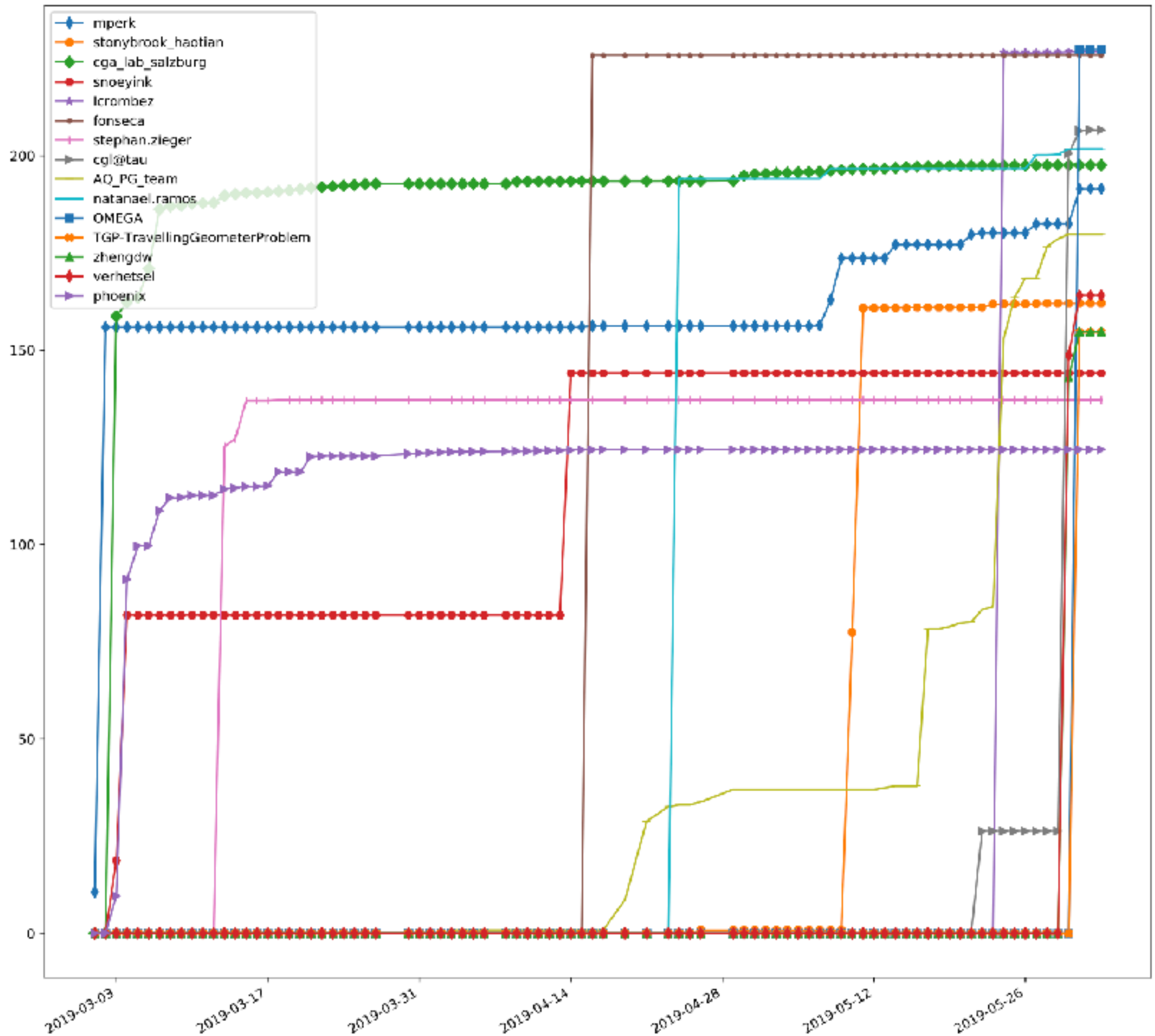

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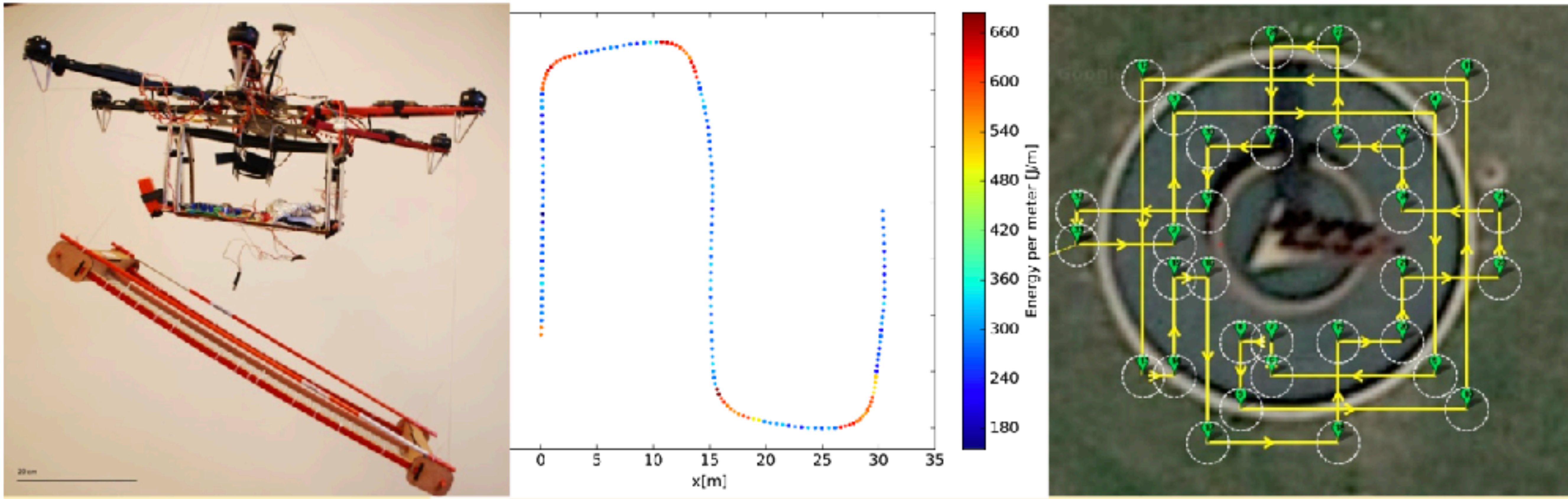
1. Introduction
2. Longest Tours
3. Stars and Matchings
4. Nonsimple Polygons
5. Optimal Area
6. Turn Cost

Video!



Zapping Zika with a Mosquito-Managing Drone: Computing Optimal Flight Patterns with Minimum Turn Cost

Aaron Becker, Mustapha Debboun, Sándor P. Fekete
Dominik Krupke, An Nguyen

Technische Universität Braunschweig

Thank you!

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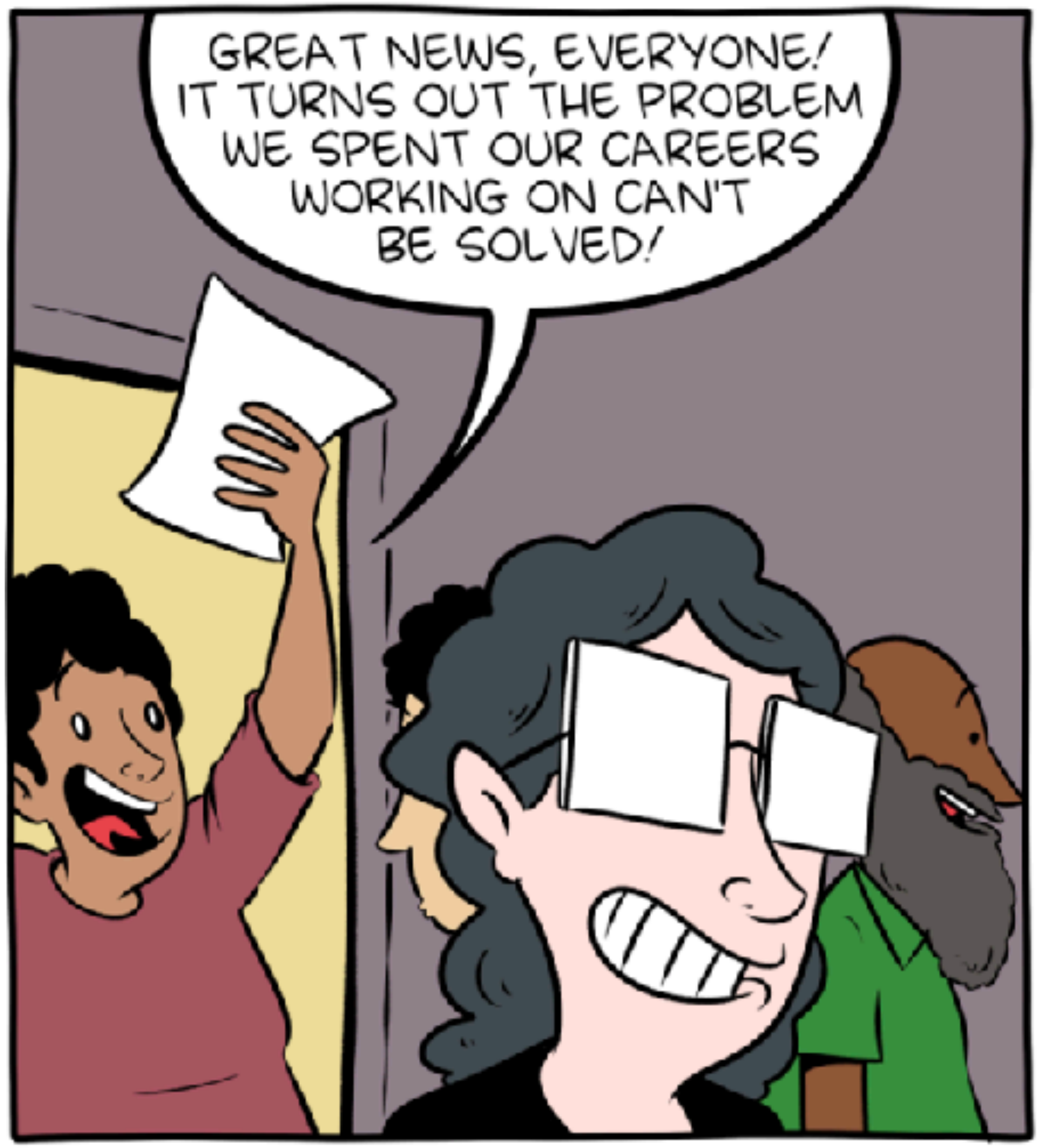
Thank you!



Mathematicians are weird.

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