Abteilung Algorithmik **Summer term 2018**
Institut für Betriebssysteme und Rechnerverbund
TU Braunschweig

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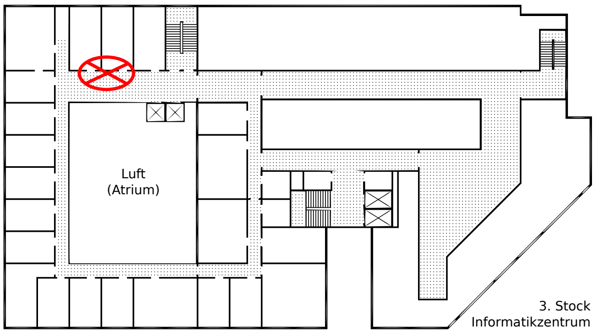
Online Algorithms
5^{th} Homework Assignment, 25^{th} of June 2018

Solutions are due Monday, the 9^{th} of July 2018, until 1:15 PM in the homework cupboard. You can also hand in your solution in person before the small tutorial begins. If you cannot hand in the homework in person, you can also hand it in via e-mail to both `j.heroldt@tu-bs.de` *and* `keldenich@ibr.cs.tu-bs.de`. Please clearly label your solutions using your name and matriculation number.



Exercise 1 (Lambert W function): This exercise serves as preparation for Exercise 2. The Lambert W function is the inverse function of $f(x) = xe^x$, i.e., $x = W(x) \cdot e^{W(x)}$. It is monotonically increasing for $x \geq 0$, unbounded, and does not have a closed-form solution.

a) Prove that $W(\log n) \in \Theta(\log \log n)$.

b) Prove that the inverse function of x^x is $\frac{\log x}{W(\log x)}$.

(5+7 points)

Exercise 2 (1D Mapping with Scan Cost): In this exercise, we consider a one-dimensional mapping problem with scan and movement costs. The 2D variant of this problem has applications in MRI and similar imaging techniques.

In our problem, we have a magnetic particle on a ray. We can move this particle at unit speed in one direction along the ray by applying a magnetic field. We know that at some unknown distance $D > 1$ from the start, there is an obstacle that will block the motion of our particle. While we are not moving the particle, we can perform a scan to determine its current position. Our goal is to determine the distance D by moving the particle into the obstacle; we only notice that our particle has hit the obstacle if we perform a scan and see that it did not move as far as it should have.

We want to minimize the time taken to move the particle as well as the number of scans. We consider the *bicriteria* version of this problem; i.e., in order to have a competitive ratio of R, a strategy must be R-competitive with respect to both the movement cost and the number of scans.

The optimal solution scans once in the beginning, then moves $D + 1$ steps, and then scans again. Thus, the optimal solution has 2 scans and movement cost $D+1$. Therefore, in order to be R-competitive, our strategy must perform at most $2R$ scans and have movement cost at most $R(D + 1)$.

a) Prove that the strategy that performs the jth scan after moving a total distance of 2^j is $\Theta(\log D)$-competitive.

b) Prove that the strategy that performs the jth scan after moving a total distance of j^j uses $\Theta(\log(D)/\log\log(D))$ scans.

c) Prove that the strategy that performs the jth scan after moving a total distance of j^j is $\Theta(\log(D)/\log\log(D))$-competitive. *Hint:* $\left(1 + \frac{1}{n}\right)^n \leq e$ for all $n \geq 1$.

(8+10+15 points)

Exercise 3 (Randomized Bin Packing with Unlimited Capacity): In this exercise, we again consider the unlimited capacity variant of Bin Packing from the last exercise sheet. In this variant of Bin Packing, we receive a sequence of items $a_i \in (0, \infty)$. Moreover we have a fixed number of m bins with unlimited capacity. As in the original Bin Packing problem, we have to pack each incoming item into a bin that we have to fix before the next item is given to us. The goal is to distribute the weight as evenly as possible. More formally, we want to minimize the total weight W packed into the fullest bin.

For the case of $m = 2$ bins, prove that there is no constant $c < \frac{4}{3}$ such that there is a randomized online algorithm that is c-competitive against an oblivious adversary. *Hint:* Consider the input sequence $a_1 = 1, a_2 = 1, a_3 = 2$ and the expected weight in the fuller bin after the first two items. **(15 points)**