Technische Universität Braunschweig

Institute of Operating Systems
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# Network Algorithms

## Tutorial 4: Matching and other stuff

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# Matching

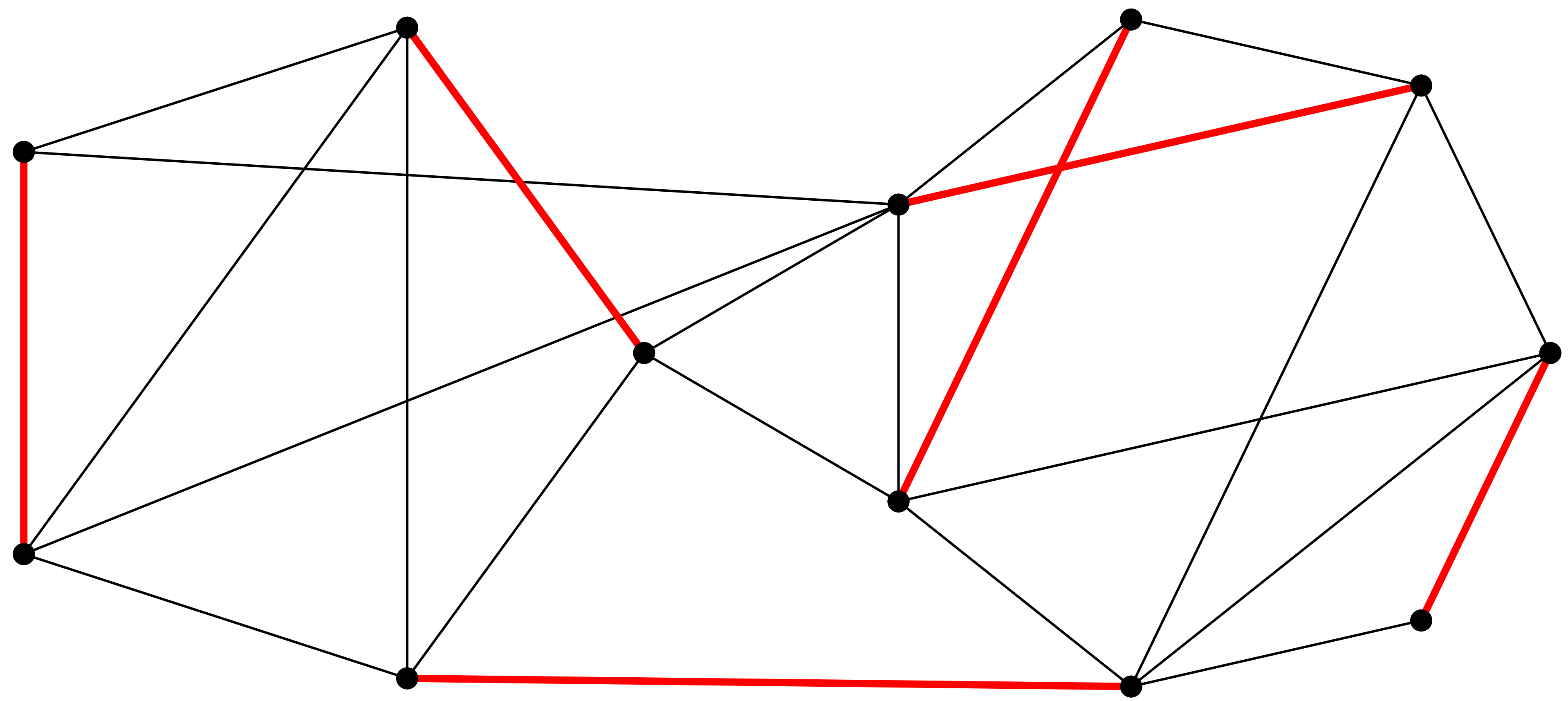
# Matching

A **matching** $M$ in a graph is a set of pairwise disjoint edges.

- perfect matching: all vertices are incident to exactly one edge in the matching
- maximum matching: a matching with the largest possible number of edges

- For weighted graphs we can also consider such matchings like min-cost perfect matching, i.e., a perfect matching that has the smallest possible weight.

# Perfect Matching



A graph with a perfect matching (in red).

# Metric TSP

# Metric TSP

**Given:** A complete, weighted graph G=(V,E) such that the triangle inequality holds for any three vertices, i.e., for all vertices i,j,k the following holds

$$d(\{i,k\}) \leq d(\{i,j\}) + d(\{j,k\})$$

**Wanted:** A Hamiltonian cycle of smallest possible length.

- Because the input graph is a complete graph, such a cycle exist.
- We can show that this problem is NP-complete.

# Metric TSP

**Definition:** An Eulerian tour in a graph G is a tour such that each edge is used exactly once.

**Theorem:** A graph G contains an Eulerian tour if and only if each vertex has even degree.

- We can construct a TSP tour based on an Eulerian tour.
- This gives an approximation on the optimal TSP tour!

**<u>Lemma:</u>** Consider a complete, weighted graph G whose edges suffices the triangle inequality. Let T=(V,E') be a spanning subgraph such that T has an Eulerian tour of length d(R). Then there is a TSP tour of length at most d(R) in G.

- if R is already a TSP tour, we are done
- else: use triangle inequality to skip doubled visited vertices in R; if the vertex with index k is visited twice, we get a new tour R' with length
  $d(R') = d(R) - [d(\{v_{k-1}, v_k\}) + d(\{v_k, v_{k+1}\})] + d(\{v_{k-1}, v_{k+1}\}) \leq d(R)$
- do this iteratively on all vertices which are visited more than once,...

# Metric TSP

The first approximation is based on doubling all edges of a minimum spanning tree.

**Theorem:** There is a simple 2-approximation for metricTSP.

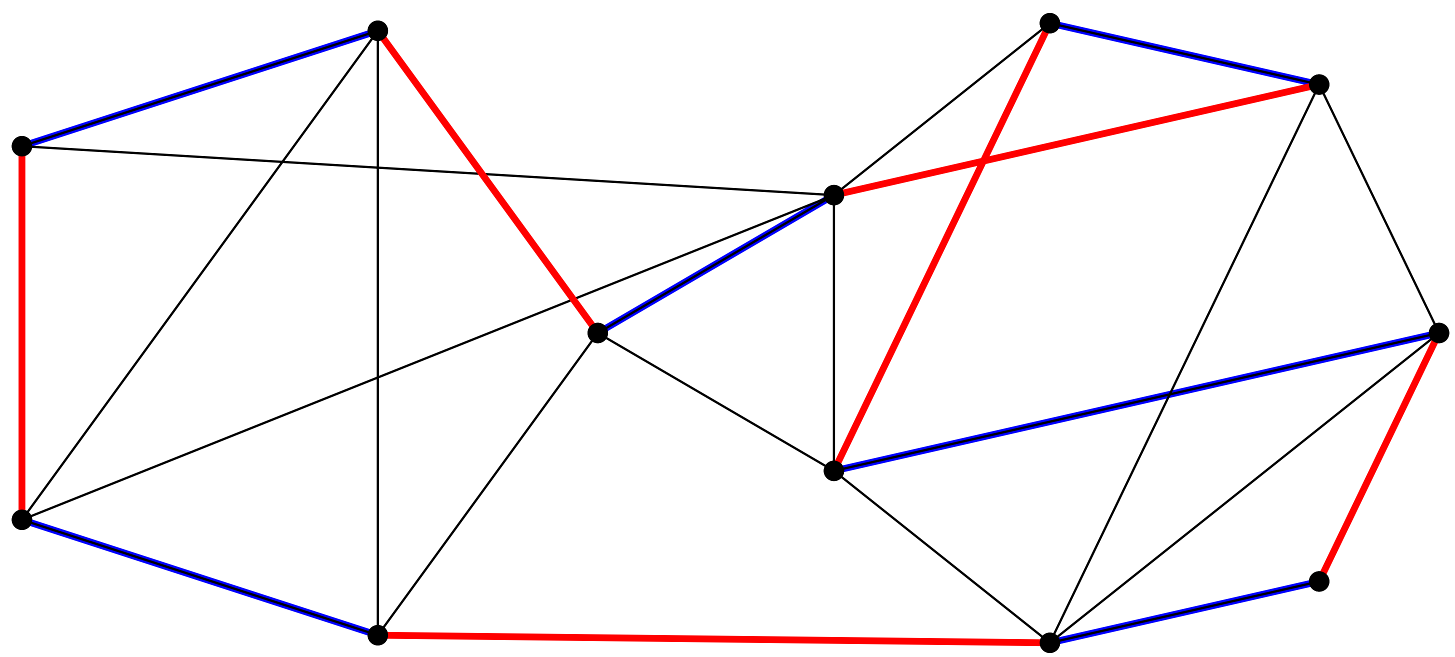
- construct a minimum spanning tree T in G
- double all edges to get an Eulerian graph
- an Eulerian tour has length 2OPT, where OPT is the optimal TSP tour in G
- skip vertices which are visited more than once
- the resulting TSP tour has length at most 2OPT

We now can use a bit more graph theory and analyze the different structures carefully to get a 1.5-approximation.

**Theorem:** In any graph, the number of vertices with odd degree is even.

- in any graph we have $\sum_{v \in V} \delta(v) = 2|E|$
- if there would be an odd number of vertices with odd degree, this sum has to be odd.

**<u>Observation:</u>** Given an optimal TSP tour R (w.l.o.g. assume that R has an even number of vertices), we can subdivide this tour into two disjoint perfect matchings.



The blue and red edges form two disjoint perfect matchings, while together they form a Hamiltonian cycle.

# Metric TSP

**Theorem:** There is a 1.5-approximation for metricTSP.

This algorithm is given by Christofides in 1976.

- construct a minimum spanning tree T of G
- perform a minimum cost perfect matching M on the odd vertices of T
- add the edges of M to T to get an Eulerian graph
- skip vertices which are visited more than once to get a TSP tour R’ in G

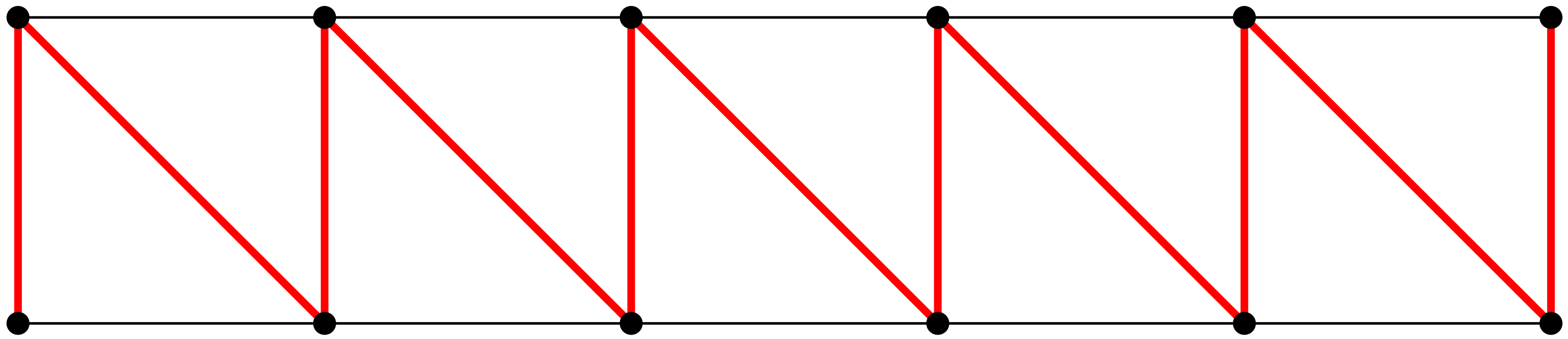
## Metric TSP

We now want to proof this theorem.
Let OPT be the length of an optimal TSP tour R.

- delete an edge from an optimal tour, we get a tree
  - thus, an MST T has length at most OPT
- we can split an optimal tour into two disjoint perfect matchings M' and M" such that $M' \cup M'' = R$
  - let M be the minimum cost perfect matching, we get $d(M) \leq d(M')$ and $d(M) \leq d(M'')$, and therefore
$$2d(M) \leq d(M') + d(M'') \leq OPT$$
- putting these together, we get
$$d(R') \leq d(M) + d(T) \leq \frac{1}{2}OPT + OPT = \frac{3}{2}OPT$$

# Metric TSP

Is there an instance were we get a tour of this length?



- all edges have unit length
- an optimal tour has length $|V|$
- let the red tree be the chosen MST of length $|V|$-1
- the min-cost perfect matching has length $|V|/2$
- together we get a tour of length $3|V|/2$ (for large $|V|$...)

# Questions?