Exercises for Geometric Algorithms SS 2012

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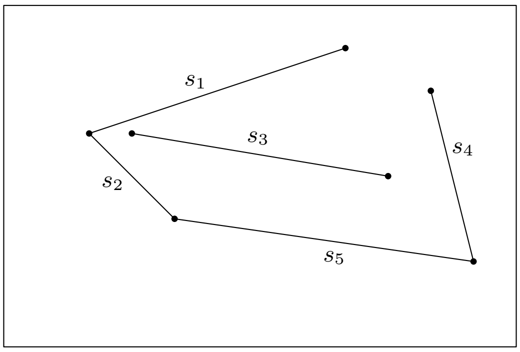
Recap - DCEL

1. Given a planar subdivision, encoded in a doubly connected edge list. How can we visit all edges incident to some vertex v? Prove the running time of your method.

Trapezoidal Maps and Point Location

2. Consider the set $S = \{s_1, \ldots, s_5\}$ of 5 line segments shown below.
   (a) Insert the segments in ascending order into a Trapezoidal Map $\mathcal{T}$. After each insertion, give both the Trapezoidal Map $\mathcal{T}_i$ and the corresponding search structure $\mathcal{D}_i$.
   (b) How does the final search structure $\mathcal{D}_5$ change if you insert s_5 before s_4?

   Hints: This is an extension of the example we did in class. For b) you may want to label the trapezoids the same as in a) to aid the comparison.



3. Give an example of a set of n line segments together with an insertion order that makes ALG 1 create a search structure of size $\Theta(n^2)$ and worst-case query time $\Theta(n)$.
4. Prove that the number of inner nodes of the search structure $\mathcal{D}$ of ALG 1 increases by $k_i - 1$ in iteration i, where k_i is the number of new trapezoids in $\mathcal{T}(S_i)$.
5. In the *single shot problem* the subdivision and the query point are given at the same time, and we have no special preprocessing to speed up the search. Solve the following problem under this assumption:
   Given a convex polygon P as an array of its n vertices in sorted order along the boundary. Show that, given a query point q, it can be tested in $O(\log n)$ time whether q lies inside P.

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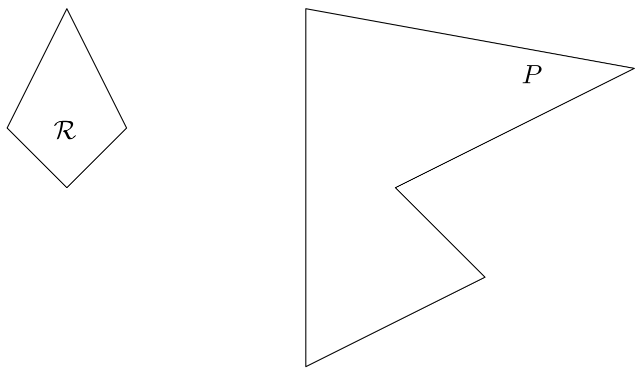
Robot Motion Planning

6. Given a robot that may translate and rotate freely in 2D Euclidean Space.
   (a) How many dimensions does the configuration space have?
   (b) How does this space look like?
7. Show that ALG 4 COMPUTE PATH is correct, i.e., all reported paths are always collision-free and if a collision-free path exists, the algorithm will always find it.
8. In the road map $\mathcal{G}_{road}$ that was constructed on the trapezoidal decomposition of the free space, we added a node in the center of each trapezoid and on each vertical wall. Is it possible to avoid adding the nodes in the center of each trapezoid? Show how the graph can be changed such that only nodes on the vertical walls are required - if possible without increasing the number of edges in the graph. How does the query algorithm need to be adapted?
9. Given the robot $\mathcal{R}$ and the polygon P in the image below. Choose some reference point of the robot.
   (a) Draw the Minkowski sum of the robot and the polygon.
   (b) Draw the configuration space obstacle of the polygon.

   What happens if you choose another reference point and redraw the Minkowski sum and configuration space obstacle?



10. Prove that the shape of a configuration space obstacle is independent of the reference point in the robot $\mathcal{R}$.

Shortest Paths

11. Give a formal proof of the following lemma:
Any shortest path between p_{start} and p_{goal} among a set S of disjoint polygonal obstacles is a polygonal path that has inner vertices at vertices of S.
Comment: You have to show that the path is polygonal AND that the vertices can only be at vertices of S.

12. Let S be a set of disjoint simple polygons in the plane with n edges in total. Prove that for any start and goal position the number of segments on the shortest path is bounded by $O(n)$. Give an example where it is $\Theta(n)$.

13. What is the maximal number of shortest paths connecting two fixed points among a set of n triangles in the plane?

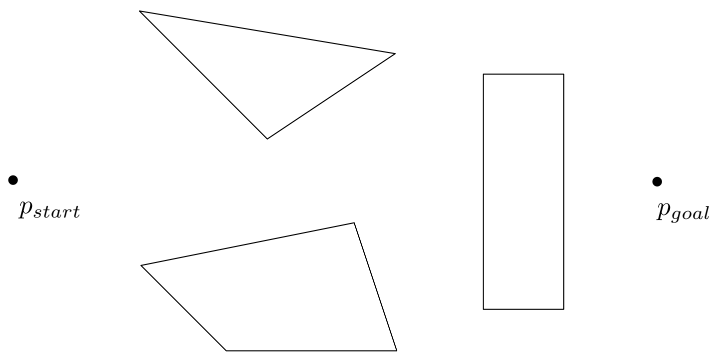
14. Design an algorithm to find a shortest path between two points inside a simple polygon. Your algorithm should run in subquadratic time.

15. There are variants of the definition of "path homotopy", depending on how we treat the endpoints of a path:

    - **tack:** the endpoints are no obstacles, i.e., the path may move over them
    - **pin:** endpoints are considered as point obstacles
    - **pushpin:** endpoints are larger obstacles with an ϵ-neighborhood that is not included in the space M

    Given these definitions, we want to find simple paths between p_{start} and p_{goal}. Consider the setting below - how many different (path-) homotopy classes can there be

    (a) in the tack model?
    (b) in the pin model?



16. What happens if p_{start} and p_{goal} are vertices of a polygon that contains all three drawn obstacles as holes? Consider both the tack- and pin model again.

Arrangements and Duality

17. Give a proof to the following theorem about the complexity of a line arrangement of n lines in the plane.
Theorem: Let L be a set of n lines in the plane, and let $\mathcal{A}(L)$ be the arrangement induced by L.
    i) The number of vertices of $\mathcal{A}(L)$ is at most $n(n-1)/2$.
    ii) The number of edges of $\mathcal{A}(L)$ is at most n^2.
    iii) The number of faces of $\mathcal{A}(L)$ is at most $n^2/2 + n/2 + 1$.

    Equality holds in these three statements if and only if $\mathcal{A}(L)$ is simple.

18. ALG 10 as described in the lecture only works for line sets that form simple arrangements. How would you need to adapt ALG 10 to allow for non-simple arrangements as well?

19. Duality in the plane:
    (a) What is the dual of a line segment?
    (b) What type of object in the primal plane would dualize to a double wedge that is open at the top and bottom?

20. Let S be a set of n points in the plane. Give an $O(n^2)$ time algorithm to find the line containing the maximum number of points in S.

21. Let L be a set of n lines in the plane. Give an $O(n \log n)$ time algorithm to compute an axis parallel rectangle that contains all the vertices of $\mathcal{A}(L)$ in its interior.

22. Let L be a set of n non-vertical lines in the plane. Suppose the arrangement $\mathcal{A}(L)$ only has vertices with level 0. What can you say about this arrangement? Next suppose that lines of L can be vertical. What can you say now about the arrangement?

Polygon Partitioning

23. Given a convex polygon P with n vertices.
    (a) Show that the medial axis of P can have a vertex of degree n.
    (b) What is the minimum and maximum number of edges in the medial axis tree?

24. Is there a non-convex polygon such that its medial axis does NOT contain any parabolic arcs, but instead is composed entirely of straight segments?

25. What is the minimum number of edges the medial axis tree $M(P)$ can have for an arbitrary polygon of n vertices?

26. Show an example for which the straight skeleton $S(P)$ has $2n-3$ edges for a polygon with n vertices.

27. Design an algorithm to construct the straight skeleton in $O(n^3)$ time.