The One-Round Voronoi Game Replayed

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Abstract

We consider the one-round Voronoi game, where the first player ("White", called "Wilma") places a set of n points in a rectangular area of aspect ratio $\rho \leq 1$, followed by the second player ("Black", called "Barney"), who places the same number of points. Each player wins the fraction of A closest to one of his points, and the goal is to win more than half of the total area. This problem has been studied by Cheong et al. who showed that for large enough n and $\rho = 1$, Barney has a strategy that guarantees a fraction of $1/2 + \alpha$, for some small fixed α.

We resolve a number of open problems raised by that paper. In particular, we give a precise characterization of the outcome of the game for optimal play: We show that Barney has a winning strategy for $n \geq 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geq 3$ and $\rho \leq \sqrt{2}/n$, for $n = 2$ and $\rho \leq \sqrt{3}/2$, and for $n = 1$. We also discuss complexity aspects of the game on more general boards, by proving that for a polygon with holes, it is NP-hard to maximize the area Barney can win against a given set of points by Wilma.

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1 Introduction

When determining success or failure of an enterprise, *location* is one of the most important issues. Probably the most natural way to determine the value of a possible position for a facility is the distance to potential customer sites. Various geometric scenarios have been considered; see the extensive list of references in the paper by Fekete, Mitchell, and Weinbrecht [6] for an overview.

One particularly important issue in location theory is the study of strategies for competing players. See the surveys by Tobin, Friesz, and Miller [7], by Eiselt and Laporte [4], and by Eiselt, Laporte, and Thisse [5].

A simple geometric model for the value of a position is used in the *Voronoi game*, which was proposed by Ahn et al. [1] for the one-dimensional scenario and extended by Cheong et al. [2] to the two- and higher-dimensional case. In this game, a site s "owns" the part of the playing arena that is closer to s than to any other site. Both considered a two-player version with a finite arena Q. The players, White ("Wilma") and Black ("Barney"), place points in Q; Wilma plays first. No point that has been occupied can be changed or reused by either player. Let W be the set of points that were played by the end of the game by Wilma,

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while B is the set of points played by Barney. At the end of the game, a Voronoi diagram of $W \cup B$ is constructed; each player wins the total area of all cells belonging to points in his or her set. The player with the larger total area wins.

Ahn et al. [1] showed that for a one-dimensional arena, i.e., a line segment $[0, 2n]$, Barney can win the *n-round game*, in which each player places a single point in each turn; however, Wilma can keep Barney's winning margin arbitrarily small. This differs from the *one-round game*, in which both players get a single turn with n points each: Here, Wilma can force a win by playing the odd integer points $\{1, 3, \ldots, 2n-1\}$; again, the losing player can make the margin as small as he wishes. The used strategy focuses on "key points". The question raised in the end of that paper is whether a similar notion can be extended to the two-dimensional scenario. We will see in Section 3 that in a certain sense, this is indeed the case.

Cheong et al. [2] showed that the two- or higher-dimensional scenario differs significantly: For sufficiently large $n \geq n_0$ and a playing surface Q, the second player has a winning strategy that guarantees at least a fixed fraction of $1/2 + \alpha$ of the total area. Their proof uses a clever combination of probabilistic arguments to show that Barney will do well by playing a random point. The paper gives rise to some interesting open questions:

- How large does n_0 have to be to guarantee a winning strategy for Barney? Wilma wins for $n = 1$, but it is not clear whether there is a single n_0 for which the game changes from Wilma to Barney, or whether there are multiple changing points.
- For sufficiently "fat" arenas, Barney wins, while Wilma wins for the degenerate case of a line. How exactly does the outcome of the game depend on the aspect ratio of the playing board?
- What happens if the number of points played by Wilma and Barney are not identical?
- What configurations of white points limit the possible gain of black points? As candidates, square or hexagonal grids were named.
- What happens for the multiple-round version of the game?
- What happens for asymmetric playing boards?

For rectangular boards and arbitrary values of n, we will give a precise characterization of when Barney can win the game. If the board Q has aspect ratio ρ with $\rho \leq 1$, we prove the following:

- Barney has a winning strategy for $n \geq 3$ and $\rho > \sqrt{2}/n$, and for $n = 2$ and $\rho > \sqrt{3}/2$. Wilma wins in all remaining cases, i.e., for $n \geq 3$ and $\rho \leq \sqrt{2}/n$, for $n = 2$ and $\rho \leq \sqrt{3}/2$, and for $n = 1$.
- If Wilma does not play her points on an orthogonal grid, then Barney wins the game.

In addition, we hint at the difficulties of more complex playing boards by showing the following:

- If Q is a polygon with holes, and Wilma has made her move, it is NP-hard to find a position of black points that maximizes the area that Barney wins.

This result is also related to recent work by Dehne, Klein, and Seidel [3] of a different type: They studied the problem of placing a single black point within the convex hull of a set of white points, such that the resulting black Voronoi cell in the unbounded Euclidean plane is maximized. They showed that there is a unique local maximum.

The rest of this paper is organized as follows. After some technical preliminaries in Section 2, Section 3 shows that Barney always wins if Wilma does not place her points on a regular orthogonal grid. This is used in Section 4 to establish our results on the critical aspect ratios. Section 5 presents some results on the computational complexity of playing optimally in a more complex board. Some concluding thoughts are presented in Section 6.

2 Preliminaries

In the following, Q is the playing board. Q is a rectangle of *aspect ratio* ρ, which is the ratio of the length of the smaller side divided by the length of the longer side. Unless noted otherwise (in some parts of Section 5), both players play n points; W denotes the n points played by Wilma, while B is the set of n points played by Barney. All distances are measured according to the Euclidean norm. For a set of points P, we denote by $V(P)$ the (Euclidean) Voronoi diagram of P. We call a Voronoi diagram $V(P)$ a regular grid if

- all Voronoi cells are rectangular, congruent and have the same orientation;
- each point $p \in P$ lies in the center of its Voronoi cell.

If e is a Voronoi edge, $C(e)$ denotes a Voronoi cell adjacent to e. If $p \in P$, then $C(p)$ denotes the Voronoi cell of p in $V(P)$. $\partial C(p)$ is the boundary of $C(p)$ and $|C(p)|$ denotes the area of $C(p)$. $|e|$ denotes the length of an edge e. Let x_p and y_p denote the x- and y-coordinates of a point p.

3 A Reduction to Grids

As a first important step, we reduce the possible configurations that Wilma may play without losing the game. The following holds for boards of any shape:

Lemma 1 *If $V(W)$ contains a cell that is not point symmetric, then Barney wins.*

Proof. Suppose there is a point $w \in W$ and a line through w that does not bisect the area of $C(w)$; say, we have $|C(w)|/2 + 2\alpha$ on one side of the line. This means that by placing a point close to w, Barney can claim at least $|C(w)|/2 + 2\alpha - \alpha/n$ of the cell $C(w)$. In each other cell $C(w)$ of $V(W)$ Barney can place a point close enough to w to claim an area of at least $|C(w)|/2 - \alpha/n$. Therefore Barney has gained at least $|Q|/2 + \alpha$.

This means that for each $w \in W$, any line $l(\varphi)$ that encloses an angle φ with the positive x-axis and passes through w bisects $C(w)$. Let $r(\varphi)$ be the distance of w to the point on the boundary of $C(w)$ that is stabbed by a ray emanating from w at angle φ. As $\partial C(w)$ is a convex curve, $r(\varphi)$ is a continuous function. Furthermore, we see that $|C(p)| = \frac{1}{2}\int_0^{2\pi} r^2(\varphi)d\varphi$, and the portion of $C(p)$ enclosed between angles φ_1 and φ_2 is $\frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2(\varphi)d\varphi$. As an infinitesimal rotation of $l(\varphi)$ about w does not change the area on either side, it follows that $r(\varphi) = r(-\varphi)$ for any φ, meaning that $C(w)$ is point symmetric. □

The following theorem is based on this observation and will be used as a key tool for simplifying our discussion in Section 4.

Theorem 2 *If the board is a rectangle and if $V(W)$ is not a regular grid, then Barney wins.*

Proof. We assume that Barney cannot win, and will show that this implies that $V(W)$ is a regular grid. By Lemma 1, we may assume that all cells of $V(W)$ are point symmetric. Let e_0 be a Voronoi edge of $V(W)$ on the top side of the board. Consider the Voronoi cell C_0 adjacent to e_0. Because C_0 is point symmetric, it contains an edge e_1 that is parallel to e_0 with $|e_0| = |e_1|$. Let C_1 be the cell adjacent to and below e_1. It contains an edge e_2 with $|e_2| = |e_1|$. Similarly define the cells $C_2, C_3, \ldots$. So cell C_i lies below C_{i-1}. Therefore there is a cell C_{k-1} such that e_k lies on the bottom edge of the board. We call $S(e_0) = \{C_0, C_1, C_2, \ldots, C_{k-1}\}$ the strip of e_0. Because C_i is convex, any horizontal line that intersects the board has an intersection with $S(e_0)$ of length $\geq |e_0|$. Consider two different Voronoi edges, e and f, on the top side of the board, with their respective strips $S(e)$ and $S(f)$. Because Voronoi cells are convex and do not have corners with angles of size π, these strips cannot intersect, i.e. do not have a cell in common. For an illustration, see Figure 1. Let S be the collection of strips of e for all Voronoi edges e of $V(W)$ on the top side of the board. The intersection of a line with S has a length which is at least as large as the sum of the lengths of the edges along the top side of the board. Because strips do not intersect, this intersection is exactly as long as the top side

of the board. This implies that S covers the whole board and that any horizontal line that intersects the board has an intersection with $S(e_0)$ of length exactly equal to $|e_0|$.

Let e be the left most edge on the top side of the board. The left hand side of $S(e)$ is the left hand side of the board. This implies that each cell in $S(e)$ is a rectangle. By the same argument, each cell in $V(W)$ is a rectangle. Let v and w be two points in W such that $C(v)$ and $C(w)$ in $V(W)$ have an edge e in common. The distance between v and e is the same as the distance between w and e. Because both $C(v)$ and $C(w)$ are point symmetric and rectangular, it follows that they have the same size. So $V(W)$ is a regular grid. □

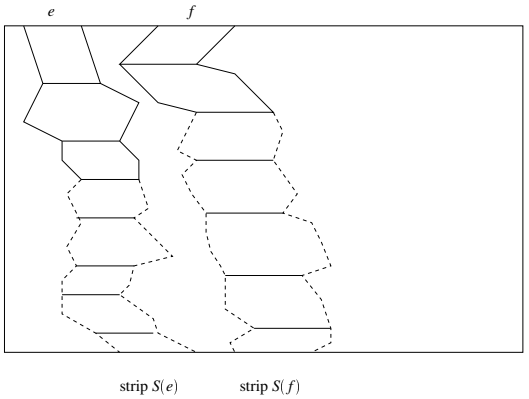

Figure 1: Playing board with two strips.

4 Critical Aspect Ratios

In this section we prove the main result of this paper: if $n \geq 3$ and $\rho > \sqrt{2}/n$, or $n = 2$ and $\rho > \sqrt{3}/2$, then Barney wins. In all other cases, Wilma wins. The proof proceeds by a series of lemmas. We start by noting the following easy observation.

Lemma 3 *Barney wins, if and only if he can place a point p that steals an area strictly larger than $|Q|/2n$ from W.*

Proof. Necessity is obvious. To see sufficiency, note that Wilma is forced to play her points in a regular grid. Barney places his first point p such that it gains an area of more than $|Q|/2n$. Let w be a point in W. If Barney places a point on the line through w and p, sufficiently close to w but on the opposite side of p, he can claim almost half of the Voronoi cell of w. By placing his remaining $n-1$ points in this fashion, he can claim a area larger than $|Q|/2n$. □

Next we take care of the case $n = 2$; this lemma will also be useful for larger n, as it allows further reduction of the possible arrangements Wilma can choose without losing.

Lemma 4 *If $n = 2$ and $\rho > \sqrt{3}/2$, then Barney wins. If the aspect ratio is smaller, Barney loses.*

Proof. Assume without loss of generality that the board has size ρ by 1. Suppose that the left bottom corner of Q lies on the origin. By Theorem 2 we know that Wilma has to place her points at $(0.5, \rho/4)$ and $(0.5, 3\rho/4)$ or at $(0.25, \rho/2)$ and $(0.75, \rho/2)$. If Wilma places her points at $(0.5, \rho/4)$ and $(0.5, 3\rho/4)$, then it is not hard to show that she will lose. So assume that Wilma places her points at $(0.25, \rho/2)$ and $(0.75, \rho/2)$. For Barney to win, he will have to gain more than $\rho/4$ with his first point. Suppose Barney places his point at location p. Without loss of generality, assume that $x_p \geq 0.5$ and $y_p \geq \rho/2$. If $y_p = \rho/2$ then Barney gains at most $\rho/4$, so we may assume that $y_p > \rho/2$. Placing a point p with $x_p > 0.75$ is not optimal for Barney: moving p in the direction of $(0.5, \rho/2)$ will increase the area gained. It is not hard to show that for $x_p = 0.75$, Barney cannot gain an area of size $\rho/4$. So we may assume that $0.5 \leq x_p < 0.75$. Let b_0 be the bisector of p and $(0.25, \rho/2)$. Let b_1 be the bisector of p and $(0.75, \rho/2)$. Let q be the intersection of b_0 and b_1. The point q lies on the vertical line through $x = 0.5$. If q lies outside the board Q, then $|C(p)| < \rho/4$, so assume that q lies in Q. Let h_0 be the length of the line segment on b_0, between q and the top or left side of the board. Let h_1 be the length of the line segment on b_1, between q and the top or right side of the board. Consider the circle C center at q which passes through p, $(0.25, \rho/2)$ and $(0.75, \rho/2)$.

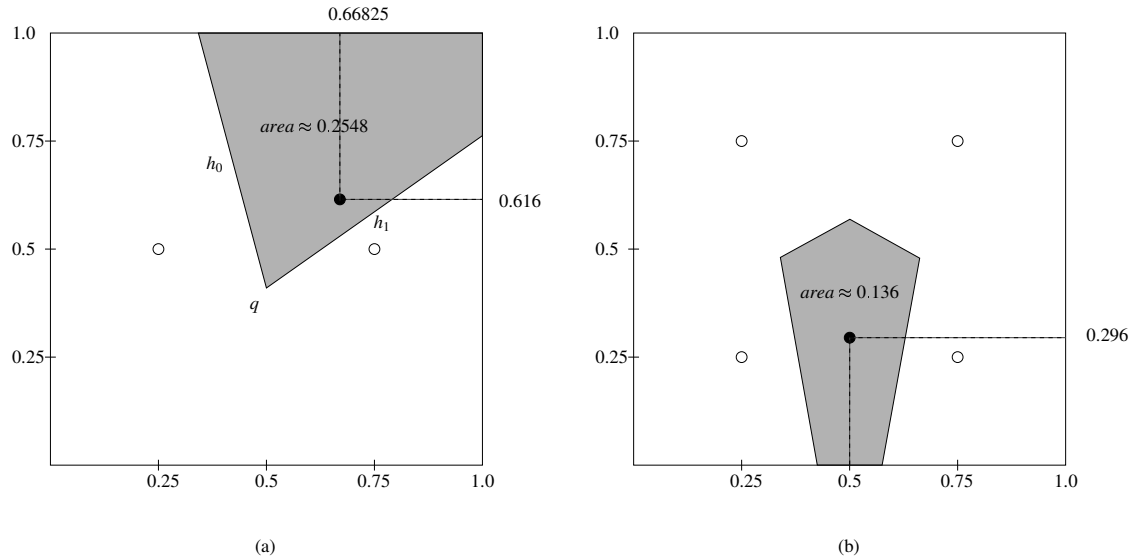

Figure 2: Barney has gained more than a quarter (a) more than an eighth (b) of the playing surface.

If b_0 does not intersect the top of the board then neither does b_1. In this case we can increase $|C(p)|$ by moving p to the left on C and we can use this to show that $|C(p)| < \rho/4$. If both b_0 and b_1 intersect the top of the board we have $h_0 \leq h_1$. We can increase h_1 and decrease h_0 by moving P to the right on C. So $|C(p)|$ can be increased until b_1 intersects the top right corner of the board. If b_0 intersect the top of the board and b_1 intersect the right top corner we have $h_0 \leq h_1$. If we move p to the right on C, both h_0 and h_1 will decrease. The area $|C(p)|$ will increase as long as $h_0 < h_1$ and reaches its maximum value when $h_0 = h_1$. Therefore the maximum exists when at the moment that p approaches $(0.75, \rho/2)$, we have $h_0 > h_1$. When $p = (0.75, \rho/2)$, we have $h_0 = \rho - y_q$ and $h_1 = \sqrt{(}1/4 + (\rho - 2y_q)^2)$. From $h_0 > h_1$ we can derive that $\rho > \sqrt{3}/2$. With his second point Barney can gain an area of size $0.25 - \varepsilon$ for an arbitrary small positive value of ε by placing the point close to $(0.25, \rho/2)$. So Barney can gain more than half the board.

If the aspect ratio is $\leq \sqrt{3}/2$, Barney can gain at most $\rho/4$ with his first move by placing his point at $(x, \rho/2)$ with $0.25 < x < 0.75$. It can be shown that with his second point he can gain almost, but not exactly a quarter. □

The gain for Barney is small if ρ is close to $\sqrt{3}/2$. We have performed computer experiments to compute the gain for Barney for values of $\rho > \sqrt{3}/2$. Not surprisingly, the largest gain was for $\rho = 1$. If the board has size 1×1, Barney can gain an area of approximately 0.2548 with his first point, by placing it at (0.66825,0.616) as illustrated in Figure 2(a).

Lemma 5 *Suppose that the board is rectangular and that $n = 4$. If Wilma places her point on a regular 2×2 grid, Barney can gain* 50.78% *of the board.*

Proof. Assume that the board has size $\rho \times 1$. By Lemma 2 we know that Wilma has to place her points on the horizontal line at height $\rho/2$, on the vertical line at $x = 0.5$ or at the points $(0.25, \rho/4)$, $(0.25, 3\rho/4)$, $(0.75, \rho/4)$ and, $(0.75, 3\rho/4)$. If Wilma does not place her points on a line, it can be computed that Barney wins at least $\rho(1/8 + 1/128)$ by placing a point at $(0.5, \rho/4)$. In addition Barney can gain a little more than $3\rho/8 - \varepsilon$ by placing his remaining three points at $(0.25 - 4\varepsilon/3, \rho/4)$, $(0.25 - 4\varepsilon/3, 3\rho/4)$, and $(0.75 + 4\varepsilon/3, 3\rho/4)$. So Barney will gain a total area of size $\rho(1/2 + 1/128) - \varepsilon$. Because of $1/2 + 1/128 = 50.78125$, the result follows. □

The value in the above lemma is not tight. For example, if Wilma places her point in a 2×2 grid on a square board, we can compute the area that Barney can gain with his first point. If Barney places it at (0.5,0.296), he gains approximately 0.136. For an illustration, see Figure 2(b). By placing his remaining three points at $(0.25 - 4\varepsilon/3, 0.25)$, $(0.25 - 4\varepsilon/3, 0.75)$, and $(0.75 + 4\varepsilon/3, 0.75)$ Barney can gain a total area of size of around $0.511 - \varepsilon$ for arbitrary small positive ε. For non-square boards, we have found larger wins for Black. This suggests that Barney can always gain more than 51% of the board if Wilma places her four points in a 2×2 grid.

The above discussion has an important implication:

Corollary 6 *If $n \geq 3$, then Wilma can only win by placing her points in a $1 \times n$ grid.*

This sets the stage for the final lemma:

Lemma 7 *Let $n \geq 3$. Barney can win if $\rho > \sqrt{2}/n$; otherwise, he loses.*

Proof. It follows from Corollary 6 that Wilma should place her points in a $1 \times n$ grid. Assume that Q has size $2r \times 2n$ and that the left bottom point of Q lies at $(-3, -r)$ and the top right point at $(2n-3, r)$. Wilma must place her points at $(-2,0), (0,0), (2,0), \ldots, (2n-4, 0)$. From Lemma 3 we know that in order to win, Barney has to find a location $p = (x,y)$ with $|V(p)| > 2r$.

If $r > \sqrt{3}$, we know from Lemma 4 that Barney can take more than a quarter from two neighboring cells of Wilma, i.e. Barney takes more than $8r/4 = 2r$ with his first point. Therefore assume that $r \leq \sqrt{3}$. We start by describing the size and area of a potential Voronoi cell for Barney's first point. Without loss of generality, we assume that $p = (x,y)$ with $y, x \geq 0$ is placed in the cell of Wilma's point $(0,0)$, so $x \leq 1$, $y \leq r$.

If $y > 0$ and if Barney gains parts of three cells of $V(W)$ with his first point, we have a situation as shown in Figure 3. It is not hard to see that he can steal from at most three cells: p has distance more than 2 from all cells not neighboring on Wilma's cells $V(-2,0)$ and $V(2,0)$, which is more than the radius of $\sqrt{r^2+1} \leq 2$ of those cells with respect to their center points. We see that

$$b_1 = \frac{y}{2} + \frac{x^2}{2y}, \quad (1)$$

$$\tan\varphi_1 = \frac{x}{y}, \quad (2)$$

$$\tan\varphi_2 = \frac{y}{2-x}. \quad (3)$$

As shown in Figure 3, the Voronoi cell of p consists of three pieces: the quadrangle R_1 (stolen from $V(0,0)$), the triangle R_0 (stolen from $V(-2,0)$), and the triangle R_2 (stolen from $V(2,0)$). Furthermore,

$$|R_1| = 2h_1 = 2(r - b_1) = 2r - y - \frac{x^2}{y}, \quad (4)$$

$$|R_2| = \frac{x_2 h_2}{2}, \quad (5)$$

$$h_2 = r - b_1 + \tan\varphi_1 \quad (6)$$

$$= r - \frac{y}{2} - \frac{x^2}{2y} + \frac{x}{y}, \quad (7)$$

$$x_2 = h_2 \tan\varphi_2 = \frac{\left(r - \frac{y}{2} - \frac{x^2}{2y} + \frac{x}{y}\right) y}{2-x}, \quad (8)$$

so

$$|R_2| = \frac{\left(ry - \frac{y^2}{2} - \frac{x^2}{2} + x\right)^2}{2y(2-x)}. \quad (9)$$

Analogously,

$$|R_0| = \frac{\left(ry - \frac{y^2}{2} - \frac{x^2}{2} - x\right)^2}{2y(2+x)}. \quad (10)$$

We first consider $r \leq \sqrt{2}$. Assume that Barney can win, i.e. can gain an area larger than $2r$ with his first point. If $y = 0$, then $|V(p)| = 2r$, so we may assume that $y > 0$. From Lemma 4, we know that Barney will not win if he only steals from two of Wilma's cells, so we may assume that Barney steals from three cells. Therefore we can use results from equations (4), (9) and (10). From

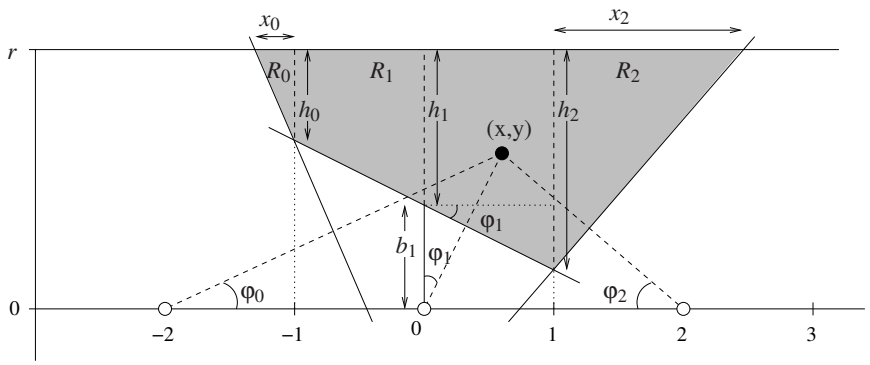

Figure 3: Wilma has placed at least three stones on a line.

$|R_0|+|R_1|+|R_2| > 2r$ we derive

$$\begin{aligned} & \left(ry-\frac{y^2}{2}-\frac{x^2}{2}-x\right)^2(2-x) \\ + & \left(ry-\frac{y^2}{2}-\frac{x^2}{2}+x\right)^2(2+x) \\ > & \ 2(y^2+x^2)(4-x^2) \end{aligned} \tag{11}$$

As the left hand side is maximized for $r=\sqrt{2}$, we conclude

$$\begin{aligned} & \left(\sqrt{2}y-\frac{y^2}{2}-\frac{x^2}{2}-x\right)^2(2-x) \\ + & \left(\sqrt{2}y-\frac{y^2}{2}-\frac{x^2}{2}+x\right)^2(2+x) \\ > & \ 2(y^2+x^2)(4-x^2), \end{aligned} \tag{12}$$

so

$$\begin{aligned} & 4\left(\sqrt{2}y-\frac{y^2}{2}-\frac{x^2}{2}\right)^2 \\ + & \ 4x^2\left(\sqrt{2}y-\frac{y^2}{2}-\frac{x^2}{2}\right)+4x^2 \\ > & \ 8y^2+8x^2-2x^2y^2-2x^4, \end{aligned} \tag{13}$$

implying

$$\begin{aligned} & 4\left(\left(\sqrt{2}y-\frac{y^2}{2}-\frac{x^2}{2}\right)+\frac{x^2}{2}\right)^2 \\ & -x^4+4x^2 \\ > & \ 8y^2+8x^2-2x^2y^2-2x^4, \end{aligned} \tag{14}$$

therefore

$$\begin{aligned} & 2\left(\sqrt{2}y-\frac{y^2}{2}\right)^2 \\ > & \ 4y^2+2x^2-\frac{x^4}{2}-x^2y^2 \end{aligned} \tag{15}$$

and thus

$$\begin{aligned} & 4y^2-2\sqrt{2}y^3+\frac{y^4}{2} \\ > & \ 4y^2+x^2\left(2-\frac{x^2}{2}-y^2\right) \end{aligned} \tag{16}$$

or

$$\begin{aligned} & y^3(\frac{y}{2}-2\sqrt{2}) \\ > & \ x^2\left(2-\frac{x^2}{2}-y^2\right). \end{aligned} \tag{17}$$

As the left hand side is negative for $0 < y \le \sqrt{2}$, we conclude that the right hand side must also be negative; clearly, it is minimized for $x=1$, so we get

$$y^3(\frac{y}{2}-2\sqrt{2}) \ > \ \left(2-\frac{1}{2}-y^2\right), \tag{18}$$

and conclude that $\sqrt{2} \ge y \ge \sqrt{3/2}$, yielding the contradiction

$$4 \ \ge \ \frac{y^4}{2}+y^2 \ > \ \frac{3}{2}+2\sqrt{2}y^3 \ > \ 4. \tag{19}$$

So the best Barney can do is gain an area of size $2r$ with all his points and tie the game. However,

notice that the contradiction in equation (19) also holds if $|R_0| + |R_1| + |R_2| = 2r$. So Barney cannot gain an area of size $2r$ if he places his point at (x,y) with $y > 0$ and steals from three cells of $V(W)$. In Lemma 4 it was shown that Barney will gain less than $2r$ if he places his point at (x,y) with $y > 0$ and steals from two cells of $V(W)$. Therefore Barney must place his points at (x,y) with $y = 0$. This reduces the problem to a one-dimensional one, and we know from [1] that in that case Barney will lose.

Secondly we consider $\sqrt{2} < r \leq \sqrt{3}$. Suppose Barney places his first point at $(0,y)$ with $y > 0$. Clearly he will steal from three cells of $V(W)$. From equations (4), (9) and (10) we derive that

$$\begin{aligned} & |R_0| + |R_1| + |R_2| \\ = \quad & \frac{r^2 y}{2} - \frac{r y^2}{2} + \frac{y^3}{8} + 2r - y. \end{aligned} \tag{20}$$

Because of $y > 0$ we have

$$|R_0| + |R_1| + |R_2| \ > \ 2r \tag{21}$$

$$\Leftrightarrow \quad \frac{r^2 y}{2} - \frac{r y^2}{2} + \frac{y^3}{8} - y \ > \ 0 \tag{22}$$

$$\Leftrightarrow \quad y^2 - 4ry + 4r^2 - 8 \ > \ 0 \tag{23}$$

$$\Leftrightarrow \quad 0 < y < 2(r - \sqrt{2}). \tag{24}$$

So Barney wins if he places a point at $(0,y)$ with $0 < y < 2(r - \sqrt{2})$. □

The value in equation (20), i.e., the total area, is maximal for $y^* = (4r - 2\sqrt{r^2+6})/3$. Computational experiments have confirmed that Barney maximizes the area with his first point at $(0,y^*)$.

Summarizing, we get:

Theorem 8 *If $n \geq 3$ and $\rho > \sqrt{2}/n$, or $n = 2$ and $\rho > \sqrt{3}/2$, then Barney wins. In all other cases, Wilma wins.*

5 A Complexity Result

The previous section resolves most of the questions for the one-round Voronoi game on a rectangular board. Clearly, there are various other questions related to more complex boards; this is one of the questions raised in [2]. Lemma 1 still applies if Wilma's concern is only to avoid a loss. Moreover, it is easily seen that all of Wilma's Voronoi cells must have the same area, as Barney can steal almost all the area of the largest cell by placing two points in it, and no point in the smallest cell. For many boards, both of these conditions may be impossible to fulfill. It is therefore natural to modify the game by shifting the critical margin that decides a win or a loss. We show in the following that it is NP-hard to decide whether Barney can beat a given margin for a polygon with holes, and all of Wilma's stones have already been placed. (In a non-convex polygon, possibly with holes, we measure distances according to the geodesic Euclidean metric, i.e., along a shortest path within the polygon.)

Theorem 9 *For a polygon with holes, it is NP-hard to maximize the area Barney can claim, even if all of Wilma's points have been placed.*

Proof. We give an outline of the proof, based on a reduction from PLANAR 3SAT, which is known to be NP-complete [8]. For clearer description, we sketch the proof for the case where Barney has fewer points to play; in the end, we hint at what can be done to make both point sets the same size. (A 3SAT instance I is said to be an instance of PLANAR 3SAT, if the following bipartite graph G_I is planar: every variable x_i and every clause c_j in I is represented by a vertex in G_I; two vertices are connected, if and only if one of them represents a variable that appears in the clause that is represented by the other vertex.) First, the planar graph corresponding to an instance I of PLANAR 3SAT is represented geometrically as a *planar rectilinear layout*, with each vertex corresponding to a horizontal line segment, and each edge corresponding to a vertical line segment that intersects precisely the line segments corresponding to the two incident vertices. There are well-known algorithms (e.g., [11]) that can achieve such a layout in linear time and linear space. See Figure 4.

Next, the layout is modified such that the line segments corresponding to a vertex and all edges incident to it are replaced by a loop – see Figure 5.

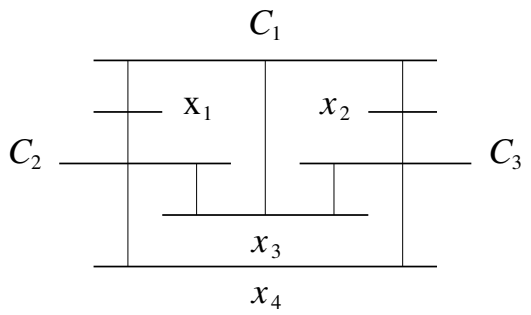

Figure 4: A geometric representation of the graph G_I for the Planar 3SAT instance $I = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_3 \vee x_4) \wedge (\bar{x}_2 \vee \bar{x}_3 \vee x_4)$.

At each vertex corresponding to a clause, three of these loops (corresponding to the respective literals) meet. Each loops gets represented by a very narrow corridor.

Now we place a sequence of extra *area gadgets* at equal distances $2d_1$ along the variable loop. Let n_i be the number of elements along the loop for variable x_i, and let $N = \sum_{i=1}^{n} n_i$, and $\varepsilon = 1/N^3$. (By construction, N is polynomially bounded.) The two basic ingredients of each such gadget is a white point "guarding" an area element of size $A = 1/N$, i.e., being at distance $d_1 + \varepsilon$ from it. Finally, for each clause, we place an extra gadget as shown in Figure 6. Similar to the area gadgets along the variable loops, it consists of a white point guarding an area element of size $A = 1/N$ at distance $d_2 + \varepsilon$. Thus, the overall number of white points is $|W| = N + m$. By making the corridors sufficiently narrow (say, $1/N^3$ wide), the overall area for the corridors is small (e.g., $O(1/N^2)$.) The total area of the resulting polygon is $1 + m/N + O(1/N^2)$. See Figure 5 for a symbolic overall picture.

Now it easy to see that for any satisfying truth assignment for I, there is a position of $N/2$ black points that steals all the area elements, i.e., claims an area of $1 + m/N$.

To see the converse, assume that Barney can claim an area of at least $q + m/N$, i.e., he can steal all area elements. Note that no position of a black point can steal more than two area elements on a variable; stealing two requires placing it at less than distance $d_1 + \varepsilon$ from both of them. As the $N/2$ black points must form a perfect matching of the N area elements, we conclude that there are only two basic ways to cover all area elements of a variable x_i by not more than $n_i/2$ black points, where each location may be subject to variations of size $O(\varepsilon)$. One of these perfect matchings corresponds to setting x_i to *true*, the other to *false*. If this truth assignment can be done in a way that also steals all area elements of clause gadgets, we must have a satisfying truth assignment. □

By adding some extra area elements (say, of size $3A$) right next to $N/2 + m$ of the white points along variable gadgets, and increasing $|B|$ to $N + m$, we can modify the proof to apply to the case in which $|W| = |B|$. Similarly, it is straighforward to shift the critical threshold such that Wilma is guaranteed a constant fraction of the board.

6 Conclusion

We have resolved a number of open problems dealing with the one-round Voronoi game. There are still several issues that remain open. What can be said about achieving a fixed margin of win in all of the cases where Barney can win? We believe that our above techniques can be used to resolve this issue. As we can already quantify this margin if Wilma plays a grid, what is still needed is a refined version of Lemma 1 and Theorem 2 that guarantees a fixed margin as a function of the amount that Wilma deviates from a grid. Eventually, the guaranteed margin should be a function of the aspect ratio. Along similar lines, we believe that it is possible to resolve the question stated by [2] on the scenario where the number of points played is not equal.

There are some real-life situations where explicit zoning laws enforce a minimum distance between stones; obviously, our results still apply for the limiting case. It seems clear that Barney will be at a serious disadvantage when this lower bound is raised, but we leave it to future research to have a close look at these types of questions.

The most tantalizing problems deal with the multiple-round game. Given that finding an opti-

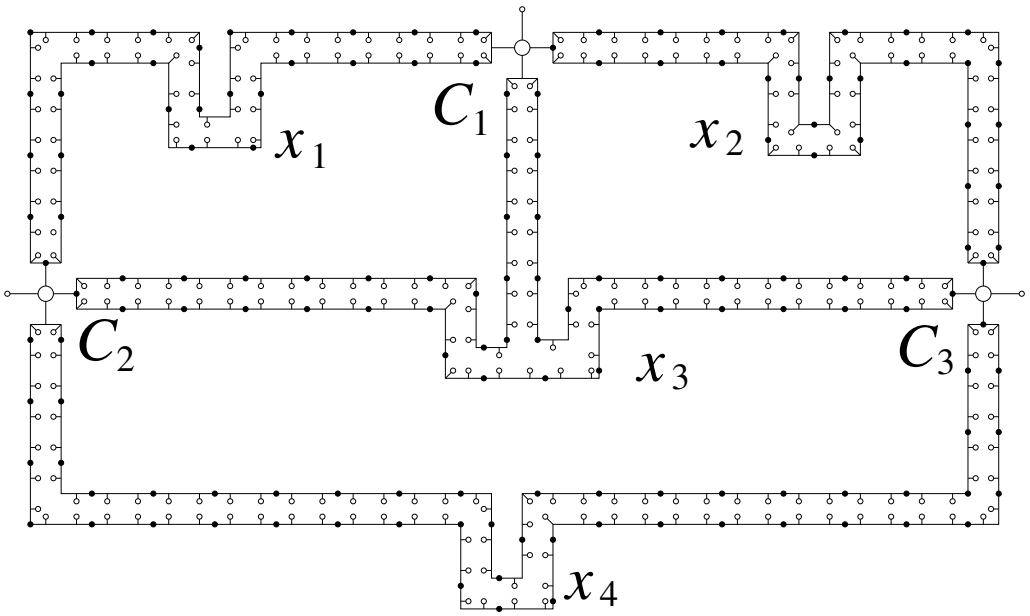

Figure 5: A symbolic picture of the overall representation: The location of white points is indicated by white dots (with area elements on variable loops not drawn for the sake of clarity). The location of black points (indicated by black dots) corresponds to the truth assignment $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 1$, which satisfies I. See Figure 6 for a closeup of the gadgets.

mal set of points for a single player is NP-hard, it is natural to conjecture that the two-player, multiple round game is PSPACE-hard. Clearly, there is some similarity to the game of Go on an $n \times n$ board, which is known to be PSPACE-hard [9] and even EXPTIME-complete [10] for certain rules.

However, some of this difficulty results from the possibility of capturing stones. It is conceivable that at least for relative simple (i.e., rectangular) boards, there are less involved winning strategies. Our results from Section 4 show that for the cases where Wilma has a winning strategy, Barney cannot prevent this by any probabilistic or greedy approach: Unless he blocks one of Wilma's key points by placing a stone there himself (which has probability zero for random strategies, and will not happen for simple greedy strategies), she can simply play those points like in the one-round game and claim a win. Thus, analyzing these key points may indeed be the key to understanding the game.

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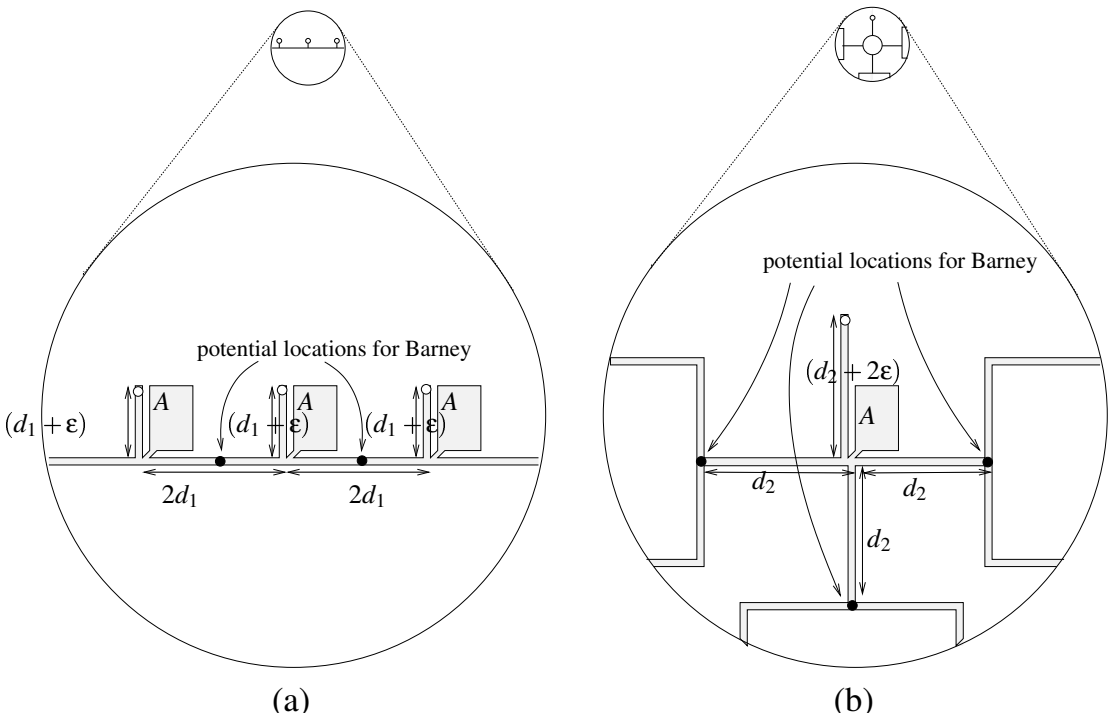

Figure 6: Area gadget (left) and clause gadgets (right)

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