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On Minimum Stars, Minimum Steiner Stars, and Maximum Matchings

by

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On Minimum Stars and Maximum Matchings

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Abstract

We discuss worst-case bounds on the ratio of maximum matching and minimum median values for finite point sets. In particular, we consider "minimum stars", which are defined by a center chosen from the given point set, such that the total geometric distance L_S to all the points in the set is minimized. If the center point is not required to be an element of the set (i. e., the center may be a Steiner point), we get a "minimum Steiner star", of total length L_{SS}. As a consequence of triangle inequality, the total length L_M of a maximum matching is a lower bound for the length L_{SS} of a minimum Steiner star, which makes the worst-case value $\rho(SS, M)$ of the value L_{SS}/L_M interesting in the context of optimal communication networks. The ratio also appears as the duality gap in an integer programming formulation of a location problem by Tamir and Mitchell.

In this paper, we show that for a finite set that consists of an even number of points in the plane and Euclidean distances, the worst-case ratio $\rho(S, M)$ cannot exceed $2/\sqrt{3}$. This proves a conjecture of Suri, who gave an example where this bound is achieved. For the case of Euclidean distances in two and three dimensions, we also prove upper and lower bounds for the worst-case value $\rho(S, SS)$ of the ratio L_S/L_{SS}, and for the worst-case value $\rho(S, M)$ of the ratio L_S/L_M. We give tight upper bounds for the case where distances are measured according to the Manhattan metric: we show that in three-dimensional space, $\rho(SS, M)$ is bounded by $3/2$, while in two-dimensional space $L_{SS} = L_M$, extending some independent observations by Tamir and Mitchell. Finally, we show that $\rho(S, SS)$ is $3/2$ in the two-dimensional case, and $5/3$ in the three-dimensional case.

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1 Introduction

The problem of finding a maximum weight *matching* for a given set of vertices in a weighted graph is to find a set of disjoint edges, such that the total weight of all the edges is maximized. Determining an optimal matching is a classical algorithmic problem, and Edmonds' famous polynomial-time algorithm [7] is one of the milestones of combinatorial optimization.

On the other hand, it has been known for quite a while [10] that the task of finding a minimum weight *Steiner tree* is an NP-hard problem: find a network of smallest total length L_{ST} that connects all given points, while allowing additional "Steiner" points for connecting edges. This algorithmic intractability differs drastically from the case where no Steiner points are allowed, so that the connected network has to be a minimum weight spanning tree (MST) of weight L_T, which can be solved very efficiently. Many aspects of optimal Steiner trees have been considered; see the book [12] for an overview. One of the most famous problems related to geometric Steiner trees deals with the worst-case value $\rho(T, ST)$ of the ratio L_T/L_{ST}. As Du and Hwang [6] managed to prove for the case of planar point sets with Euclidean distances, $\rho(T, ST) = 2/\sqrt{3}$.

A special type of Steiner tree problems arises in the context of location theory: The so-called *Weber problem* asks for the location of a single center point, such that the sum of distances from the given points to the center is minimized. It was shown by Bajaj [1] that even for the simple case of 5 points in the Euclidean plane, a solution can in general not be expressed by radicals. (In particular, it is impossible to construct an optimal solution by means of ruler and compass). In the context of communication networks, the resulting tree has been called a *star* [9]. As in the case of general tree networks, we can distinguish the Steiner case (where the center point can be chosen anywhere) from the more restricted case, where the center point is required to be chosen from the given set. In the following, we will speak of "minimum Steiner stars" (with a total edge length denoted by L_{SS}) and "minimum stars" (with a total edge length denoted by L_S).

When dealing with algorithmically hard problems like the task of designing optimal communication networks, it is of great importance to provide good upper and lower bounds for an optimal solution. It has been pointed out by Fingerhut, Suri, and Turner [9] that L_M is a lower bound for L_{SS}, which is an upper bound for L_{ST}. This makes it interesting to consider the worst-case value $\rho(SS, M)$ of the ratio L_{SS}/L_M. It was conjectured by Suri [15] that for the case of points in the plane with Euclidean distances, $\rho(SS, M)$ is $2/\sqrt{3}$ – the Steiner tree ratio. Proving this conjecture is one of the main results of this paper. In addition, we consider the behavior of the worst-case bounds $\rho(S, SS)$ for L_S/L_{SS}, and $\rho(S, M)$ for L_S/L_M. The ratio L_S/L_{SS} appears in location problems, where the sum of distances from a depot to a set of positions is to be minimized. (See the book by Drezner [5] for an overview in location theory.) If the depot may be chosen arbitrarily, we get total cost L_{SS}. If the depot has to be chosen from the given set of positions, we get a cost of L_S. Thus, the worst-case ratio can be interpreted as a bound on the possible payoff from opening a new depot.

For the case of Euclidean distances in two and three dimensions, we prove upper and lower bounds for all these worst-case values.

The above problems are also of interest when distances are not measured according to the Euclidean metric. Of particular relevance is the case of rectilinear (or "Manhattan") distances, which arises in the context of VLSI layout. Tamir and Mitchell [16] have considered $\rho(SS, M)$ for the case of rectilinear distances, motivated by questions from cost allocation for cooperative matching games. They prove that in the case of points in the plane, $L_{SS}/L_M = 1$, which implies that the core of a related matching game is nonempty. By establishing a tight upper bound on $\rho(SS, M)$, we can establish the largest possible value of the duality gap for their integer programming formulation for the case of Euclidean distances. In Section 5, we will prove that for the case of Manhattan distances, $\rho(SS, M) = 3/2$, and $\rho(S, SS) = 3/2$. Section 6 deals with Manhattan distances in three-dimensional space. We show that $\rho(SS, M) = 3/2$, $\rho(S, M) = 5/3$, while $\rho(S, M)$ lies between $5/3$ and 2. (See Table 1 at the end of this paper for an overview.)

Finally, we would like to note some further algorithmic implications: The result by Tamir and Mitchell [16] yields an $O(n)$ time algorithm for finding a maximum weight matching for a planar point set with Manhattan distances. With some extra work, some of the underlying properties of minimum Steiner stars have been used by Fekete [8] to construct an $O(n)$ time algorithm for finding a Traveling Salesman tour of maximum total length. (See the paper by Barvinok et al. [2] for more results on this problem.)

The rest of this paper is organized as follows. In Section 2 we introduce some basic notation and some general results. Section 3 deals with Euclidean distances in two-dimensional space, while Section 4 contains results for the case of Euclidean distances in three-dimensional space. In Sections 5 and 6, we consider Manhattan distances in two- and three-dimensional space. The concluding Section 7 contains a discussion of remaining open problems.

2 Preliminaries

Let $G = (V, E)$ be a graph with non-negative edge weights $w(e)$. Throughout this paper, the vertex set V of G will be represented by a point set $P = \{p_0, p_1, \ldots, p_{n-1}\}$ from Euclidean space, and edge weights correspond to geometric distances, according to some metric. A *star* of P is a set of $n-1$ edges (represented by line segments) connecting an element of P with all other elements of P. A *Steiner star* of P with center point c is a set of n edges (represented by line segments), connecting each point of P to c. A (perfect) matching of P is a set of $n/2$ edges that pair each point of P with another unique point of P. In the remainder of this paper, any star, Steiner star or matching is assumed to be a star, Steiner star or matching of P, denoted by the symbols E_S, E_{SS} and E_M. Their lengths are denoted by $\|E_S\|$, $\|E_{SS}\|$, and $\|E_M\|$, for a specified metric $\|...\|$. Let E_S^*, E_{SS}^* and E_M^* denote a star, Steiner star and matching of minimal, minimal and maximal length respectively. For easier notation, we write L_S, L_{SS}, and L_M for $\|E_S^*\|$, $\|E_{SS}^*\|$, and $\|E_M^*\|$; furthermore, we denote by $\rho(X, Y)$ (with $X, Y = S, SS, M$) the worst-case ratio $\sup_{P \in R^d} L_X/L_Y$.

Before we consider various geometric instances, we note a general bound on ratios that holds for all weight functions on the edges, even if we do not have triangle inequality.

Observation 1 *For arbitray weighted graphs G, we have $\rho(S, M) = 2$.*

Proof: Let matrix A be the distance matrix of the points in P, so $A(i, j)$ is the distance between p_i and p_j. Let S be the sum of all entries in A. Since L_S is the minimal row sum of A, we have by the pigeonhole principle that $L_S \leq S/n$. The maximal matching consists of $n/2$ elements of A, so again by the pigeonhole principle we have $L_M \geq S/(2n)$. Hence,

$$L_S \leq S/n \leq 2L_M.$$

To see that the ratio L_S/L_M can be arbitrarily close to 2, even if we assume triangle inequality, consider the complete graph on n vertices, with all edge weights being 1. Then $L_S = (n-1)$, and $L_M = n/2$. ∎

For the value $\rho(S, SS)$, we note the following:

Observation 2 *Assuming triangle inequality, we have $\rho(S, SS) = 2$. Without triangle inequality, $\rho(S, SS)$ is unbounded.*

Proof: Let c be the center of an optimal Steiner star. Let $v_0 \in V$ be a vertex closest to c, and let $w(c, v_0) = d$. Then by triangle inequality, $w(v_0, v_i) \leq w(v_0, c) + w(c, v_i)$, so the star $St(v_0)$ with center v_0 satisfies $L_S \leq \|St(v_0)\| \leq (n-1)d + L_{SS} \leq 2L_{SS}$.

To see that the bound of 2 is tight, let G be the complete graph on n vertices, with all edge weights being 2. Let $\overline{G} = (V \cup \{c\}, \overline{E})$ be a complete graph on (n+1) vertices, with all edges adjacent to c having weight 1. Then $L_S = 2(n-1)$, and $L_{SS} = n$.

For a class of examples without triangle inequality and unbounded $\rho(S, SS)$, let $G = (V, E)$ be the complete graph on n vertices, with each edge having weight $2N$. Let $\overline{G} = (V \cup \{c\}, \overline{E})$ be the complete graph on $(n+1)$ vertices, with all edges adjacent to c having weight 1. Then $L_S = 2N(n-1) > Nn = NL_{SS}$. ∎

In a geometric setting, distances in an arrangement of points are far more restricted, so the above bounds may no longer be best possible. It is the main purpose of this paper to provide tight estimates for geometric scenarios.

3 Euclidean Distances in Two-dimensional Space

Throughout this and the following section, we will consider arrangements of points in two- and three-dimensional space, with distances measured according to the Euclidean metric. At several occasions, we make use of the following lemma:

Lemma 1 *Given a triangle with edge lengths a, b and c, where the angle γ opposite c is bounded from below by $\phi > 0$. Then we have $a + b \leq \sqrt{\frac{2}{1-\cos\phi}} \cdot c$. For the special case $\phi = 2\pi/3$, we get $a + b \leq 2/\sqrt{3} \cdot c$; for $\phi = \pi/2$, we get $a + b \leq \sqrt{2} \cdot c$.*

Proof: By triangle inequality, we have $a + b \geq c$, so $(a+b)^2/c^2$ is bounded from below by 1. By the cosine theorem, we have $\frac{(a+b)^2}{c^2} = \frac{(a+b)^2}{a^2+b^2-2ab\cos\gamma}$. This expression is symmetric in a and b and attains the lower bound of 1 for $a = 0$ or $b = 0$, and the maximum is attained for $a = b$, from which the claim follows. ∎

3.1 Minimum Steiner Stars and Maximum Matchings

In this subsection, we give a proof of Suri's conjecture [15]. Throughout the section, distances are measured according to the Euclidean metric. The key idea is to make use of Lemma 1 and construct a matching that guarantees that the angle between the connections of the matched points and the origin has a good lower bound.

Let l be a directed line in the plane. We say that l is a *halving line* of P if at most half of the points of P are to the right l and at most half of the points of P are to the left of l.

Lemma 2 *For point sets P in two-dimensional space we can find three directed lines l_0, l_1 and l_2 such that the three lines intersect in a common point, all three lines are halving lines of P and the smallest angle between any two lines is $\pi/3$.*

Proof:

See Figure 1. The collection of halving lines for a given direction α form a directed closed strip which we call S_α. Consider the strips S_0, $S_{\pi/3}$ and $S_{2\pi/3}$. If these three strips have a point in common, we are done. Therefore assume without loss of generality that $S_0 \cap S_{\pi/3}$ lies to the left of $S_{2\pi/3}$. It follows that $S_\pi \cap S_{4\pi/3}$ lies to the right of $S_{5\pi/3}$.

We now consider the strips S_α, $S_{\alpha+\pi/3}$ and $S_{\alpha+2\pi/3}$, where α increases from $\alpha = 0$ to $\alpha = \pi$. The three strips move in a continuous manner. Suppose that for no value of α the three strips have a point in common. Then $S_\alpha \cap S_{\alpha+\pi/3}$ stays to the left of $S_{\alpha+2\pi/3}$, which contradicts the fact that $S_\pi \cap S_{4\pi/3}$ lies to the right of $S_{5\pi/3}$. Therefore there is a value of α for which the three strips S_α, $S_{\alpha+\pi/3}$ and $S_{\alpha+2\pi/3}$ have a point in common, which proves the lemma. ∎

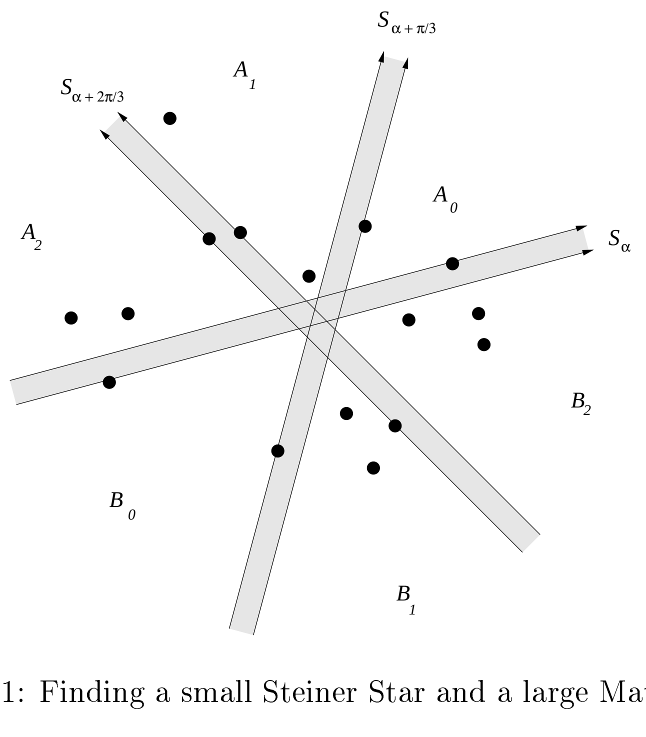

Figure 1: Finding a small Steiner Star and a large Matching

Theorem 1 *For point sets P of even cardinality in two-dimensional space with Euclidean distances, we have $\rho(SS, M) = 2/\sqrt{3}$.*

Proof: The following example with $L_{SS}/L_M = 2/\sqrt{3}$ was given by Suri. Suppose n is divisible by 6. Place $n/3$ points on each corner of an equilateral triangle with sides of length $2\sqrt{3}$. Then $L_{SS} = 2n$ and $L_M = n\sqrt{3}$.

To see that $\rho(SS, M) \leq 2/\sqrt{3}$, find three lines l_0, l_1 and l_2 as shown in Figure 1, such that the three lines intersect in a common point, all three lines are halving lines for P and the smallest angle between any two lines is $\pi/3$. These lines divide P into six sets A_0, A_1, A_2, B_0, B_1 and B_2, where A_i lies opposite B_i for all i, as shown in Figure 1. By assigning the points of P on the three lines to only one of the sets it belongs to, we can assume that $|A_0| + |A_1| + |A_2| = n/2 = |B_0| + |B_1| + |B_2|$, as well as $|A_0| + |A_1| + |B_2| = n/2 = |B_0| + |B_1| + |A_2|$, and $|A_0| + |B_1| + |B_2| = n/2 = |B_0| + |A_1| + |A_2|$. This implies $|A_i| = |B_i|$ for all i.

Let O be the intersection of l_0, l_1 and l_2. Construct a Steiner Star E_{SS} with O as its center. Now choose an (arbitrary) perfect matching E_M of the points in A_i with the points in B_i for all i.

We know that by construction, for each edge (p, q) in E_M, the angle $\angle(p, O, q)$ is bounded from below by $\phi = 2\pi/3$. By Lemma 1, this implies $\|p - O\| + \|O - q\| \leq 2/\sqrt{3} \cdot \|p - q\|$. So $L_{SS} \leq \|E_{SS}\| \leq 2/\sqrt{3} \cdot \|E_M\| \leq 2/\sqrt{3} \cdot L_M$.

∎

3.2 Minimum Stars and Minimum Steiner Stars

Let us turn to the possible values of $\rho(S, SS)$. As we noted in the introduction, this ratio appears in location problems like the Weber problem, where the sum of distances from a depot to a set of positions is to be minimized. Here we give bounds on the worst-case value of the possible payoff from opening a new depot.

Since $L_S \geq L_{SS}$, we assume without loss of generality that the star center c of an optimal Steiner star E^*_{SS} is not an element of P.

Let the line segments, or rays, of a Steiner star be r_i, and denote their lengths by a_i. Let α_i be the angle between the positive x-axis and ray r_i.

Lemma 3 *For the angles α_i of E^*_{SS}, and any θ, we have the local optimality condition*

$$\sum_{i=0}^{n-1} \cos(\alpha_i + \theta) \;=\; 0.$$

Proof: For the given Steiner star center, the sum of distances is locally minimal, so the projection of the gradient onto any line must disappear; by linearity of the derivative, $\sum_{i=0}^{n-1} \cos(\alpha_i + \theta)$ is the projection of the derivative onto the line that encloses an angle of θ with the origin. ∎

Lemma 4 *Let E_{SS} be a Steiner star of $P = \{p_0, \ldots, p_{n-1}\}$ and let a_i, r_i and α_i be defined as above. Let b_i be the distance between p_0 and p_i. In the special case that all points of P lie on a circle centered at c, of radius $a_0 = a_i$ for all i, the local optimality condition*

$$\sum_{i=0}^{n-1} \cos(\alpha_i + \theta) \;=\; 0 \quad \textit{implies that} \quad \sum_{i=0}^{n-1} b_i \;\leq\; a_0 n\sqrt{2}.$$

Proof: We have $\sum_{i=0}^{n-1} \cos(\alpha_i + \theta) = 0$ for all θ, so without loss of generality we can assume that P is rotated around the origin so that $p_0 = (a_0, 0)$. We have

$$b_i \;=\; 2a_0 \sin\left(\frac{\alpha_i}{2}\right) \;=\; a_0\sqrt{2(1 - \cos\alpha_i)}$$

(see also Figure 2). The function $f(\alpha) = \sqrt{1 - \cos\alpha}$, where $0 \leq \alpha < 2\pi$, is a concave function of $\cos\alpha$. By Jensen's inequality [14] we have

$$\sum_{i=0}^{n-1} \frac{1}{n}\sqrt{1 - \cos\alpha_i} \;\leq\; \sqrt{1 - (\sum_{i=0}^{n-1} \frac{1}{n} \cos\alpha_i)} \;=\; 1,$$

from which the result follows. ∎

Now we can prove the following upper bound:

Theorem 2 *For point sets P in two-dimensional space with Euclidean distances, we have $4/\pi \leq \rho(S, SS) \leq \sqrt{2}$.*

Proof: For the lower bound, choose P to be a set of points spaced evenly on the unit circle. Assume that P includes the point $(x, y) = (0, -1)$. The center of E^*_{SS} is the center of the circle, so $L_{SS} = n$. Consider the star E_S centered at $(0, -1)$. Denote the rays of E_S by r_i and their lengths by a_i. Let α_i be the angle between the positive x-axis and ray r_i. We have $a_i \;=\; 2\sin\alpha_i$. Therefore the average ray length in the limit is

$$\frac{1}{\pi}\int_0^{\pi/2} 4\sin\alpha d\alpha \;=\; 4/\pi.$$

Now consider the upper bound. If the center of E^*_{SS} is an element of P, the bound holds, so assume that the center of the E^*_{SS} is not an element of P. Without loss of generality assume that the center of E^*_{SS} is the origin, that r_0 is a shortest ray and that r_0 runs along the positive x-axis, as shown in Figure 2.

Consider the star E_S centered at the end-point of r_0. Denote the rays of E_S by r'_i and their length by a'_i. Using triangle inequality we have

$$a'_i \;\leq\; (a_i - a_0) + 2a_0\sin(\alpha_i/2).$$

So from Lemma 4 we have

$$\begin{aligned} L_S \;\leq\; \|E_S\| \;&=\; \sum_{i=0}^{n-1} a'_i \;\leq\; \sum_{i=0}^{n-1} a_i - na_0 + na_0\sqrt{2} \\ &\leq\; L_{SS} + L_{SS}(\sqrt{2} - 1) \;=\; \sqrt{2}\cdot L_{SS}. \end{aligned}$$ ∎

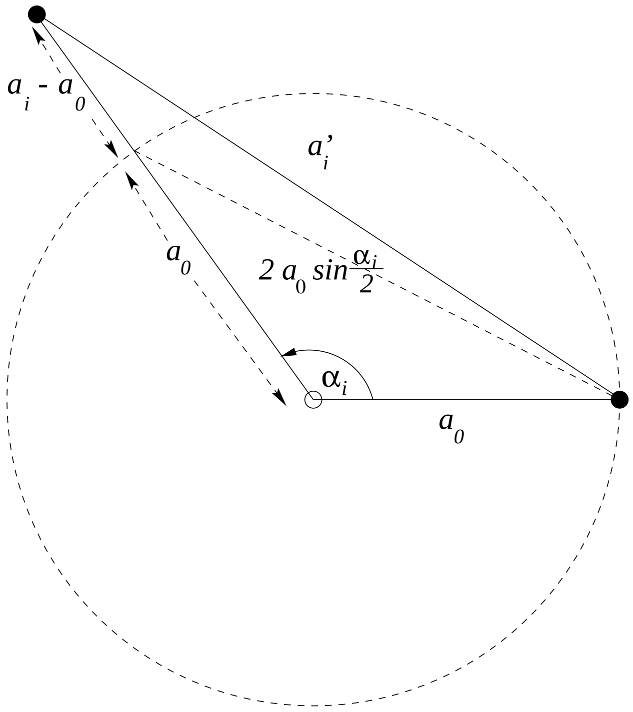

Figure 2: Length of a chord in a circle

3.3 Minimum Stars and Maximum Matchings

It is not hard to derive upper and lower bounds for $\rho(S, M)$ by using the previous results of this section:

Theorem 3 *For point sets P of even cardinality in two-dimensional space with Euclidean distances, we have $4/3 \leq \rho(S, M) \leq 2\sqrt{2}/\sqrt{3}$.*

Proof: The upper bound follows immediately from Theorems 1 and 2.

For the lower bound, suppose n is divisible by 6, and place $n/3$ points on each corner of an equilateral triangle with sides of length 6. Then $L_S = 4n$ and $L_M = 3n$. ∎

4 Euclidean Distances in Three-dimensional Space

Now we turn to point sets in three-dimensional space under Euclidean distances. Again, we start by providing tools for the ratio $\rho(SS, M)$.

Following an idea similar to the one from Section 3.1, we show that there always exist three orthogonal planes that partition P into 8 octants such that opposite octants contain the same number of points.

Let p be a plane. We say that p is a halving plane of P if at most half of the points of P are on one side of p and at most half of the points of P are on the other side of p.

Let p_0, p_1 and p_2 be three orthogonal planes, each of which is a halving plane of P. Each plane Π_i divides P into points above Π_i, on Π_i and below Π_i. Let Q_0^i be the points in P below Π_i and Q_1^i be the points in P above Π_i. We assign the points on Π_i to either Q_0^i and Q_1^i in such a way that $|Q_0^i| = |Q_1^i|$. We will call $|Q_0^i|$ and $|Q_1^i|$ the set of points below and above Π_i respectively, even though some of these points may in fact lie on Π_i. We define for $i, j, k \in \{0, 1\}$:

$$Q_{ijk} \;=\; Q_i^2 \cap Q_j^1 \cap Q_k^0.$$

For example, Q_{110} is the set of points in P above Π_2, above Π_1 and below Π_0. Since $|Q_0^i| = |Q_1^i|$, we can derive the following equalities:

$$\begin{aligned}
|Q_{000}| + |Q_{001}| &= |Q_{110}| + |Q_{111}| \\
|Q_{100}| + |Q_{101}| &= |Q_{010}| + |Q_{011}| \\
|Q_{000}| + |Q_{010}| &= |Q_{101}| + |Q_{111}| \\
|Q_{001}| + |Q_{011}| &= |Q_{100}| + |Q_{110}| \\
|Q_{000}| + |Q_{100}| &= |Q_{011}| + |Q_{111}| \\
|Q_{001}| + |Q_{101}| &= |Q_{010}| + |Q_{110}|
\end{aligned}$$

We show that we can always find three orthogonal planes such that opposite octants, i.e. Q_{ijk} and $Q_{(1-i)(1-j)(1-k)}$ have the same cardinality.

Lemma 5 *For point sets P in three-dimensional space, we can find three orthogonal planes such that $|Q_{ijk}| = |Q_{(1-i)(1-j)(1-k)}|$ for all i, j, k.*

Proof: Notice that it suffices to find three orthogonal planes such that $|Q_{000}| = |Q_{111}|$, since this implies that $|Q_{ijk}| = |Q_{(1-i)(1-j)(1-k)}|$ for all i, j, k. We first assume that the points are in general position, in the sense that if we project all points in P onto the $(z = 0)$ plane, then no three points are collinear and no line through two points is perpendicular to another line through two points. Let Π_2 be a halving plane parallel to the $(z = 0)$ plane. We map all points from P onto the $(z = 0)$ plane, and call the projected points from Q_1^2 and Q_0^2 the black and white points respectively.

The problem is now a two-dimensional one. Orthogonal halving planes Π_0 and Π_1 will become halving lines l_0 and l_1 in the projection. Let a directed line l be a halving line if at most half of the black and white points are to the left of l and at most half lie to the right of l. Because of the non-degeneracy assumption there are at most two points on a halving line l. The collection of halving lines for a given direction α form a directed closed strip which we call S_α. Let l_α be the halving line in the middle of S_α. Consider the strips S_α and $S_{\alpha+\pi/2}$. The corresponding lines l_α and $l_{\alpha+\pi/2}$ divide the points into 4 subsets Q_{ij} for $i, j \in \{0, 1\}$, where Q_{ij} is the projection of $Q_{0ij} \cup Q_{1ij}$. Therefore $|Q_{00}| = |Q_{11}|$ and $|Q_{01}| = |Q_{10}|$, as illustrated in Figure 3.

As α increases, at most one point is added to and at most one point is removed from Q_{ij} at any one time. Changes only occur when S_α or $S_{\alpha+\pi/2}$ is a line. Because of the non-degeneracy assumption it is not possible that both S_α and $S_{\alpha+\pi/2}$ are lines. Either

- the number of points in two opposite quadrants such as Q_{00} and Q_{11} both increase or both decrease, or

- two neighboring quadrants exchange a point.

In the first case, $|Q_{000}|$ and $|Q_{111}|$ either both increase, both decrease, or only one changes. It is not possible for $|Q_{000}|$ to decrease and for $|Q_{111}|$ to increase or vice-versa. In the latter case, at most one of $|Q_{000}|$ and $|Q_{111}|$ changes. Therefore if for some value of α_0 we have $|Q_{000}| < |Q_{111}|$, while for α_1 we have $|Q_{000}| > |Q_{111}|$, then there is an α with $\alpha_0 < \alpha < \alpha_1$ for which $|Q_{000}| = |Q_{111}|$.

Consider first the strips S_0 and $S_{\pi/2}$. If $|Q_{000}| = |Q_{111}|$ then we are done. Therefore suppose that $|Q_{000}| < |Q_{111}|$. It follows that for strips S_π and $S_{3\pi/2}$ we have $|Q_{000}| > |Q_{111}|$. Therefore there is a value of α with $0 < \alpha < \pi$ for which $|Q_{000}| = |Q_{111}|$.

If the non-degeneracy assumption does not hold, we can move all points by an infinitesimal small distance, in such a way that the assumption does hold. The construction shown above gives three orthogonal halving planes of the perturbed set. The same planes partition P in the correct way, whereby the perturbation of a point that lies on a halving plane determines to which sides of the plane this point should be assigned. ∎

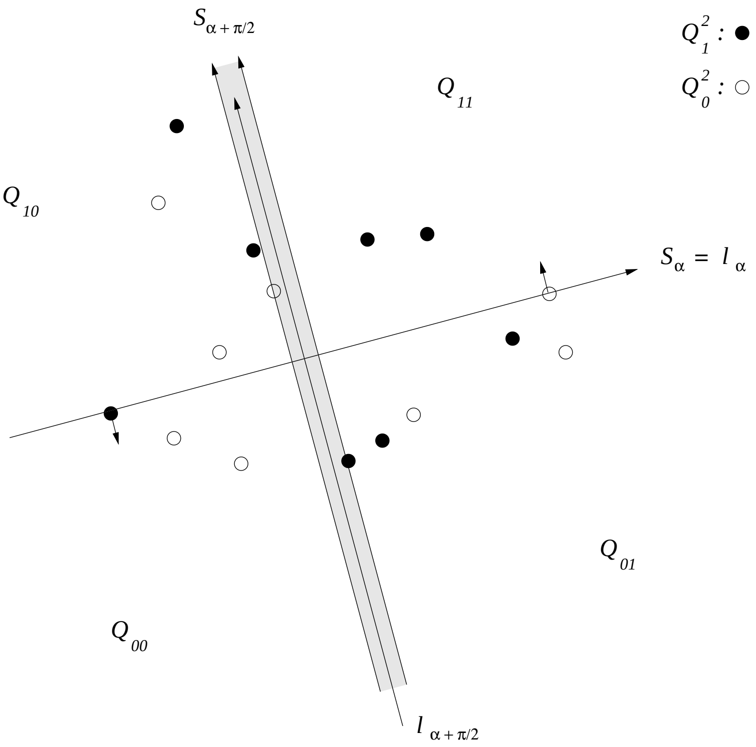

Figure 3: Orthogonal halving strips

Theorem 4 *For sets of points P in three-dimensional space with Euclidean distances, we have the inequality $\sqrt{3}/\sqrt{2} \leq \rho(SS, M) \leq \sqrt{2}$.*

Proof: For the lower bound, suppose n is divisible by 4. Place $n/4$ points on each corner of a tetrahedron with sides of length 2. Then $L_{SS} = n\sqrt{3}/\sqrt{2}$ and $L_M = n$.

To prove the upper bound, find three orthogonal halving planes that satisfy the condition $\|Q_{ijk}\| = \|Q_{(1-i)(1-j)(1-k)}\|$ for all i, j, k. Let O be the point of intersection of these three planes. Construct a Steiner Star E_SS with O as its center. Construct a matching E_M by connecting points in each octant to points in the opposite octant.

For each edge (p, q) in E_M, the angle between the ray from O to p and the ray from O to q is bounded from below by $\pi/2$, from which we derive by Lemma 1 that $\|p-O\|+\|O-q\| \leq \sqrt{2} \cdot \|p-q\|$. So $L_{SS} \leq \|E_{SS}\| \leq \sqrt{2} \cdot \|E_M\| \leq \sqrt{2} \cdot L_M$. ∎

Next we discuss the worst-case value $\rho(S, SS)$.

Theorem 5 *For sets of points P in three-dimensional space with Euclidean distances, we have the inequality $4/3 \leq \rho(S, SS) \leq \sqrt{2}$.*

Proof:
The proof for the upper bound is the same as the proof for the two-dimensional case shown in Theorem 2.

To see the lower bound, let P be a set of points evenly distributed over the unit sphere. Assume P includes the point $(x, y, z) = (0, -1, 0)$. The center of E^*_{SS} is the center of the sphere, so $L_{SS} = n$. Consider the star E_S centered at the point $(0, -1, 0)$. Denote the rays of E_S by r_i and their lengths by a_i. Let α_i be the angle between the $(y = 0)$-plane and ray r_i. We have $a_i = 2 \sin \alpha_i$.

The average ray length can be computed as follows. Let θ be the angle between a ray and the $(y = 0)$ plane and ϕ the angle between the projection of the ray onto the $(y = 0)$ plane and the positive x-axis. If $\Delta\theta = \theta_1 - \theta_0$ and $\Delta\phi = \phi_1 - \phi_0$ are small, then the surface area covered by all rays with angles in this range is approximately equal to $4 \sin\theta \cos\theta \Delta\phi\Delta\theta$, so the area of the surface of the sphere is

$$\int_0^{\pi/2} \int_0^{2\pi} 4 \sin\theta \cos\theta \; d\phi d\theta \;=\; 4\pi$$

and the average ray length is

$$\frac{1}{4\pi} \int_0^{\pi/2} \int_0^{2\pi} (4 \sin\theta \cos\theta) \; (2 \sin\theta) \; d\phi d\theta \;=\; 4/3.$$

∎

We conclude our discussion on Euclidean distances:

Observation 3 *For point sets in three-dimensional space with Euclidean distances, $3/2 \leq \rho(S, M) \leq 2$.*

Proof: We know from Observation 1 that $\rho(S, M)$ cannot exceed 2. For the lower bound, suppose n is divisible by 8. Place $n/4$ points on each corner of a tetrahedron with sides of length 4. Then $L_S = 3n$ and $L_M = 2n$. ∎

5 Manhattan Distances in Two-dimensional Space

5.1 Minimum Steiner Stars and Maximum Matchings

Independent from our work, the following proposition was also noted by Tamir and Mitchell [16]. Since some of the steps are of importance for our further results, we give a sketch of the proof.

Proposition 1 *For point sets P of even cardinality in two-dimensional space with Manhattan distances, we have $L_M = L_{SS}$.*

Sketch: It is not hard to see that the center for an optimal Steiner star must both be a median of the x-coordinates and the y-coordinates of the points in P. Assume without loss of generality that an optimal center is located at $(0, 0)$ and consider the numbers n_1, n_2, n_3, and n_4 of points in each of the four quadrants, with points on the boundary of two quadrants assigned in a suitable way. Using $n_1 + n_2 = n_3 + n_4$ and $n_2 + n_3 = n_4 + n_1$, we get $n_1 = n_3$ and $n_2 = n_4$, i. e., diagonally opposite quadrants must contain the same number of points. This allows us to match points from opposite quadrants. It is straightforward to see that to each edge of the matching, we have a corresponding pair of edges of the Steiner star with the same total length, implying that the total length of the matching is equal to the total length of the Steiner star. ∎

5.2 Minimum Stars and Minimum Steiner Stars

In the following, we consider $\rho(S, SS)$. For any point $p_i = (x_i, y_i) \in P$, $L_S(p_i)$ is the total length of the star centered at p_i. We denote by ρ_n the supremum of the values L_S/L_{SS} for point sets of cardinality n. Without loss of generality, we may assume that $L_{SS} > 0$, and thus $L_{SS} = 1$. Furthermore, we may assume that the origin is an optimal Steiner center.

We will make use of the following lemma:

Lemma 6 *For any n, there are point sets for which L_S/L_{SS} attains the value ρ_n.*

Proof: For any fixed n, the set of point arrangements with $L_{SS} = 1$ and optimal Steiner center O is a compact subset of R^{2n}. Since L_S is a continuous function on R^{2n}, the claim holds. ∎

Lemma 7 *Let P be a set of n points with $L_S/L_{SS} = \rho_n$. Then for all $p_i \in P$, $L_S(p_i) = \rho_n$.*

Proof: Suppose there is a point p_i with $L_S(p_i) > L_S$. For sufficiently small ε, replacing p_i by the point $p_i' = (1 - \varepsilon)p_i$ does not turn p_i' into an optimal star center. Thus, the replacement reduces L_{SS} by some small ε', but L_S by not more than ε'. Therefore, the new arrangement has a ratio of at least $\frac{L_S - \varepsilon'}{L_{SS} - \varepsilon'} > \frac{L_S}{L_{SS}} = \rho_n$, a contradiction. ∎

It is straightforward to see that this implies the following:

Corollary 1 *For an arrangement with* $L_S/L_{SS} = \rho_n$*, we cannot move a vertex such that* L_{SS} *decreases by* ε *and* L_S *by not more than* ε*.*

Corollary 2 *For an arrangement with* $L_S/L_{SS} = \rho_n$*, we cannot move a vertex such that* L_{SS} *remains the same, one or more of the* $L_S(p_i)$ *increase, and none of them decrease.*

The following implication allows a further reduction of arrangements that can achieve the worst-case ratio; it will be used for the proof of Lemma 10.

Corollary 3 *For any arrangement in two-dimensional space with Manhattan distances and with* $L_S/L_{SS} = \rho_n$*, there cannot be two points* $p_1 = (x_1, y_1) \neq p_2 = (x_2, y_2)$*, such that* p_1 *lies between* p_2 *and the origin, i.e.,* $0 \leq (x_1, y_1) \leq (x_2, y_2)$*, or* $0 \leq (-x_1, y_1) \leq (-x_2, y_2)$*, or* $0 \leq (x_1, -y_1) \leq (x_2, -y_2)$*, or* $0 \leq (-x_1, -y_1) \leq (-x_2, -y_2)$*.*

Proof: In any of the four cases, it is straightforward to see that $L_S(p_1) < L_S(p_2)$. ∎

The next lemma shows that we may restrict our attention to arrangements with extreme points on the coordinate axes:

Lemma 8 *For any arrangement in two-dimensional space with Manhattan distances and with* $L_S/L_{SS} = \rho_n$*, any point* $p_i = (x_i, y_i)$ *with minimal or maximal* x_i *among the points in* P *must have* $y_i = 0$*. Conversely, minimal or maximal* y_i *implies* $x_i = 0$*.*

Proof: Without loss of generality consider a point p_i with maximal y_i and assume $x_i > 0$. Let $x_k = \max\{x_j | j \neq i\}$. If $x_i - x_k = \delta > 0$, replace p_i by $(x_k, y_i + \delta)$; this does not change L_S or L_{SS}, and allows us to consider without loss of generality the case $x_i - x_k \leq 0$.

For sufficiently small ε, replace p_i by $p_i' = p_i + (-\varepsilon, \varepsilon)$. This does not change L_{SS}, increases $L_S(p_k)$, and does not decrease $L_S(p_j)$ for any $j \neq k$, contradicting Corollary 2. ∎

Lemma 9 *For any* n*, we have* $\rho_n \leq \rho_{kn}$*.*

Proof: Suppose we have an arrangement of n points with $L_S/L_{SS} = \rho_n$. Replace each point by k copies; this yields an arrangement of kn points with $L_S/L_{SS} = \rho_n$. ∎

Lemma 10 *For any arrangement in two-dimensional space with Manhattan distances and with* $L_S/L_{SS} = \rho_n$*, there can be at most four points* p_i *with* $x_i, y_i \neq 0$*.*

Proof: Assume that there are at least five points not on coordinate axes. Then we may assume without loss of generality that two of them (say, $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$) are in the positive quadrant. Because of Corollary 3, we may assume that $0 < x_2 < x_1$ and $0 < y_1 < y_2$. Define

$n_0 =$ the number of vertices i with $x_1 < x_i$
$n_1 =$ the number of vertices $i \neq 1, 2$ with $x_1 \leq x_i \leq x_2$
$n_2 =$ the number of vertices i with $x_i < x_2$
$m_0 =$ the number of vertices i with $y_2 < y_i$
$m_1 =$ the number of vertices $i \neq 1, 2$ with $y_1 \leq y_i \leq y_2$
$m_2 =$ the number of vertices i with $y_i < y_1$

So $n_2 > n_0$ and $m_2 > m_0$. Let ε_x and ε_y be such that $\varepsilon_x/\varepsilon_y = (m_2 - m_0)/(n_2 - n_0)$. Also let ε_x and ε_y be positive but smaller than the smallest non- zero difference between the x-coordinates and y-coordinates of any two points respectively. Now replace p_1 by $p_1' = p_1 + (\varepsilon_x, -\varepsilon_y)$ and p_2 by $p_2' = p_2 + (-\varepsilon_x, +\varepsilon_y)$ It is not hard to see that these replacements do not decrease $L_S(p_i)$ for any $i \neq 1, 2$. Furthermore, L_{SS} does not change. The value $L_S(p_1)$ changes by $\Delta_1 = (n_2 + n_1 - n_0)\varepsilon_x + (-m_2 + m_1 + m_0)\varepsilon_y + 2\varepsilon_x + 2\varepsilon_y$. So $\Delta_1 > (n_2 - n_0)\varepsilon_x + (-m_2 + m_0)\varepsilon_y = (m_2 - m_0)\varepsilon_y + (-m_2 + m_0)\varepsilon_y = 0$. Similarly, the value $L_S(p_2)$ changes by $(-n_2 + n_1 + n_0)\varepsilon_x + (m_2 + m_1 - m_0)\varepsilon_y + 2\varepsilon_x + 2\varepsilon_y$, which is also positive. This contradicts Corollary 2. ∎

In order to analyze the limit of the sequence ρ_n, we define a sequence β_n. This is the supremum of the values L_S/L_{SS} for all arrangements of n points, such that any point lies on a coordinate axis.

With the help of Corollary 3, Lemma 9, and Lemma 10, it is not hard to prove that for arrangements with many points, the bounded number of points not on coordinate axes becomes negligible for the worst-case ratio:

Lemma 11 $\limsup_{n\to\infty} \rho_n = \limsup_{n\to\infty} \beta_n$.

Proof: For any n, consider a point arrangement P_n with $L_{SS} = 1$ and $L_S = \rho_n$. By Lemma 10, for any P_n, there can be at most 4 points not on coordinate axes; by Corollary 3, we conclude that the points on the axes are positioned at $p_1 = (d_1, 0)$, $p_2 = (0, d_2)$, $p_3 = (-d_3, 0)$, $p_4 = (0, -d_4)$, with multiplicities n_1, n_2, n_3, n_4 and $\sum_{i=1}^4 n_i \geq n - 4$.

Because of Lemma 9, we are done if there are only finitely many P_n with a point not on an axis. So assume there are infinitely many P_n with a point $p_0 = (x_0, y_0) > (0, 0)$. By Corollary 3, we conclude that for any such P_n, $d_1 > x_0$ and $d_2 > y_0$. If d_1 and d_2 tend to zero as n becomes large, the contribution of p_0 to L_{SS} and L_S becomes arbitrarily small, and we are done.

So suppose without loss of generality that $d_1 = \max\{d_j | j = 1, \ldots, 4\}$ and that d_1 remains bounded from below. Since $L_{SS} = 1$, this means that n_1 remains bounded. As n becomes large, it follows that some n_i becomes arbitrarily large. Then $L_{SS} = 1$ implies that d_i tends to zero. Consider the star $E_{S(j)}$ centered at p_j. Using 0 as a lower and $3d_1$ as an upper bound for the distance of points not on axes to p_1 and p_i, respectively, we get

$$\|E_{S(1)}\| \geq n_2 d_2 + n_3 d_3 + n_4 d_4 + (n_2 + n_3 + n_4) d_1 = \sum_{j=1}^{4} n_j d_j + (\sum_{j=1}^{4} n_j - 2n_1) d_1,$$

whereas

$$\|E_{S(i)}\| \leq 12 d_1 + \sum_{j=1}^{4} n_j d_j + (\sum_{j=1}^{4} n_j - 2n_i) d_i.$$

This means that for sufficiently large n, we have

$$\|E_{S(1)}\| - \|E_{S(i)}\| \geq (d_1 - d_i) \sum_{j=1}^{4} n_j - 2n_1 d_1 + 2n_i d_i - 12 d_1 > 0,$$

since $(d_1 - d_i) \sum_{j=1}^{4} n_j$ gets arbitrarily large for increasing n, while all other terms are bounded from below. This contradicts Lemma 7, and we are done. ∎

In order to establish an upper bound of the sequence β_n, we need the following lemma:

Lemma 12 *Let $0 \leq \lambda_1, \ldots, \lambda_{2d} < 1/2$, such that $\sum_{i=1}^{2d} \lambda_i = 1$. Then*

$$\sum_{i=1}^{2d} \frac{\lambda_i}{1 - 2\lambda_i} \geq \frac{d}{d-1}.$$

Proof: Since $f(x) = \frac{x}{1-2x}$ is a convex function on the interval $[0, \frac{1}{2})$, we have from Jensen's inequality [14]

$$f(\sum_{i=1}^{2d} \frac{1}{2d}\lambda_i) \;\leq\; \frac{1}{2d}\sum_{i=1}^{2d} f(\lambda_i).$$

So

$$\frac{1}{2d-2} \;\leq\; \frac{1}{2d}\sum_{i=1}^{2d} \frac{\lambda_i}{1-2\lambda_i}$$

from which the lemma follows. ∎

Now we can proceed to prove the following:

Theorem 6 *For point sets P in two-dimensional space with Manhattan distances, we have $\rho(S, SS) = 3/2$.*

Proof: To see the lower bound, suppose n is divisible by 4. Place $n/4$ points on each corner of the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. Then $L_S = 3n/2$ and $L_{SS} = n$.

By Lemma 11, we only have to show that $\beta_n \leq 3/2$ for the upper bound. Similar to Lemma 6, we can assume that there are sets of n points for which $L_S/L_{SS} = \beta_n$. By Lemma 3, these sets consist of n_1 points p_1 at position $(d_1, 0)$, of n_2 points p_2 at position $(0, d_2)$, of n_3 points p_3 at position $(-d_3, 0)$, and of n_4 points p_4 at position $(0, -d_4)$, with $d_i \geq 0$. We assume that

$$L_{SS} = \sum_{i=1}^{4} n_i d_i = 1. \tag{1}$$

Furthermore, $L_S(p_i) = \sum_{j \neq i} n_j(d_i + d_j) = (\sum_{j=1}^{4} n_j d_j) + (\sum_{j=1}^{4} n_j d_i) - 2n_i d_i = 1 + (n - 2n_i)d_i$. By Lemma 7, we have $L_S(p_i) = \beta_n$, which implies

$$d_i = \frac{\beta_n - 1}{(n - 2n_i)}. \tag{2}$$

Equations 1 and 2 yield

$$\frac{1}{\beta_n - 1} = \sum_{i=1}^{4} \frac{n_i}{(n - 2n_i)}. \tag{3}$$

With $n_i/n = \lambda_i$, Lemma 12 implies $\frac{1}{\beta_n - 1} \geq 2$, so $\beta_n \leq 3/2$, and we are done. ∎

Together with Theorem 1, this implies the following:

Corollary 4 *For point sets P of even cardinality in two-dimensional space with Manhattan distances, we have $\rho(S, M) = 3/2$.*

6 Manhattan Distances in Three-dimensional Space

It was noted by Tamir and Mitchell in [16] that the equality $L_M = L_{SS}$ that holds in two-dimensional rectilinear space is no longer valid in three-dimensional space. In fact, their example $P = \{(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\}$ shows that $\rho(SS, M) \geq 3/2$. The following result shows that this is a worst-case example:

Theorem 7 *For point sets P of even cardinality in three-dimensional space with Manhattan distances, we have $\rho(SS, M) = 3/2$.*

Proof: Assume that $O = (0,0,0)$ is the center of an optimal Steiner star, so $|\{p_i|x_i < 0\}| \leq n/2$, $|\{p_i|x_i > 0\}| \leq n/2$, $|\{p_i|y_i < 0\}| \leq n/2$, etc. Then $L_{SS} = \sum_i |x_i| + |y_i| + |z_i|$. Without loss of generality, assume that $\sum_i |z_i| \leq \sum_i |x_i| \leq \sum_i |y_i|$. Now consider the point set $P' = \{p'_i | i = 1, \dots, n\}$, where $p'_i = (x_i, y_i, 0)$. Because of the above conditions, O is the center of an optimal Steiner star $E^{*'}_{SS}$ for P'. We have $\|E^{*'}_{SS}\| = \sum_i |x_i| + |y_i| \geq 2/3 L_{SS}$. By Proposition 1, P' has a matching E'_M of size $\|E^{*'}_{SS}\|$, and the claim follows. ∎

Using the same tools as in the two-dimensional case, we can show the following for $\rho(S, SS)$:

Theorem 8 *For point sets P in three-dimensional space with Manhattan distances, we have $\rho(SS, S) = 5/3$.*

Proof: For a lower bound, suppose n is divisible by 6. Place $n/6$ points on each of the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$. Then $L_S = 5n/3$ and $L_{SS} = n$.

For the upper bound we proceed similarly to the proof of Theorem 6. Note that Lemmas 6, 7, and Corollaries 1 and 2 stay valid without any change, as well as Lemmas 9 and 12. It is straightforward to modify Corollary 3 and Lemma 8 to higher dimensions. Lemma 10 is replaced by the following three-dimensional version:

For any arrangement with $L_S/L_{SS} = \rho_n$, there can be at most eight points p_i not on coordinate axes.

This is shown as follows: Suppose there are nine points not on coordinate axes, then there must be at least two points $p_1 = (x_1, y_1, z_1)$ and $p_2 = (x_2, y_2, z_2)$ in the same octant, say the positive one. By the analogue of Corollary 3, we cannot have $0 \leq (x_1, y_1, z_1) \leq (x_2, y_2, z_2)$ or $0 \leq (x_2, y_2, z_2) \leq (x_1, y_1, z_1)$ with $p_1 \neq p_2$. This allows us to consider without loss of generality $0 < x_1 < x_2$, $0 < y_2 < y_1$ and apply the same modification to the x- and y-coordinates of p_1 and p_2 as in the proof of Lemma 10.

With the help of these lemmas, the claim of Lemma 11 still holds. Using Lemma 12 for $d = 3$, we get

$$\frac{1}{\beta_n - 1} = \sum_{i=1}^{6} \frac{n_i}{(n - 2n_i)} \geq \frac{d}{d-1} = \frac{3}{2},$$

implying $\beta_n \leq 5/3$. This concludes the proof. ∎

For $\rho(S, M)$, we only have the following easy observation:

Observation 4 *For point sets P of even cardinality in three-dimensional space under Manhattan distances, we have $5/3 \leq \rho(S, M) \leq 2$.*

Proof: The upper bound follows from Observation 1. For the lower bound, suppose n is divisible by 6. Place $n/6$ points on each of the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,0,0)$, $(0,-1,0)$, $(0,0,-1)$. Then $L_S = 5n/3$ and $L_M = n$. ∎

7 Conclusion

We have derived a number of upper and lower bounds for the largest possible value of the ratios between the size of Minimum stars, Minimum Steiner stars, and Maximum matchings. A summary of our bounds is given in Table 1. Some of these bounds are not tight; in all cases, we suspect that the upper bounds can be improved to match the lower bounds. This belief is strengthened by the fact that some of the tools we used for the case of Manhattan distances (in particular, Lemma 7) are true for Euclidean distances as well. We note the following conjectures:

Conjecture 1 *For point sets P in two-dimensional space with Euclidean distances, we have $\rho(S, SS) = 4/\pi$.*

Conjecture 2 *For point sets P in three-dimensional space with Euclidean distances, we have $\rho(S, SS) = 4/3$.*

Table 1: Lower and upper bounds for worst-case ratios

Distances	Dimension	Ratio	Lower bound	Upper bound
Euclidean	2D	$\rho(SS, M)$	$\frac{2}{\sqrt{3}} = 1.15...$	$\frac{2}{\sqrt{3}} = 1.15...$
		$\rho(S, SS)$	$\frac{4}{\pi} = 1.27...$	$\sqrt{2} = 1.41...$
		$\rho(S, M)$	$\frac{4}{3} = 1.33...$	$\frac{2\sqrt{2}}{\sqrt{3}} = 1.63...$
	3D	$\rho(SS, M)$	$\frac{\sqrt{3}}{\sqrt{2}} = 1.22...$	$\sqrt{2} = 1.41...$
		$\rho(S, SS)$	$\frac{4}{3} = 1.33...$	$\sqrt{2} = 1.41...$
		$\rho(S, M)$	$\frac{3}{2} = 1.5$	2
Manhattan	2D	$\rho(SS, M)$	1	1
		$\rho(S, SS)$	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$
		$\rho(S, M)$	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$
	3D	$\rho(SS, M)$	$\frac{3}{2} = 1.5$	$\frac{3}{2} = 1.5$
		$\rho(S, SS)$	$\frac{5}{3} = 1.66...$	$\frac{5}{3} = 1.66...$
		$\rho(S, M)$	$\frac{5}{3} = 1.66...$	2

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