Online Searching with Turn Cost

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Abstract

We consider the problem of searching for an object on a line at an unknown distance OPT from the original position of the searcher, in the presence of a cost of d for each time the searcher changes direction. This is a generalization of the well-studied linear-search problem. We describe a strategy that is guaranteed to find the object at a cost of at most $9 \cdot \mathrm{OPT} + 2d$, which has the optimal competitive ratio 9 plus the minimum additive term. Our argument for upper and lower bound uses an infinite linear program, which we solve by experimental solution of an infinite series of approximating finite linear programs, guessing the limits, and solving the resulting recurrences. We feel that this technique is interesting in its own right and should help solve other searching problems. In particular, we consider the *star search* or *cow-path problem* with turn cost, where the hidden object is placed on one of m rays emanating from the original position of the searcher. For this problem we give a tight bound of $\left(1 + 2\frac{m^m}{(m-1)^{m-1}}\right) \mathrm{OPT} + m\left(\left(\frac{m}{m-1}\right)^{m-1} - 1\right) d$. Our results also establish the notion of a competitive ratio with an additive term.

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Keywords: search games, linear search problem, star search, cow-path problem, turn cost, competitive ratio, two-term competitive ratios, infinite linear programs.

1 Introduction

Searching for an object is one of the fundamental issues of everyday life, and one of the basic algorithmic problems in the context of computing. Thus, determining an optimal competitive ratio for search problems is a natural task. For the linear search problem, the optimal competitive ratio is 9, as was first shown by Beck and Newman [3]: The searcher should alternate between going to the right and to the left, at each iteration doubling his step size. By placing the object at one of the points just beyond a turning point of the searcher, the hider can assure that this ratio of 9 is best possible.

The linear search problem (i.e., searching for a target on the line) has been rediscovered, re-solved, and generalized independently by a number of researchers. One such generalization is the *star search* or *cow-path problem* (first solved by Gal [4]), where the searcher has to locate the object on one of m rays emanating from the origin; thus, the linear search problem is a special case for $m = 2$. See [2] for a rediscovery and some extensions, as well as [5, 6]. More recent results and references can be found in [7], which focuses on an estimate of lower-order terms if there is a known upper bound on the distance to the object.

In this paper, we study a clean and simple way to extend previous research to cover the more realistic scenario of turn cost. This assumption is natural and realistic, as any reasonable scenario incurs some such cost for turning. We describe optimal trajectories for this scenario. As it turns out, the resulting

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$$
\begin{array}{llcccccccccccccccc}
\text{minimize} & B \\
\text{subject to} & 2x_1 & & & & & & & & & & & + & d & \leq & B \\
& -6x_1 & + & 2x_2 & & & & & & & & & + & 2d & \leq & B \\
& 2x_1 & - & 6x_2 & + & 2x_3 & & & & & & & + & 3d & \leq & B \\
& \vdots & + & \vdots & - & \vdots & + & \vdots & & & & & + & \vdots & \leq & B \\
& 2x_1 & + & 2x_2 & + & \cdots & + & 2x_{i-2} & - & 6x_{i-1} & + & 2x_i & + & id & \leq & B \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & & & & & & & & & x_i & & & \geq & 0
\end{array}
$$

Table 1: The infinite linear program describing the optimal turn-cost coefficient.

trajectories are more realistic than the purely geometric series for the linear search problem that start with infinite "wiggling". This solves an open problem from [1]. Another interesting aspect is that we explore a competitive ratio with an additive term.

2 An Infinite Linear Program

In the presence of a positive turn cost d, the hider can make it impossible for the searcher to achieve any competitive ratio at all, by simply placing the object arbitrarily close to the origin, on the side that is not picked first by the searcher. Clearly, this requires a minimum cost of d, regardless of OPT. Moreover, the searcher will be forced to make a second turn if he starts with a too small (or infinitesimal) first step. This increases the minimum cost to $2d$, although only for certain values of OPT.

We show that the optimal competitive ratio, c, remains the same even if we add turn cost. Thus, the worst-case time to reach the target is $c \cdot \mathrm{OPT}$ plus some additive term, denoted B, that we wish to minimize. Determining the minimal value of B is one of the main objectives of this paper.

After establishing that there has to be a first step in any good online search trajectory, suppose that the searcher carries out a sequence of step lengths $x_1, x_2, \ldots$ from the origin, where $x_1, x_3, \ldots$ are increasing distances to the right, while $x_2, x_4, \ldots$ are increasing distances to the left. In the following, we denote these turn positions by $p_i = (-1)^{i+1} x_i$.

For a given sequence of step lengths x_i, the hider can choose the possible set of locations $y_i = (-1)^{i+1}(x_i + \varepsilon)$ for an arbitrarily small ε. If the object is placed at y_n, the searcher will only encounter it after traveling a distance of $(\sum_{i=1}^{n+1} 2x_i) + x_n + \varepsilon$, and making $n+1$ turns. Note that for arbitrarily small d, this approaches the linear search problem, with a competitive ratio of 9. In order to guarantee this competitive ratio plus an additive cost of B, B must be big enough that y_i and the corresponding search trajectories satisfy the infinite linear program shown in Table 1.

Each subsystem yields a valid lower bound on the optimal B for a given d. Solving a number of finite subsystems with CPLEX 7.1 (for $d = 1$), we make several experimental observations:

1. Even small subsystems may have "ugly" solutions, indicating relatively fast-increasing effort for manually analyzing subsystems to establish lower bounds.

2. Convergence is rather slow, roughly logarithmic.

3. Actually closing the remaining gap and proving that the lower bound of $9 \cdot \mathrm{OPT} + 2d$ is tight requires considering the whole infinite linear program.

We will demonstrate in the following section how the latter point can be carried out analytically.

3 Provably Optimal Strategies

We establish optimal solutions for the infinite linear program and its dual, and thus a pair of optimal strategies for searcher and adversary. These solutions are based on extrapolation from our experimental data. We start by describing a feasible dual solution that yields a lower bound of $2d$ for the objective value B.

Consider $c_i = \frac{1}{2^i}$. With these dual multipliers, the linear combination of the first n coefficients for any variable x_j yields the coefficient $-\frac{8}{2^{j+1}} + \sum_{i=j}^{n} \frac{2}{2^i} = \frac{1}{2^j}\left(-4 + 2\sum_{i=0}^{n-j} \frac{1}{2^i}\right)$. Clearly, this tends to zero as n grows. Similarly, the coefficient of d becomes $\sum_{i=1}^{n} \frac{i}{2^i}$, which tends to 2 as n grows. Finally, the coefficient of B on the right-hand side of the inequality is the geometric series $\sum_{i=1}^{n} \frac{1}{2^i}$, which tends to 1. In other words, using an infinite linear combination with the described dual variables, we get the bound $2d \leq B$.

To see that $2d$ is also an upper bound, and thus an optimal solution for B, consider $x_j = d(2^j - 1)/2$. For this particular set, the ith inequality $\left(\sum_{h=1}^{i} 2x_h\right) - 8x_{i-1} + id \leq B$, simplifies to $2d \leq B$, implying that these x_i indeed characterize an optimal strategy for the searcher. It should be noted that the primal and dual solutions satisfy complementary slackness, as all constraints hold with equality.

Note that our solution to the linear search problem with turn cost achieves a competitive ratio 9, which must be the optimal value because the time to find the target increases with $d > 0$.

Theorem 1 *In the presence of turn cost d for the linear search problem, the searcher can guarantee a solution within time* $9 \cdot \mathrm{OPT} + 2d$ *by choosing the search strategy* $x_i = d(2^i - 1)/2$ *for* $i = 1, 2, \ldots$. *The additive term* $2d$ *is optimal subject to the optimal competitive ratio,* 9.

4 Star Search

In the presence of more than two rays emanating from the origin, we obtain the following result:

Theorem 2 *In the presence of turn cost d for the star search problem on* m *rays, the searcher can guarantee a solution within time* $\left(1 + 2\frac{m^m}{(m-1)^{m-1}}\right)\mathrm{OPT} + m\left(\left(\frac{m}{m-1}\right)^{m-1} - 1\right)d$ *by choosing the search strategy* $x_i = \frac{d}{2}\left(\left(\frac{m}{m-1}\right)^{i+1} - 1\right)$ *for* $i = 1, 2, \ldots$. *The additive term is optimal given the optimal competitive ratio.*

To prove that this is tight, we can work out the explicit values of the dual variables c_i by considering that the coefficient of x_{j+1} becomes $\left(2\sum_{i=j}^{\infty} c_i\right) + 2\frac{m^m}{(m-1)^{m-1}} c_{j+m-1}$, which must be zero. Subtracting these conditions for x_j and x_{j+1}, we get a recursive formula for c_n. Technical details are somewhat involved and omitted from this short abstract.

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