# A Note on the Traveling Preacher Problem


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**Abstract**

In this paper, we consider a problem of cost allocations for shortest roundtrips. Given a weighted graph $G = (V, E)$, we are to find a subset $S \subseteq V$ with a maximum weight core element for the Traveling Preacher game, i.e., a subset $S \subseteq V$ with a maximum cost allocation $x(S)$ to the vertices, such that there is no nontrivial subset $S' \subseteq S$ of vertices with a total cost allocation $x(S')$ exceeding the cost of a Traveling Salesman tour visiting the vertices in the subset $S$. This game can be considered as a variant of the so-called Traveling Salesman game, with the difference that there is no specified central root node for the salesman. We show that this "Traveling Preacher Problem" can be solved in polynomial time, showing that the difficulty of finding a core allocation for a combinatorial optimization problem may be caused by the existence of a special depot node, rather than being a consequence of the hardness of the optimization problem itself.



**Keywords:** Traveling Salesman Problem, combinatorial optimization, cooperative games, core allocation, polynomial algorithm, computational complexity.

**AMS Classification:** 90C27, 91A12


## 1 Introduction

The field of game theory deals with the economic behavior of individuals, companies, and nations. What are the prices for goods and services in a market economy? What strategies should be followed to choose from a variety of options in order to optimize profit? These are questions that were studied in the book by von Neumann and Morgenstern [18]. They suggest that rational entities (the "players") interact in a way that maximizes their individual benefit. Instances of these interactions are called "games"; while *Competitive* Game Theory studies the strategies of conflict, *Cooperative* Game Theory deals with aspects of cooperations between a finite set $N$ of players: How should they divide cost or profit in a fair way?

Formally, let $v : 2^N \to \mathbb{R}$ be a function that describes the cost for any subset of players. This function characterizes the game; typical examples arise from the cost for a connected cable network, or a roundtrip that visits all players in the set. It is reasonable to assume that no subset $S$ is willing to contribute more than the amount of $v(S)$, i.e., the cost of their own network or roundtrip. Since we have a finite number of subsets, the set of cost allocations satisfying all these conditions is a polyhedron, i.e., the *core* or set of fair allocations.

**Definition 1.1** *Let $(N, v)$ be a cooperative game. The* core *of the game is the polyhedron consisting of all vectors $x \in \mathbb{R}^n$ satisfying*

$$x(N) := \sum_{i \in N} x_i \quad = \quad v(N), \tag{1}$$

$$x(S) := \sum_{i \in S} x_i \quad \leq \quad v(S) \text{ for all } S \subset N. \tag{2}$$

Note that in the case of distributing savings or profit instead of cost, the inequalities have to be reversed. The concept of *stability* was already postulated by von Neumann and Morgenstern [18]. The first to define the core explicitly was Gillies (see [13]).

The geometry of the core is the subject of a number of questions: Is the core of a game non-empty? What is to be done if it is? What allocation should be chosen if there is more than one core element? How can this be achieved algorithmically?

It should be noted that over the years, game theory has been strong on concepts; these concepts include the Shapley value, bargaining sets, the kernel, and the nucleolus. On the other hand, until recently, there has been less emphasis on aspects of computation; this may be in part because the description of a general cooperative game already takes space that is exponential in the number of players, as there are exponentially many values of $v(S)$ that need to be listed. However, typical structures where cost needs to be allocated have been considered by combinatorial optimizers in the context of finding a best solution for a particular set of players: Scenarios where the cost for a set of nodes are given by a minimum cost spanning tree (MCST), a shortest roundtrip, or an optimal matching are examples of *implicit* descriptions of the value $v(S)$ of a subset. In more recent years, problems of this type have been considered: for Traveling Salesman (or TSP) games by Tamir [16], Potters and Curiel [12], and Faigle, Fekete, Hochstättler, and Kern [6]; for routing games (where the order of nodes to be visited by a traveling salesman is fixed and induced by the permutation on the full set of nodes) by Curiel, Potters, and Tijs [3], Derks and Kuipers [5], Solymósi, Aarts, and Driessen [15]. For MCST games the number is even larger, see the references in [7] for an overview. Similarly, see the references in [8] for matching games. Other results related to network games include Tamir [17], and Skorin-Kapov [14]. Moreover, there has been work on a number of other combinatorial optimization games; see the articles by Deng, Ibaraki, and Nagamochi [4], and Goemans and Skutella [9], to name just two recent works.

In this paper, we show that an analogue to TSP games without a depot can be solved in polynomial time, even though the related polytope is described by an exponential number of facets, each of which is described by a problem that is computationally hard.

## 2 The Traveling Preacher Problem

For a combinatorial optimizer, it is no surprise that dealing with TSP games is a difficult task. But even an "easy" combinatorial optimization problem like the MCST (where the core is never empty) gives rise to hard questions on cooperative games: It was shown by Faigle, Fekete, Hochstättler, and Kern [7] that it is CoNP-hard to decide whether a cost allocation is fair or not, implying that it is computationally hard to optimize a linear function over the core of an MCST game.

We will see in the following that this is not necessarily because of the fact that there are exponentially many restriction, but mostly due to the existence of a depot, creating a Steiner tree type problem.

TSP games are defined by the following cost function for each subset:

**Definition 2.1** *Consider the complete undirected graph $G = (V_0, E)$ with vertex set $V_0 = V \cup \{0\}$, $V = \{1, \ldots, n\}$, and non-negative edge weights $d_{ij}$, $i, j \in V_0$. Define for every coalition $S \subseteq V$,*

$$\begin{aligned} v(S) &= \text{minimum length of a TSP tour visiting all nodes in } S \cup \{0\}, \text{i.e.,} \\ &= \min_{\pi(S\cup\{0\})} \left( \sum_{i \in S \cup \{0\}} d_{i,\pi(i)} \right), \text{ where } \pi \text{ is a permutation of } S \cup \{0\}. \end{aligned}$$

The following game arises from TSP games by dropping the depot 0:

**Definition 2.2** *Consider the complete undirected graph $G = (V, E)$ with vertex set $V$, and non-negative edge weights $d_{ij}$, $i, j \in V$. Define for every coalition $S \subseteq V$, $|S| \geq 3$:*

$$\begin{aligned} v(S) &= \text{minimum length of a TSP tour visiting all nodes in } S, \text{i.e.,} \\ &= \min_{\pi(S)} \left( \sum_{i \in S} d_{i,\pi(i)} \right), \text{ where } \pi \text{ is a permutation of } S. \end{aligned}$$

Instead of a salesman with a fixed home vertex, we consider a traveler who is willing to leave his home behind in order to serve a set of followers, without staying in one place, and without backtracking on the same road. (These natural requirements eliminate the easy cases $|S| = 1$ and $|S| = 2$.) Consequently, we call this scenario the *Traveling Preacher Problem.* Trying to find a subset of vertices that allows a largest fair cost allocation can be formulated as the following optimization problem:

$$\max \textstyle\sum_{i=1}^{n} x_i$$

$(TPP)$ **subject to the constraints**

$$x(S) \quad \leq \quad v(S) \quad \text{for all} \quad S \subseteq V, |S| \geq 3$$

Each of the exponentially many constraints involves the solution of an instance of the TSP—seemingly a hopeless task.

It should be noted that in her thesis [2], Cameron gives a "Coflow Polyhedron Theorem", which is based on a related problem for a directed graph, where both variables and costs are assigned to the nodes. For this case, an integrality result (TDI) holds. No such result holds for the case we study here.

However, we have the following theorem:

**Theorem 2.3** *The Traveling Preacher Problem can be solved in polynomial time.*

**Proof.** By the polynomial equivalence of linear optimization and separation over a polyhedron [11], it suffices to prove that separation over the set of feasible solutions can be done in polynomial time.

Therefore, consider a cost allocation $y = (y_1, \ldots, y_n)$ and for any edge $e = (i, j)$, define the following auxiliary edge weights:

$$w(e) := d_{i,j} - \frac{y_i}{2} - \frac{y_j}{2}.$$

In the complete graph with vertex set $V$ and edge weights $w(e)$, the existence of a negative cycle can be decided in polynomial time, see [1]. Suppose we have such a negative cycle, without loss of generality let it be given by $S = \{1, \ldots, s\}$. Then

$$w(s, 1) + \sum_{i=1}^{s-1} w(i, i+1) < 0 \Leftrightarrow d_{s,1} + \sum_{i=1}^{s-1} d_{i,i+1} < \sum_{i=1}^{s} y_i.$$

Clearly, the last inequality can hold if and only if there is a subset $S$ with $y(S) > c(S)$. Thus, detection of a negative cycle can be used as a polynomial separation oracle, completing the proof.

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It easy to modify the separation oracle to incorporate core restrictions arising from coalitions of two players; this correspond to locating an edge $e$ for which $w(e)$ is negative, which is a necessary condition for the existence of a negative cycle. Including restrictions by individual players makes the game trivial, since no cost can be allocated.

# 3 Conclusion

In this paper, we have shown that the Traveling Preacher Problem of allocating the cost for a shortest roundtrip can be solved in polynomial time. Our result contrasts to the known computational difficulty of finding core allocation for other combinatorial optimization games, including the closely related Traveling Salesman Problem. This shows that the difficulty of finding a core allocation for a combinatorial optimization problem may be caused by the existence of a special depot node, rather than being a consequence of the hardness of the optimization problem itself, or having to consider an exponential number of subsets.

There are other NP-hard problems for which the related game has a nice core structure, e.g., for some variants of the facility location problem (see [9]). On the other hand, the main reason for hardness of the

Steiner tree game and the TSP game is the fact that we can modify any instance by adding a constant amount to the cost for any set of players. This is also the case for the MCST game in the presence of a depot node; it was shown in [7] that separation over the core of MCST games is NP-hard.

It is conceivable that there is a strongly polynomial method for solving the Traveling Preacher Problem, possibly using a primal-dual approach. However, this appears to require a lot more technical work [10], clearly beyond the scope of this short note. Therefore, we leave this to future investigation.

## Acknowledgment

We thank Kamal Jain for very helpful feedback, in particular concerning the reason for NP-hardness mentioned above and the existence of a strongly polynomial algorithm. We also thank Bernhard Stengel for his assistance in this process.

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