# On the Reflexivity of Point Sets

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## Abstract

We introduce a new measure for planar point sets $S$. Intuitively, it describes the combinatorial distance from a convex set: The *reflexivity* $\rho(S)$ of $S$ is given by the smallest number of reflex vertices in a simple polygonization of $S$. We prove various combinatorial bounds and provide efficient algorithms to compute reflexivity, both exactly (in special cases) and approximately (in general). Our study naturally takes us into the examination of some closely related quantities, such as the *convex cover* number $\kappa_1(S)$ of a planar point set, which is the smallest number of convex chains that cover $S$, and the *convex partition* number $\kappa_2(S)$, which is given by the smallest number of disjoint convex chains that cover $S$. We prove that it is NP-complete to determine the convex cover or the convex partition number, and we give logarithmic-approximation algorithms for determining each.

## 1 Introduction

In this paper, we study a fundamental combinatorial property of a discrete set, $S$, of points in the plane: What is the minimum number, $\rho(S)$, of *reflex vertices* among all of the *simple polygonalizations* of $S$? A *polygonalization* of $S$ is a closed tour on $S$ whose straight-line embedding in the plane defines a connected cycle without crossings, i.e., a simple polygon. A vertex of a simple polygon is *reflex* if it has interior angle greater than $\pi$. We refer to $\rho(S)$ as the *reflexivity* of $S$.

In general, there are many different polygonalizations of a point set, $S$. There is always at least one: simply connect the points in angular order about the center of mass. A set $S$ has precisely one polygonization, iff it is in convex position, but usually there is a great number of them. Studying the set of polygonizations (e.g., counting them, enumerating them, or generating a random element) is a challenging active area of investigation in computational geometry.

The reflexivity $\rho(S)$ quantifies, in a combinatorial sense, the degree to which the set of points $S$ is in convex position. See Figure 1 for an example.

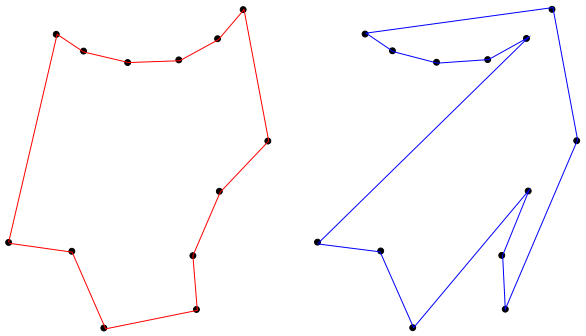

Figure 1: Two polygonalizations of a point set, one (left) using 7 reflex vertices and one (right) using only 3 reflex vertices.

We conduct a formal study of reflexivity, both in terms of its combinatorial properties and in terms of an algorithmic analysis of the complexity of computing it, exactly or approximately. Some of our attention is focussed on the closely related *convex cover number* of $S$, which gives the minimum number of convex chains (subsets of $S$ in convex position) that are required to cover all points of $S$. For this question, we distinguish two cases: The *convex cover number*, $\kappa_1(S)$, is the smallest number of convex chains to cover $S$; the *convex partition number*, $\kappa_2(S)$, is the smallest number of convex chains with pairwise-disjoint convex hulls to cover $S$. (Note that nested chains are feasible for a convex cover, but not for a convex partition.)

**Motivation.** In addition to the fundamental nature of the questions and problems we address, we are also motivated to study reflexivity for several other

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reasons:

(1). An application motivating our original investigation is that of meshes of low stabbing number and their use in performing ray shooting efficiently. If a point set $S$ has low reflexivity or convex partition number, then it has a triangulation of low stabbing number, which is much lower than the general $O(\sqrt{n})$ upper bound guaranteed to exist ([1, 11, 19]). (e.g., a constant reflexivity implies a logarithmic stabbing number triangulation)

(2). Classifying point sets by their reflexivity may give us some structure for dealing with the famously difficult question of counting and exploring the set of all polygonalizations of $S$. See [10] for some references to this problem.

(3) There are several applications in computational geometry in which the number of reflex vertices of a polygon can play an important role in the complexity of algorithms. If one or more polygons are *given* to us, there are many problems for which more efficient algorithms can be written with complexity in terms of "$r$" (the number of reflex vertices), instead of "$n$" (the total number of vertices), taking advantage of the possibility that we may have $r << n$ for some practical instances. (See, e.g., [12, 14].) The number of reflex vertices also plays an important role in convex decomposition problems for polygons (see, e.g., [15]).

(4) Reflexivity is intimately related to the issue of convex cover numbers, which has *roots in classical work* of Erdős and Szekeres [7, 8], and has been studied more recently by Urabe [17, 18].

(5) Our problems are related to some problems in curve (surface) reconstruction, where the goal is to obtain a "good" polygonalization of a set of sample points. (See [3, 4, 5].)

**Related Work.** The study of convex chains in finite planar point sets is the topic of classical papers by Erdős and Szekeres [7, 8], who showed that any point set of size $n$ has a convex subset of size $t = \Omega(\log n)$. This is closely related to the convex cover number $\kappa_1$, since it implies an asymptotically tight bound on $\kappa_1(n)$. There are still a number of open problems related to the exact relationship between $t$ and $n$; see, for example, [16] for recent developments. Other issues have been considered, such as the existence and computation ([6]) of large "empty" convex subsets (i.e., with no points of $S$ interior to their hull); this is related to the convex partition number, $\kappa_2(S)$. It was shown by Horton [13] that there are sets with no empty convex chain larger than 6, so this implies that $\kappa_2(n) \geq n/6$. Tighter worst-case bounds were given by Urabe [17, 18].

Another possibility is to consider a simple polygon having a given set of vertices, that is "as convex as possible". This has been studied in the context of TSP tours of a point set $S$, where convexity of $S$ provides a trivial optimal tour. Convexity of a tour can be characterized by two conditions. If we drop the global condition (i.e., no crossing edges), but keep the local condition (i.e., no reflex vertices), we get "pseudoconvex" tours. In [9] it was shown that any set with $|S| \geq 5$ has such a pseudoconvex tour. It is natural to require the global condition of simplicity instead, and minimize the number of local violations – i.e., the number of reflex vertices. As in the paper [9], this draws a close connection to angles in a tour, a problem that has also been studied by Aggarwal et al. [2]. We observe in the full paper that there are further connections.

The number of polygonizations on $n$ points is in general exponential in $n$ and to give tight bounds on the maximum value attainable for a given $n$ has also been object of intensive research ([10]). The minimum number of reflex vertices among all the polygonizations of a point set $S$ is the *reflexivity* of $S$, a concept we introduce in this work.

**Main Results** The main results of this work include:

- Tight combinatorial bounds for the worst-case reflexivity in a number of cases, including the general case and the case of onion depth 2.
- Upper and lower combinatorial bounds for reflexivity, convex cover number, convex partition number, and their relative behavior. We obtain exact worst-case values for small cardinalities.
- Proofs of NP-completeness for computing convex cover and convex partition numbers.
- Algorithmic results that give logarithmic approximations for convex cover number, convex partitioning number, and (Steiner) reflex-

ivity. We also give efficient exact algorithms for cases of low reflexivity.

## 2 Conclusion and Future Work

We have introduced a new class of combinatorial and algorithmic problems related to simple polygonizations of a planar point set. We have given lower and upper combinatorial bounds, settled the complexity status of some problem variants, and given several efficient approximation algorithms. Clearly, there are a number of interesting open problems, some of combinatorial type, some of algorithmic.

Furthermore, it may be of interest to study the generalization of our problem to three-dimensional simple surfaces for a given set of points. This may be of particular interest in the context of surface reconstruction.

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