Mathematical Methods of Algorithms **Winter 2023/24**

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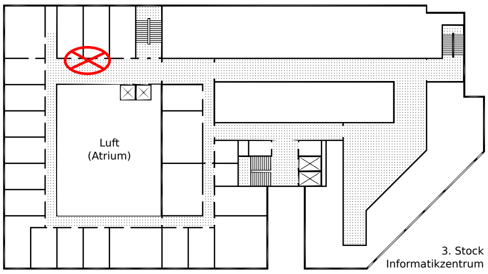
Due: 07.02.2023
Discussion: 07.02.2023

Homework 5

Solutions are to be left in the dedicated cupboard (see the pic) until 15:00 on the due date. Please put your name on all pages.



Exercise 1 (Integer Programming Modeling): $1 + 2$ ✓

For a given graph $G = (V, E)$, an *independent set* is a subset $S \subseteq V$ of the vertices such that the vertices in S are pairwise non-adjacent in G, i.e., $\forall v, w \in S, vw \notin E$. An independent set S is called *maximum independent set* if there is no independent set $S' \subseteq V$ with more vertices, i.e., with $|S'| > |S|$.

(a) Model finding a maximum independent set as 0-1-program, i.e., describe a polynomial-time transformation of a graph G into an integer program $P(G)$ with $\{0, 1\}$ variables such that an optimal solution to $P(G)$ can be transformed into a maximum independent set in G. Use exactly one variable per vertex, and exactly one constraint per edge (not counting variable bounds).

(b) A *clique* in a graph G is a subset $C \subseteq G$ of vertices that are pairwise adjacent, i.e., $\forall v, w \in C, v \neq w \Rightarrow vw \in E$.

An *edge clique cover* of G is a set $\mathcal{C}$ of cliques of G such that, for each edge $e = vw \in E$, there is a clique $C \in \mathcal{C}$ with $\{v, w\} \subseteq C$, i.e., a clique that contains both endpoints of e.

Assuming that you are given a graph G and an edge clique cover $\mathcal{C}$, model finding a maximum independent set as 0-1-program with at most $|\mathcal{C}|$ constraints (again not counting variable bounds).

Furthermore, give an example graph G and an edge clique cover $\mathcal{C}$ of G for which an optimal basic solution to the linear relaxation of your model from (a) is not integral, but an optimal basic solution to the linear relaxation of your model from (b) is integral.

Exercise 2 (Warm Starting): 2 + 1 ✓
Consider the problem

$$\max -3x_1 - 18x_2 + 9x_3 \text{ s.t.}$$
$$5x_1 - 3x_2 \leq 5$$
$$-2x_1 + 3x_2 - 2x_3 \leq 12$$
$$2x_1 + 2x_2 - 2x_3 \leq 4$$
$$x_3 \leq 10$$

Let's suppose we solved this, arriving at the following optimal dictionary.

$$\begin{array}{rrrrr} \zeta = & 90 - & 3x_1 - & 18x_2 - & 9w_4 \\ \hline w_1 = & 5 - & 5x_1 + & 3x_2 & \\ w_2 = & 32 + & 2x_1 - & 3x_2 - & 2w_4 \\ w_3 = & 24 - & 2x_1 - & 2x_2 - & 2w_4 \\ x_3 = & 10 & & - & 1w_4 \end{array}$$

(a) Suppose we want to add the constraint $2x_1 - 9x_2 + 2x_3 \leq 14$ to the problem. Update the optimal dictionary by adding a row according to this constraint (with a new slack variable w_5.)

Use the appropriate (either primal or dual) simplex algorithm, starting from the updated dictionary, to solve the new problem with the added constraint.

(b) Suppose you wanted to add a new variable x_4 instead, which has non-zero coefficients in some of the existing constraints, and a non-zero objective coefficient. Describe how you could use the existing dictionary to warm-start the appropriate (either primal or dual) simplex with the basic solution it provides, instead of re-solving from scratch.

Hint: It may be a good idea to change to matrix notation!