**Mathematical Methods of Algorithms** **Winter 2023/24**

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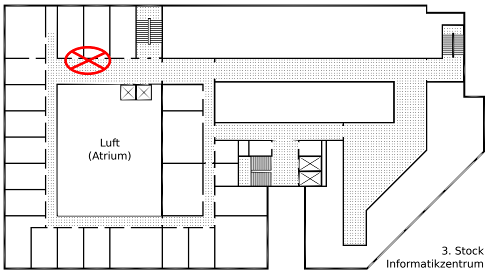
**Due:** 29.11.2023
**Discussion:** 29.11.2023

# Homework 1

**Solutions are to be left in the dedicated cupboard (see the pic) until 15:00 on the due date. Please put your name on all pages.**

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**Exercise 1 (Modeling as LP):** 1 ✓
A 10-acre slum in New York City is to be cleared. The officials of the city must decide on the redevelopment plan. Two housing plans are to be considered: low-income housing and middle-income housing. These types of housing can be developed at 20 and 15 units per acre, respectively. The unit costs of the low- and middle-income housing are $17,000 and $25,000. The lower and upper limits set by the officials on the number of low-income housing units are 80 and 120. Similarly, the number of middle-income housing units must lie between 40 and 90. The combined maximum housing market potential is estimated to be 190 (which is less than the sum of the individual market limits due to the overlap between the two markets). The total mortgage committed to the renewal plan is not to exceed $2.5 million. Finally, it was suggested by the architectural adviser that the number of low-income housing units be at least 50 units greater than one-half the number of the middle-income housing units.

a) Formulate the minimum cost renewal planning problem as a linear program and solve it graphically.

b) Resolve the problem if the objective is to maximize the number of houses to be constructed.

**Exercise 2 (Solve the LP):** 1 ✓
Consider the following problem:

$$\begin{array}{llll} \max & +2x_1 & +3x_2 & \\ & +1x_1 & +x_2 \leq & 3 \\ & +4x_1 & +6x_2 \leq & 9 \end{array}$$

$$x_1, x_2 \geq 0$$

a) Sketch the feasible region.

b) Find two alternative optimal extreme (corner) points.

c) Find an infinite class of optimal solutions.

**Exercise 3 (Entering Variables):** 1 ✓
Give an example linear program in which a variable that becomes a basic variable in a pivot step immediately leaves the basis in the next pivot step of the simplex algorithm. You may pick an arbitrary method to choose which variable with positive coefficient in the objective function becomes basic.

**Exercise 4 (Leaving Variables):** 1 ✓
Show that a variable that becomes non-basic in a pivot step cannot become basic again in the next pivot step.

**Exercise 5 (Solving an Abstract Linear Program):** 2 ✓
Consider the following linear program.

$$\max \sum_{j=1}^{n} p_j x_j$$

$$\text{s.t.} \sum_{j=1}^{n} q_j x_j \leq \beta$$

$$0 \leq x_j \leq 1, j \in \{1, \ldots, n\}.$$

In this program, let the coefficients $p_j$ and $q_j$ all be strictly positive. Furthermore, assume that they satisfy

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < \cdots < \frac{p_n}{q_n},$$

and let $\beta > 0$.

Let $t \geq 1$ be the smallest index with

$$\sum_{j=t}^{n} q_j \leq \beta,$$

and assume that such an index exists. Prove that the optimal solution is

$$x_j = 1, \text{ for } t \leq j \leq n,$$

$$x_{t-1} = \frac{\beta - \sum_{j=t}^{n} q_j}{q_{t-1}}, \text{ and}$$

$$x_j = 0, \text{ for } 1 \leq j < t - 1.$$