Linear Programming

[V. ch5]: Duality Theory

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Associated with every linear program is another called its *dual*. The dual of this dual linear program is the *original linear program* (which is then referred to as the *primal* linear program).

- $\rightsquigarrow$ linear programs come in primal/dual pairs.
- $\rightsquigarrow$ every feasible solution for one of these two linear programs gives a bound on the optimal value for the other.

These ideas are important and form a subject called *duality theory*, the topic of this chapter.

We begin with an example:

$$\begin{array}{rrrrl}
\max\limits_{x} & 4x_1 + & x_2 + & 3x_3 & \\
\text{subject to} & x_1 + & 4x_2 & & \leq 1 \\
& 3x_1 - & x_2 + & x_3 & \leq 3 \\
& x_1, & x_2, & x_3 & \geq 0
\end{array}$$

Observe that

→ every feasible solution to this LP provides a *lower bound* on the optimal value, ζ^*.

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But, *How good is this solution?*

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To answer:

we need to give upper bounds

$$\begin{array}{rrrrl} \max\limits_{x} & 4x_1 + & x_2 + & 3x_3 & \\ \text{subject to} & x_1 + & 4x_2 & & \leq 1 \\ & 3x_1 - & x_2 + & x_3 & \leq 3 \\ & x_1, & x_2, & x_3 & \geq 0 \end{array}$$

We can find a bound as follows:

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\end{array}$$

We can find a bound as follows:

→ Multiply the first constraint by 2 and *add* that to 3 times the second constraint

$$\begin{array}{rrrrrl}
& 2 \times (& x_1 + & 4x_2 & & \leq 1 \quad) \\
+ & 3 \times (& 3x_1 - & x_2 + & x_3 & \leq 3 \quad) \\
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Since each variable is nonnegative, we can compare the sum against the objective function

$$\zeta = 4x_1 + x_2 + 3x_3 \leq 11x_1 + 5x_2 + 3x_3 \leq \boxed{11}$$

$$\Rightarrow \quad \zeta^* \leq \boxed{11}$$

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These bounds leave a gap, $\boxed{9} \leq \zeta^* \leq \boxed{11}$. Now,

better insight on the quality of feasible solutions!

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→ Multiply the first constraint by $y_1 (\geq 0)$ and *add* that to $y_2 (\geq 0)$ times the second constraint

$$
\begin{array}{rrrrl}
 & y_1 \times (1x_1 + & 4x_2 & & \leq 1) \\
+ & y_2 \times (3x_1 - & x_2 + & x_3 & \leq 3) \\
\hline
 & (y_1 + 3y_2)x_1 + & (4y_1 - y_2)x_2 + & (y_2)x_3 & \leq y_1 + 3y_2
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$\rightarrow$ Enforce that each of the coefficients of the x_i 's be *at least as large as* the corresponding coefficient in the objective function, i.e.

$$\begin{array}{rrl} y_1 & + 3y_2 & \geq 4 \\ 4y_1 & - y_2 & \geq 1 \\ & y_2 & \geq 3 \end{array}$$

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$$\begin{array}{lll} y_1 & + 3y_2 & \geq 4 \\ 4y_1 & - y_2 & \geq 1 \\ & y_2 & \geq 3 \end{array}$$

→ then we can compare the objective function against this sum (and its bound).

$$\zeta = 4x_1 + x_2 + 3x_3 \leq (y_1 + 3y_2)x_1 + (4y_1 - y_2)x_2 + (y_2)x_3 \leq \boxed{y_1 + 3y_2}$$

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We are naturally led to the following optimization problem:

$$\begin{array}{lrrl} \min\limits_{y} & y_1 + & 3y_2 & \\ & y_1 + & 3y_2 & \geq 4 \\ & 4y_1 - & y_2 & \geq 1 \\ & & y_2 & \geq 3 \\ & y_1, & y_2 & \geq 0 \end{array}$$

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$\leadsto$ This problem is called the *dual* LP associated with the given LP.

GMPL Code 1: Primal model.

```
var x1 >=0;
var x2 >=0;
var x3 >=0;

subject to con1:   x1 + 4*x2      <= 1;
subject to con2: 3*x1 -   x2 + x3 <= 3;

maximize z: 4*x1 + x2 + 3*x3;

solve;

display x1.val, x2.val, x3.val, z.val;

end;
```

```
GLPSOL: GLPK LP/MIP Solver, v4.65
Reading model section from ex1_inSlides.mod...
.
.
.
Model has been successfully generated
GLPK Simplex Optimizer, v4.65
.
.
.
OPTIMAL LP SOLUTION FOUND
Time used:   0.0 secs
Memory used: 0.1 Mb (102265 bytes)
Display statement at line 12
x1.val = 0
x2.val = 0.25
x3.val = 3.25
z.val = 10
Model has been successfully processed
```

GMPL Code 2: Dual model.

```
var y1 >=0;
var y2 >=0;

subject to con1:   y1 + 3*y2 >= 4  ;
subject to con2: 4*y1 -    y2 >= 1  ;
subject to con3:          y2 >= 3 ;

minimize z: y1 + 3*y2 ;

solve;

display y1.val, y2.val, z.val;

end;
```

```
GLPSOL: GLPK LP/MIP Solver, v4.65
Reading model section from ex1_dual_inSlides.mod...
.
.
.
Model has been successfully generated
GLPK Simplex Optimizer, v4.65
.
.
.
OPTIMAL SOLUTION FOUND BY LP PREPROCESSOR
Time used:   0.0 secs
Memory used: 0.1 Mb (94214 bytes)
Display statement at line 12
y1.val = 1
y2.val = 3
z.val = 10
Model has been successfully processed
```

As a another example, consider the Resource Allocation Problem.

Recall that

$$
\begin{aligned}
\max_{x} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\
& x_1, x_2, \cdots, x_n \geq 0
\end{aligned}
$$

where

c_j = profit per unit of product j produced

b_i = unit of raw material i on hand

a_{ij} = units raw material i required to produce one unit of product j

If we produce one unit less of product j, then:

for each i, we free up a_{ij} units of raw material i.

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Selling these unused raw materials for $y_1, y_2, \cdots, y_m$ dollars/unit yields:

$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m$ dollars.

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Only interested if this revenue exceeds lost profit on each product j:

$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j, \quad j = 1, 2, \cdots, n..$

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$$a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m \geq c_j, \quad j = 1, 2, \cdots, n..$$

Now, consider a buyer offering to purchase our entire inventory. Subject to above constraints, buyer wants to minimize cost:

$$\min_y \quad b_1y_1 + b_2y_2 + \cdots + b_my_m$$

And the following linear program needs to be solved

$$
\begin{aligned}
\min_{y} \quad & b_1y_1 + b_2y_2 + \cdots + b_my_m \\
\text{subject to} \quad & a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1 \\
& \vdots \\
& a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_n \geq c_n \\
& y_1, y_2, \cdots, y_m \geq 0
\end{aligned}
$$

$\rightsquigarrow$ This problem is called the *dual* LP associated with the given LP.

Given a linear programming problem in standard form,

$$\begin{aligned}
\max_{x} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \cdots, m \\
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the associated *dual* linear program is given by

$$\begin{aligned}
\min_{y} \quad & \sum_{i=1}^{m} b_i y_i \\
\text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, \cdots, n \\
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\end{aligned}$$

Primal Problem

$$\max_{x} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \cdots, m$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

⇝ *Original problem is called the primal problem.*

Dual in "Standard" Form

$$-\max_{y} \quad \sum_{i=1}^{m} -b_i y_i$$

$$\text{subject to} \quad \sum_{i=1}^{m} -a_{ij} y_i \leq -c_j, \quad j = 1, 2, \cdots, n$$

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⇝ Dual is "negative transpose" of primal.

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⇝ Dual is "negative transpose" of primal.

Theorem *Dual of dual is primal.*

Theorem (Weak Duality)

if $(x_1, \cdots, x_n)$ *is feasible for the primal problem and* $(y_1, \cdots, y_m)$ *is feasible for the dual problem, then*

$$\sum_j c_j x_j \leq \sum_i b_i y_i$$

Proof.

$$
\begin{aligned}
\sum_j c_j x_j &\leq \sum_j (\sum_i y_i a_{ij}) x_j \\
&= \sum_i \sum_j y_i a_{ij} x_j \\
&= \sum_i (\sum_j a_{ij} x_j) yi \\
&\leq \sum_i b_i y_i
\end{aligned}
$$

□

An important question:
Is there a gap between the largest primal value and the smallest dual value?

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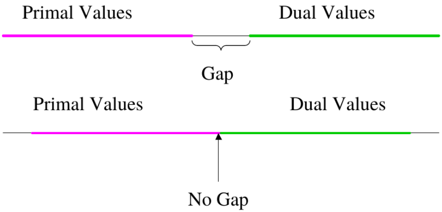

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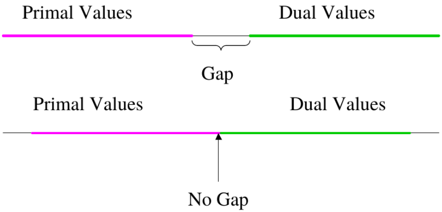

$\rightsquigarrow$ Answer is provided by the Strong Duality Theorem (coming later).

Consider the exmple:
Initial primal and dual dictionaries are provided.

Primal

$$
\begin{array}{rrrrr}
\zeta = & 0 - & 3x_1 + & \boxed{2}\, x_2 + & \boxed{1}\, x_3 \\
\hline
w_1 = & 0 - & 0x_1 + & x_2 - & 2x_3 \\
w_2 = & 3 + & 3x_1 - & 4x_2 - & x_3
\end{array}
$$

Dual

$$
\begin{array}{rrrr}
-\xi = & 0 + & 0y_1 - & 3y_2 \\
\hline
z_1 = & 3 - & 0y_1 - & 3y_2 \\
z_2 = & \boxed{-2} - & 1y_1 + & 4y_2 \\
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$\rightsquigarrow$ *Dual is negative transpose of primal.*

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⇝ Dual is negative transpose of primal.

⇝ Primal is feasible, dual is not.

Dual

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\hline
z_1 = & 3 - & 0y_1 - & 3y_2 \\
z_2 = & \boxed{-2} - & 1y_1 + & 4y_2 \\
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⇝ *Dual is negative transpose of primal.*

⇝ *Primal is feasible, dual is not.*

⇝ *Use primal to choose pivot: x_2 enters, w_2 leaves.*

Consider the exmple:
Initial primal and dual dictionaries are provided.

Primal

$$\begin{array}{rcrrr} \zeta = & 0 - & 3x_1 + & \boxed{2}\, x_2 + & \boxed{1}\, x_3 \\ \hline w_1 = & 0 - & 0x_1 + & x_2 - & 2x_3 \\ w_2 = & 3 + & 3x_1 - & 4x_2 - & x_3 \end{array}$$

Dual

$$\begin{array}{rrrr} -\xi = & 0 + & 0y_1 - & 3y_2 \\ \hline z_1 = & 3 - & 0y_1 - & 3y_2 \\ z_2 = & \boxed{-2} - & 1y_1 + & 4y_2 \\ z_3 = & \boxed{-1} + & 2y_1 + & 1y_2 \end{array}$$

$\rightsquigarrow$ *Dual is negative transpose of primal.*

$\rightsquigarrow$ *Primal is feasible, dual is not.*

$\rightsquigarrow$ *Use primal to choose pivot: x_2 enters, w_2 leaves.*

$\rightsquigarrow$ *Make analogous pivot in dual: z_2 leaves, y_2 enters.*

After first pivot

Primal

$$
\begin{array}{rrrrr}
\zeta = & \frac{3}{2} - & \frac{3}{2}x_1 - & \frac{1}{2}w_2 + & \boxed{\frac{1}{2}}\,x_3 \\
\hline
w_1 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{9}{4}x_3 \\
x_2 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{1}{4}x_3
\end{array}
$$

Dual

$$
\begin{array}{rrrr}
-\xi = & -\frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
\hline
z_1 = & \frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
y_2 = & \frac{1}{2} + & \frac{1}{4}y_1 + & \frac{1}{4}z_2 \\
z_3 = & \boxed{-\frac{1}{2}} + & \frac{9}{4}y_1 + & \frac{1}{4}z_2
\end{array}
$$

After first pivot

Primal

$$
\begin{array}{rrrrr}
\zeta = & \frac{3}{2} - & \frac{3}{2}x_1 - & \frac{1}{2}w_2 + & \boxed{\frac{1}{2}}\,x_3 \\
\hline
w_1 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{9}{4}x_3 \\
x_2 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{1}{4}x_3
\end{array}
$$

$\rightsquigarrow$ *negative transpose property intact.*

Dual

$$
\begin{array}{rrrr}
-\xi = & -\frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
\hline
z_1 = & \frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
y_2 = & \frac{1}{2} + & \frac{1}{4}y_1 + & \frac{1}{4}z_2 \\
z_3 = & \boxed{-\frac{1}{2}} + & \frac{9}{4}y_1 + & \frac{1}{4}z_2
\end{array}
$$

After first pivot

Primal

$$
\begin{array}{rrrrr}
\zeta = & \frac{3}{2} - & \frac{3}{2}x_1 - & \frac{1}{2}w_2 + & \boxed{\frac{1}{2}}\, x_3 \\
\hline
w_1 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{9}{4}x_3 \\
x_2 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{1}{4}x_3
\end{array}
$$

$\rightsquigarrow$ *negative transpose property intact.*

$\rightsquigarrow$ *Primal is feasible, dual is not.*

Dual

$$
\begin{array}{rrrr}
-\xi = & -\frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
\hline
z_1 = & \frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\
y_2 = & \frac{1}{2} + & \frac{1}{4}y_1 + & \frac{1}{4}z_2 \\
z_3 = & \boxed{-\frac{1}{2}} + & \frac{9}{4}y_1 + & \frac{1}{4}z_2
\end{array}
$$

After first pivot

Primal

$$\begin{array}{rrrrr} \zeta = & \frac{3}{2} - & \frac{3}{2}x_1 - & \frac{1}{2}w_2 + & \boxed{\frac{1}{2}}x_3 \\ \hline w_1 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{9}{4}x_3 \\ x_2 = & \frac{3}{4} + & \frac{3}{4}x_1 - & \frac{1}{4}w_2 - & \frac{1}{4}x_3 \end{array}$$

Dual

$$\begin{array}{rrrr} -\xi = & -\frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\ \hline z_1 = & \frac{3}{2} - & \frac{3}{4}y_1 - & \frac{3}{4}z_2 \\ y_2 = & \frac{1}{2} + & \frac{1}{4}y_1 + & \frac{1}{4}z_2 \\ z_3 = & \boxed{-\frac{1}{2}} + & \frac{9}{4}y_1 + & \frac{1}{4}z_2 \end{array}$$

$\rightsquigarrow$ *negative transpose property intact.*

$\rightsquigarrow$ *Primal is feasible, dual is not.*

$\rightsquigarrow$ *Use primal to pick pivot: x_3 enters, w_1 leaves.*

After first pivot

Primal

$$
\begin{array}{rcrrrr}
\zeta & = & \frac{3}{2} & - \frac{3}{2}x_1 & - \frac{1}{2}w_2 & + \boxed{\frac{1}{2}}x_3 \\
\hline
w_1 & = & \frac{3}{4} & + \frac{3}{4}x_1 & - \frac{1}{4}w_2 & - \frac{9}{4}x_3 \\
x_2 & = & \frac{3}{4} & + \frac{3}{4}x_1 & - \frac{1}{4}w_2 & - \frac{1}{4}x_3
\end{array}
$$

Dual

$$
\begin{array}{rcrrr}
-\xi & = & -\frac{3}{2} & - \frac{3}{4}y_1 & - \frac{3}{4}z_2 \\
\hline
z_1 & = & \frac{3}{2} & - \frac{3}{4}y_1 & - \frac{3}{4}z_2 \\
y_2 & = & \frac{1}{2} & + \frac{1}{4}y_1 & + \frac{1}{4}z_2 \\
z_3 & = & \boxed{-\frac{1}{2}} & + \frac{9}{4}y_1 & + \frac{1}{4}z_2
\end{array}
$$

⇝ *negative transpose property intact.*

⇝ *Primal is feasible, dual is not.*

⇝ *Use primal to pick pivot: x_3 enters, w_1 leaves.*

⇝ *Make analogous pivot in dual: z_3 leaves, y_1 enters.*

After second pivot

Primal

$$
\begin{array}{rrrrr}
\zeta = & \frac{5}{3} - & \frac{4}{3}x_1 - & \frac{5}{9}w_2 - & \frac{2}{9}w_1 \\
\hline
x_3 = & \frac{1}{3} + & \frac{1}{3}x_1 - & \frac{1}{9}w_2 - & \frac{4}{9}w_1 \\
x_2 = & \frac{2}{3} + & \frac{2}{3}x_1 - & \frac{2}{9}w_2 + & \frac{1}{9}w_1
\end{array}
$$

Dual

$$
\begin{array}{rrrr}
-\xi = & -\frac{5}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\
\hline
z_1 = & \frac{4}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\
y_2 = & \frac{5}{9} + & \frac{1}{9}z_3 + & \frac{2}{9}z_2 \\
y_1 = & \frac{2}{9} + & \frac{4}{9}z_3 - & \frac{1}{9}z_2
\end{array}
$$

After second pivot

Primal

$$\begin{array}{rrrrr} \zeta = & \frac{5}{3} - & \frac{4}{3}x_1 - & \frac{5}{9}w_2 - & \frac{2}{9}w_1 \\ \hline x_3 = & \frac{1}{3} + & \frac{1}{3}x_1 - & \frac{1}{9}w_2 - & \frac{4}{9}w_1 \\ x_2 = & \frac{2}{3} + & \frac{2}{3}x_1 - & \frac{2}{9}w_2 + & \frac{1}{9}w_1 \end{array}$$

$\rightsquigarrow$ *negative transpose property remains intact.*

Dual

$$\begin{array}{rrrr} -\xi = & -\frac{5}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\ \hline z_1 = & \frac{4}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\ y_2 = & \frac{5}{9} + & \frac{1}{9}z_3 + & \frac{2}{9}z_2 \\ y_1 = & \frac{2}{9} + & \frac{4}{9}z_3 - & \frac{1}{9}z_2 \end{array}$$

After second pivot

Primal

$$
\begin{array}{rrrrr}
\zeta = & \frac{5}{3} - & \frac{4}{3}x_1 - & \frac{5}{9}w_2 - & \frac{2}{9}w_1 \\
\hline
x_3 = & \frac{1}{3} + & \frac{1}{3}x_1 - & \frac{1}{9}w_2 - & \frac{4}{9}w_1 \\
x_2 = & \frac{2}{3} + & \frac{2}{3}x_1 - & \frac{2}{9}w_2 + & \frac{1}{9}w_1
\end{array}
$$

$\leadsto$ *negative transpose property remains intact.*

$\leadsto$ *Primal and dual are both optimal.*

Dual

$$
\begin{array}{rrrr}
-\xi = & -\frac{5}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\
\hline
z_1 = & \frac{4}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\
y_2 = & \frac{5}{9} + & \frac{1}{9}z_3 + & \frac{2}{9}z_2 \\
y_1 = & \frac{2}{9} + & \frac{4}{9}z_3 - & \frac{1}{9}z_2
\end{array}
$$

After second pivot

Primal

$$\begin{array}{rcccc}
\zeta = & \frac{5}{3} - & \frac{4}{3}x_1 - & \frac{5}{9}w_2 - & \frac{2}{9}w_1 \\
\hline
x_3 = & \frac{1}{3} + & \frac{1}{3}x_1 - & \frac{1}{9}w_2 - & \frac{4}{9}w_1 \\
x_2 = & \frac{2}{3} + & \frac{2}{3}x_1 - & \frac{2}{9}w_2 + & \frac{1}{9}w_1
\end{array}$$

Dual

$$\begin{array}{rccc}
-\xi = & -\frac{5}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\
\hline
z_1 = & \frac{4}{3} - & \frac{1}{3}z_3 - & \frac{2}{3}z_2 \\
y_2 = & \frac{5}{9} + & \frac{1}{9}z_3 + & \frac{2}{9}z_2 \\
y_1 = & \frac{2}{9} + & \frac{4}{9}z_3 - & \frac{1}{9}z_2
\end{array}$$

$\rightsquigarrow$ *negative transpose property remains intact.*

$\rightsquigarrow$ *Primal and dual are both optimal.*

$\rightsquigarrow$ *Simplex Alg. applied to primal, solves both the primal and the dual.*

THEOREM (STRONG DUALITY)

If the primal problem has an optimal solution,

$$x^* = (x_1^*, x_2^*, \cdots, x_n^*)$$

then the dual also has an optimal solution,

$$y^* = (y_1^*, y_2^*, \cdots, y_m^*)$$

and

$$\sum_j c_j x_j^* = \sum_i c_i y_i^*$$

$\rightsquigarrow$ If primal has an optimal solution, then there is no duality gap.

THEOREM (STRONG DUALITY)

If the primal problem has an optimal solution,

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then the dual also has an optimal solution,

$$y^* = (y_1^*, y_2^*, \cdots, y_m^*)$$

and

$$\sum_j c_j x_j^* = \sum_i c_i y_i^*$$

$\leadsto$ If primal has an optimal solution, then there is no duality gap.

$\leadsto$ Let's prove it.

The strong duality theorem tells us that:

If the primal has an optimal solution $\rightarrow$ the dual also has one and there is no duality gap

What if the primal problem does not have an optimal solution?

$\rightsquigarrow$ In case of unbounded primal, weak duality shows that the dual must be infeasible.

$\rightsquigarrow$ Similarly, an unbounded dual will have an infeasible primal.

there is still another possibility:

$\rightsquigarrow$ both the primal and the dual problems could be infeasible.
(strong duality theorem does not hold globally)

Four possibilities:

- Primal optimal, dual optimal (no gap).
- Primal unbounded, dual infeasible (no gap).
- Primal infeasible, dual unbounded (no gap).
- Primal infeasible, dual infeasible (infinite gap).

$\rightsquigarrow$ Example LP with infinite gap.

$$\begin{aligned} \max_{x} \quad & 2x_1 - x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 1 \\ & -x_1 + x_2 \leq -2 \\ & x_1, \; x_2 \geq 0 \end{aligned}$$

Lets check primal and dual infeasiblity.

Sometimes it is necessary to recover an optimal dual solution when only an optimal primal solution is known. (*without having access to the optimal primal dictionary*)

THEOREM (COMPLEMENTARY SLACKNESS)

suppose $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_m)$ *are primal and dual feasible solutions. They are optimal for their respective problems iff*

$$x_j z_j = 0 \quad \forall j = 1, 2, \cdots, n,$$
$$w_i y_i = 0 \quad \forall i = 1, 2, \cdots, m.$$

where $w = (w_1, \ldots, w_m)$ *and* $z = (z_1, \ldots, z_n)$ *are the corresponding primal and dual slack variables.*

$\rightsquigarrow$ Let's prove it.

Now, knowing this theorem, suppose that we have a *nondegenerate* optimal primal basic solution

$$(x_1^*, \ldots, x_n^*)$$

we wish to find a corresponding optimal dual solution. Note that if the primal slack values $(w_1^*, \ldots, w_m^*)$ are not given they could be easily computed. (how?)

Now the dual constraints are

$$\sum_i y_i a_{ij} - z_j = c_j, \qquad j = 1, \ldots, n$$

n equations in $m + n$ unknowns. But m of which are known to be 0 through complementary slackness theorem (why?)

we are left with just n equations in n unknowns.

One could simply apply the simplex method to the dual problem.

$\rightsquigarrow$You could do that on primal side, without writing down dual dictionaries
(negative transpose property provides you all needed data)

It could be seen simply as

$\rightsquigarrow$ a new way of picking the entering / leaving variables in a sequence of primal dictionaries,

The algorithm is called *dual simplex algorithm*.

Lets see dual simplex algorithm in an example.

Consider the following example:

$$
\begin{aligned}
\max_x \quad & -x_1 - x_2 \\
\text{subject to} \quad & -2x_1 - x_2 \leq 4 \\
& -2x_1 + 4x_2 \leq -8 \\
& -x_1 + 3x_2 \leq -7 \\
& x_1, \quad x_2 \geq 0
\end{aligned}
$$

and its dual

$$
\begin{aligned}
\min_y \quad & 4y_1 - 8y_2 - 7y_3 \\
\text{subject to} \quad & -2y_1 - 2y_2 - y_3 \geq -1 \\
& -y_1 + 4y_2 + 3y_3 \geq -1 \\
& y_1, \quad y_2, \quad y_3 \geq 0
\end{aligned}
$$

Introducing primal/dual slacks w_i/z_j, the initial dictionaries look like

P

$$\begin{array}{rrrr} \zeta = & & - \; 1x_1 - & x_2 \\ \hline w_1 = & 4 + & 2x_1 + & x_2 \\ w_2 = & \boxed{-8} + & 2x_1 - & 4x_2 \\ w_3 = & \boxed{-7} + & x_1 - & 3x_2 \end{array}$$

D

$$\begin{array}{rrrrr} -\xi = & & - \; 4y_1 + & \boxed{8}\, y_2 + & \boxed{7}\, y_3 \\ \hline z_1 = & 1 - & 2y_1 - & 2y_2 - & y_3 \\ z_2 = & 1 - & y_1 + & 4y_2 + & 3y_3 \end{array}$$

Note that:

dual dictionary is feasible, whereas the primal one is not.
How to proceed: phase I with primal or directly apply simplex to with dual

P

$$
\begin{array}{rrrr}
\zeta = & - & 1x_1 - & x_2 \\
\hline
w_1 = & 4 + & 2x_1 + & x_2 \\
w_2 = & \boxed{-8} + & 2x_1 - & 4x_2 \\
w_3 = & \boxed{-7} + & x_1 - & 3x_2
\end{array}
$$

D

$$
\begin{array}{rrrrr}
-\xi = & - & 4y_1 + & \boxed{8}\, y_2 + & \boxed{7}\, y_3 \\
\hline
z_1 = & 1 - & 2y_1 - & 2y_2 - & y_3 \\
z_2 = & 1 - & y_1 + & 4y_2 + & 3y_3
\end{array}
$$

$\rightsquigarrow$ (y_2, z_1) entring/leaving pair in the dual dictionary
$\rightsquigarrow$ their complementary variables w_2 and x_1 come in leaving/entering pair for primal dictionary.

how do you select this pair *without looking at dual dictionary*?

after doing the pivot:

P

$$\begin{array}{rrrr} \zeta = & -4 - & 0.5w_2 - & 3x_2 \\ \hline w_1 = & 12 + & w_2 + & 5x_2 \\ x_1 = & 4 + & 0.5w_2 + & 2x_2 \\ w_3 = & \boxed{-3} + & 0.5w_2 - & 1x_2 \end{array}$$

D

$$\begin{array}{rrrrr} -\xi = & 4 - & 12y_1 - & 4z_1 + & \boxed{3}\, y_3 \\ \hline y_2 = & 0.5 - & y_1 - & 0.5z_1 - & 0.5y_3 \\ z_2 = & 3 - & 5y_1 - & 2z_1 + & 1y_3 \end{array}$$

Negative transpose property and Dual feasibility preserved.

dual: (y_3, y_2) entring/leaving
primal: (w_3, w_2) leaving/entring

after doing the pivot:

P

$$
\begin{array}{rrrr}
\zeta = & -7 - & w_3 - & 4x_2 \\
\hline
w_1 = & 18 + & 2w_3 + & 7x_2 \\
x_1 = & 7 + & w_3 + & 3x_2 \\
w_2 = & 6 + & 2w_3 + & 2x_2
\end{array}
$$

D

$$
\begin{array}{rrrrr}
-\xi = & 7 - & 18y_1 - & 7z_1 - & 6y_2 \\
\hline
y_3 = & 1 - & 2y_1 - & z_1 - & 2y_2 \\
z_2 = & 4 - & 7y_1 - & 3z_1 - & 2y_3
\end{array}
$$

Negative transpose property and Dual feasibility preserved.

both dictionaries are optimal

Dual simplex method can be entirely described in terms of the *primal dictionaries*:

Note that the dictionary must be dual feasible (coefficients of the nonbasic variables in the primal objective row must be nonpositive). Given this:

Leaving variable selection:
Pick the basic variable whose constant term in the dictionary is the most negative
(if no one is negative, the dictionary is optimal)

Entering variable selection:
Scan the row selected above and pick the column with largest negated ratio

Lets illustrate it using an example:

$$
\begin{aligned}
\max_{x} \quad & -x_1 + 4x_2 \\
\text{subject to} \quad & -2x_1 - x_2 \leq +4 \\
& -2x_1 + 4x_2 \leq -8 \\
& -x_1 + 3x_2 \leq -7 \\
& x_1, \quad x_2 \geq 0
\end{aligned}
$$

Lets look at initial primal and dual dictionaries:

P

$$
\begin{array}{rcrrr}
\zeta & = & - & 1x_1 + & \boxed{4}\, x_2 \\
\hline
w_1 & = & 4 + & 2x_1 + & x_2 \\
w_2 & = & \boxed{-8} + & 2x_1 - & 4x_2 \\
w_3 & = & \boxed{-7} + & x_1 - & 3x_2
\end{array}
$$

D

$$
\begin{array}{rcrrrr}
-\xi & = & - & 4y_1 + & \boxed{8}\, y_2 + & \boxed{7}\, y_3 \\
\hline
z_1 & = & 1 - & 2y_1 - & 2y_2 - & y_3 \\
z_2 & = & \boxed{-4} - & y_1 + & 4y_2 + & 3y_3
\end{array}
$$

Neither the primal nor the dual dictionary is feasible $\rightarrow$ we need to do Phase I.

A new idea for Phase I :
Change the primal objective function so we can produce a dual feasible dictionary and proceed with dual simplex.

let us temporarily change the primal objective function to

$$\eta = -x_1 - x_2$$

performing dual simplex to this modified problem, we get the final optimal dictionary as

$$\begin{array}{rrrr} \eta = & -7 - & w_3 - & 4x_2 \\ \hline w_1 = & 18 + & 2w_3 + & 7x_2 \\ x_1 = & 7 + & w_3 + & 3x_2 \\ w_2 = & 6 + & 2w_3 + & 2x_2 \end{array}$$

Phase I is done: bring the original objective function and continue with primal simplex.

$$\begin{aligned}
\zeta &= -x1 + 4x_2 \\
&= -(7 + w_3 + 3x_2) + 4x_2 \\
&= -7 + w_3 + x_2
\end{aligned}$$

Hence, the starting dictionary for Phase II is:

$$\begin{array}{rrrr}
\zeta = & -7 - & w_3 + & \boxed{1}\, x_2 \\
\hline
w_1 = & 18 + & 2w_3 + & 7x_2 \\
x_1 = & 7 + & w_3 + & 3x_2 \\
w_2 = & 6 + & 2w_3 + & 2x_2
\end{array}$$

and immediately, we detect unboundedness.

How could one detect infeasibility using this new Phase I algorithm?

The primal problem is infeasible if and only if the modified problem is dual unbounded.