LINEAR PROGRAMMING

[V. CH2]: THE SIMPLEX METHOD

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Some examples first

The Simplex Algorithm

Initialization/Infeasibility

Unboundedness

geometry

Simplex Algorithm

In this chapter, we are going to learn a *method to solve* general linear programs. The method, called *Simplex algorithm*, will be developed for a general linear program (LP) in *standard form*.

SIMPLEX ALGORITHM

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Consider a simple example:

EXAMPLE

$$\begin{aligned} \max_x \quad & 5x_1 + 4x_2 + 3x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + x_3 \leq 5 \\ & 4x_1 + x_2 + 2x_3 \leq 11 \\ & 3x_1 + 4x_2 + 2x_3 \leq 8 \\ & x_1, \quad x_2, \quad x_3 \geq 0 \end{aligned}$$

Equalities and Slacks

Start by adding the so-called slack variables and convert *inequality* constraints to *equality* ones.

For each of the less-than inequalities: Introduce a slack variable that represents the difference between the right-hand side and the left-hand side.

- $\rightsquigarrow$ Introducing slack variable w_1

$$2x_1 + 3x_2 + x_3 \leq 5 \quad \Longleftrightarrow \quad w_1 = 5 - 2x_1 - 3x_2 - x_3, \quad w_1 \geq 0$$

- $\rightsquigarrow$ Introducing w_2

$$4x_1 + x_2 + 2x_3 \leq 11 \quad \Longleftrightarrow \quad w_2 = 11 - 4x_1 - x_2 - 2x_3, \quad w_2 \geq 0$$

- $\rightsquigarrow$ Introducing w_3

$$3x_1 + 4x_2 + 2x_3 \leq 8 \quad \Longleftrightarrow \quad w_3 = 8 - 3x_1 - 4x_2 - 2x_3, \quad w_3 \geq 0$$

Equalities and Slacks

We get the following *equivalent* LP

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\max\limits_{x} & \zeta = & & 5x_1 + & 4x_2 + & 3x_3 \\
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 & & \multicolumn{4}{l}{x_1, x_2, x_3, w_1, w_2, w_3 \geq 0}
\end{array}
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The simplex method is an iterative process in which:

- $\rightsquigarrow$ we start with a less-than-optimal solution $(\dot{x}_1, \dot{x}_2, \cdots, \dot{w}_3)$ that satisfies the equations and non-negativities and then

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- ⇝ we look for a new solution $(\bar{x}_1, \bar{x}_2, \cdots, \bar{w}_3)$, which is better in the sense that it has a larger objective function value:

$$5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 > 5\dot{x}_1 + 4\dot{x}_2 + 3\dot{x}_3$$

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- ⇝ We continue this process until we arrive at a solution that cannot be improved.

This final solution is then an *optimal* solution.

Initial Solution

Consider our example problem.

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To start the iterative process, we need an *initial feasible solution*.

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Simply set all the *original* variables to *zero*:

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0.$$

Now, use the equations to determine the slack variables:

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Luckily, we found a *feasible* solution:

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{w}_1, \dot{w}_2, \dot{w}_3) = (0, 0, 0, 5, 11, 8)$$

with objective function value $\zeta = 0$.

SOLUTION IMPROVEMENT

We now ask whether this solution can be improved.

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Observation.
Since the coefficient of x_1 in the objective function is *positive*, if we increase the value of x_1 from zero to some positive value, we will increase ζ.

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Observation.
As we change x_1's value, the values of the slack variables will also change. We must make sure that *we do not let any of them go negative.*

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x_2 and x_3 are currently set to 0, we see that

$$w_1 = 5 - 2x_1,$$

and so keeping w_1 non-negative imposes

$$w_1 \geq 0 \iff 5 - 2x_1 \geq 0 \iff x_1 \leq \frac{5}{2}.$$

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- ⇝ Non-negativity of w_2 imposes the bound that $x_1 \leq \frac{11}{4}$.
- ⇝ Non-negativity of w_3 imposes the bound that $x_1 \leq \frac{8}{3}$.

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Since *all of these non-negativity conditions* must be met, we see that x_1 cannot be made larger than the smallest of these bounds: $x_1 \leq \frac{5}{2}$.

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Our new solution then is

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (\frac{5}{2}, 0, 0, 0, 1, \frac{1}{2})$$

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We found an improved solution!

RECAPITULATION

Lets capture what we have done up to now.

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Only this easy because of the special layout!

Continuing

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What made the first step easy was the fact that we had one group of variables that were initially zero and we had the rest explicitly expressed in terms of these.

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– This special layout is called a dictionary.
In a dictionary, objective and variables on the left are *defined* by variables on the right.

– Dependent variables (on the left) are called basic variables.

– Independent variables (on the right) are called nonbasic variables.

– Setting variables on the right to zero and reading off the values of the variables on the left gives us a dictionary solution.

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Now, use the r.h.s. to describe w_2, w_3 and ζ only with the new set of independent variables: w_1, x_2 and x_3 as

$$\begin{array}{rrrrr} \zeta = & 12.5 - & 2.5w_1 - & 3.5x_2 + & 0.5x_3 \\ \hline x_1 = & 2.5 - & 0.5w_1 - & 1.5x_2 - & 0.5x_3 \\ w_2 = & 1 + & 2w_1 + & 5x_2 & \\ w_3 = & 0.5 + & 1.5w_1 + & 0.5x_2 - & 0.5x_3 \end{array}$$

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Note.
We can recover our current solution by setting the *independent* (non-basic) variables to zero and using the equations to read off the values for the dependent (basic) variables.

Next Improvement

Having the current (dictionary) solution and its corresponding dictionary, we can look for any further improvement.

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Again, we need to determine how much x_3 can be increased without violating the requirement that all the dependent variables remain nonnegative.

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\begin{array}{rrrrr}
\zeta = & 12.5 - & 2.5w_1 - & 3.5x_2 + & 0.5x_3 \\
\hline
x_1 = & 2.5 - & 0.5w_1 - & 1.5x_2 - & 0.5x_3 \\
w_2 = & 1 + & 2w_1 + & 5x_2 & \\
w_3 = & 0.5 + & 1.5w_1 + & 0.5x_2 - & 0.5x_3
\end{array}
$$

Now x_3 is the only variable with a positive coefficient.

Again, we need to determine how much x_3 can be increased without violating the requirement that all the dependent variables remain nonnegative.

This time, we see that the equation for w_2 is not affected by changes in x_3, but the equations for x_1 and w_3 do impose bounds, namely $x_3 \leq 5$ and $x_3 \leq 1$, respectively.

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$\rightarrow x_3$ could be increased up to 1.

Next Improvement

Set $x_3 = 1$ and re-compute dependent (basic) variable values according to the defining equations:

$$\begin{aligned} x_1 &= 2.5 - 0.5x_3 \\ w_2 &= 1 \\ w_3 &= 0.5 - 0.5x_3 \end{aligned}$$

we get

$$x_1 = 2, \quad w_2 = 1, \quad w_3 = 0.$$

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Our new solution then is

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3) = (2, 0, 1, 0, 1, 0)$$

with objective function value

$$\zeta = 5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 = 13.$$

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$$\zeta = 5\bar{x}_1 + 4\bar{x}_2 + 3\bar{x}_3 = 13.$$

We found an improved solution!

RETAINING THE DICTIONARY

In order to retain a dictionary layout for this solution, use w_2's defining equation and re-write it as

$$w_3 = 0.5 + 1.5w_1 + 0.5x_2 - 0.5x_3 \iff \boxed{x_3 = 1 + 3w_1 + x_2 - 2w_3}.$$

Now, use the right-hand side to describe x_1, w_2 and ζ only with the new set of independent variables: w_1, x_2 and w_3 as

$$\begin{array}{rrrrr} \zeta = & 13 - & w_1 - & 3x_2 - & w_3 \\ \hline x_1 = & 2 - & 2w_1 - & 2x_2 + & w_3 \\ w_2 = & 1 + & 2w_1 + & 5x_2 & \\ x_3 = & 1 + & 3w_1 + & x_2 - & 2w_3 \end{array}$$

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Note.
There is *no independent variable* for which an increase in its value would produce a corresponding increase in ζ and the algorithm stops.

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Claim: The current dictionary solution is *optimal*! The objective value ζ is at most 13. Why?

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Claim: The current dictionary solution is *optimal*! The objective value ζ is at most 13. Why?
We got to the equation

$$\zeta = 13 - w_1 - 3x_2 - w_3$$

by equivalence-preserving steps using only the constraints of our linear program!

Another Example

Consider another example:

Example

$$
\begin{aligned}
\max_{x} \quad & -x_1 + 3x_2 - 3x_3 \\
\text{s.t.} \quad & 3x_1 - x_2 - 2x_3 \leq 7 \\
& -2x_1 - 4x_2 + 4x_3 \leq 3 \\
& x_1 - 2x_3 \leq 4 \\
& -2x_1 + 2x_2 + x_3 \leq 8 \\
& 3x_1 \leq 5 \\
& x_1, x_2, x_3 \geq 0
\end{aligned}
$$

Slack Variables

Rewrite examples with slack variables:

$$
\begin{array}{lrrrrr}
\max_x & \zeta = & - & x_1 + & 3x_2 - & 3x_3 \\
\text{s.t.} & w_1 = & 7 - & 3x_1 + & x_2 + & 2x_3 \\
& w_2 = & 3 + & 2x_1 + & 4x_2 - & 4x_3 \\
& w_3 = & 4 - & x_1 & + & 2x_3 \\
& w_4 = & 8 + & 2x_1 - & 2x_2 - & x_3 \\
& w_5 = & 5 - & 3x_1 & & \\
& \multicolumn{5}{c}{x_1, x_2, x_3, w_1, w_2, w_3, w_4, w_5 \geq 0.}
\end{array}
$$

We obtain

– an initial dictionary with

as non-basic (independent) variables

as basic (dependent) variables

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We obtain

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- and dictionary solution

$$(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{w}_1, \dot{w}_2, \dot{w}_3, \dot{w}_4, \dot{w}_5) = (0, 0, 0, 7, 3, 4, 8, 5)$$

First Iteration

$$
\begin{array}{rrrrr}
\zeta = & - & x_1 + & 3\,x_2 - & 3x_3 \\
\hline
w_1 = & 7 - & 3x_1 + & x_2 + & 2x_3 \\
w_2 = & 3 + & 2x_1 + & 4x_2 - & 4x_3 \\
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– If x_2 increases, ζ goes up. Which variable reaches 0 first?

First Iteration

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– As x_2 increases, w_4 reaches zero; all other basic variables stay or go up! We say: x_2 becomes basic and w_4 becomes nonbasic.

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- Algebraically rearrange equations to retain the corresponding dictionary. This is called a pivot.

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- As x_2 increases, w_4 reaches zero; all other basic variables stay or go up! We say: x_2 becomes basic and w_4 becomes nonbasic.
- Algebraically rearrange equations to retain the corresponding dictionary. This is called a pivot.
- This basically means: Rearrange the linear equation defining the leaving variable w_4 to isolate the entering variable x_2, and substitute the new definition of x_2 in all other equations.

PIVOT STEP

A pivot: x_2 gets basic (enters the basis) and w_4 gets nonbasic (leaves the basis).

$$
\begin{array}{rrrrr}
\zeta = & 12 + & 2x_1 - & 1.5w_4 - & 4.5x_3 \\
\hline
w_1 = & 11 - & 2x_1 - & 0.5w_4 + & 1.5x_3 \\
w_2 = & 19 + & 6x_1 - & 2w_4 - & 6x_3 \\
w_3 = & 4 - & x_1 & + & 2x_3 \\
x_2 = & 4 + & x_1 - & 0.5w_4 - & 0.5x_3 \\
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– Now, let x_1 increase. Which basic variable becomes nonbasic?

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– Now, let x_1 increase. Which basic variable becomes nonbasic?

– Of the basic variables, w_5 hits zero first at $x_1 = \frac{5}{3}$. x_1 enters and w_5 leaves the basis.

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- Now, let x_1 increase. Which basic variable becomes nonbasic?
- Of the basic variables, w_5 hits zero first at $x_1 = \frac{5}{3}$. x_1 enters and w_5 leaves the basis.
- Rearrange equations accordingly.

Resulting Dictionary

$$
\begin{array}{rrrrr}
\zeta = & \frac{46}{3} - & \frac{2}{3}w_5 - & \frac{3}{2}w_4 - & \frac{9}{2}x_3 \\
\hline
w_1 = & \frac{23}{3} + & \frac{2}{3}w_5 - & \frac{1}{2}w_4 + & \frac{3}{2}x_3 \\
w_2 = & 29 - & 2w_5 - & 2w_4 - & 6x_3 \\
w_3 = & \frac{7}{3} + & \frac{1}{3}w_5 & + & 2x_3 \\
x_2 = & \frac{17}{3} - & \frac{1}{3}w_5 - & \frac{1}{2}w_4 - & \frac{1}{2}x_3 \\
x_1 = & \frac{5}{3} - & \frac{1}{3}w_5 & &
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– no improvement is possible and the dictionary solution

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\end{array}
$$

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$$(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4, \bar{w}_5) = (\tfrac{5}{3}, \tfrac{17}{3}, 0, \tfrac{23}{3}, 29, \tfrac{7}{3}, 0, 0)$$

is optimal with $\zeta = \frac{46}{3}$.

Some examples first

The Simplex Algorithm

Initialization/Infeasibility

Unboundedness

geometry

INPUT

We now try to describe the simplex algorithm to solve a general linear program.
Given an LP in standard form:

$$\begin{aligned} \max_{x} \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad , \quad i = 1, 2, \cdots, m \\ & x_j \ge 0 \quad , \quad j = 1, 2, \cdots, n. \end{aligned}$$

Our first task is to introduce

INPUT

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Given an LP in standard form:

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Our first task is to introduce *slack variables* and a *name* for the objective function value:

$$\begin{aligned} \zeta = & \qquad \sum_{j=1}^{n} c_j x_j \\ w_i = & \quad b_i - \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, 2, \cdots, m \end{aligned}$$

FIRST DICTIONARY

As we saw in our examples, as the simplex method proceeds, the slack variables become *intertwined* with the original variables, and the whole collection is treated the same.

So lets rewrite

$$(x_1, \cdots, x_n, w_1, \cdots, w_m) = (x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+m})$$

That is, we let $x_{n+i} = w_i, \quad i = 1, 2, \cdots, m$.

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With this notation, our *first* dictionary looks like

$$\begin{aligned} \zeta &= \qquad\quad \sum_{j=1}^{n} c_j x_j \\ x_{n+i} &= \quad b_i - \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, 2, \cdots, m \end{aligned}$$

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As the simplex method progresses, it moves from one dictionary to another in its search for an optimal solution. Each dictionary has m basic variables and n nonbasic variables.

Basis

Let

$\mathscr{B}$ denote the set of indices, from $\{1, 2, \ldots, n + m\}$, corresponding to the basic variables,

$\mathscr{N}$ denote the set of indices corresponding to the nonbasic variables

BASIS

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$\mathscr{B}$ denote the set of indices, from $\{1, 2, \ldots, n+m\}$, corresponding to the basic variables,

$\mathscr{N}$ denote the set of indices corresponding to the nonbasic variables

Initially, we have

$$\mathscr{N} = \{1, 2, ..., n\}$$
$$\mathscr{B} = \{n+1, n+2, ..., n+m\}$$

but this of course changes after the first iteration.

Basis

Let

$\mathscr{B}$ denote the set of indices, from $\{1, 2, \ldots, n+m\}$, corresponding to the basic variables,

$\mathscr{N}$ denote the set of indices corresponding to the nonbasic variables

Initially, we have

$$\begin{aligned} \mathscr{N} &= \{1, 2, ..., n\} \\ \mathscr{B} &= \{n+1, n+2, ..., n+m\} \end{aligned}$$

but this of course changes after the first iteration.
Down the road, the current dictionary will look like:

$$\begin{aligned} \zeta &= \bar{\zeta} + \sum_{j \in \mathscr{N}} \bar{c}_j x_j \\ x_i &= \bar{b}_i - \sum_{j \in \mathscr{N}} \bar{a}_{ij} x_j, \qquad i \in \mathscr{B} \end{aligned}$$

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Note.
We have put bars over the coefficients to indicate that they *change* as the algorithm progresses.

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⇝ For now, suffice it to pick an index k having the largest coefficient (which again could leave us with a choice).
⇝ Technically, any choice works; in practice, the choice has a strong influence on the number of steps we have to do.

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We must *ensure* that each of these variables *remains non-negative*. Hence, we require that

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Note.
Of these expressions, the only ones that can go negative (as x_k increases) are those for which $\bar{a}_{ik}$ is *positive*; the rest remain fixed or increase.

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Hence, we can restrict our attention to those i's for which $\bar{a}_{ik}$ is positive. And for such an i, the value of x_k at which the expression becomes zero is

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Since we do not want any of these to go negative, we must raise x_k only to the smallest of all of these values:

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pick l from $\{i \in \mathscr{B}$: $\bar{a}_{ik} > 0$ and $\frac{\bar{b}_i}{\bar{a}_{ik}}$ is minimal$\}$

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What special situation occurs if we have a choice regarding the leaving variable?

⇝ We end up with a basic variable that is 0, a so-called *degeneracy*.

⇝ This can cause problems! Can you think of any? We will deal with that in the next lecture!

PIVOTING

Once the *leaving basic* and *entering nonbasic* variables have been selected,

- the move from the current dictionary to the new dictionary involves appropriate row operations to achieve the *interchange*.
- In general, this means rearranging the defining equation of the leaving variable to make it define the entering variable instead, and substituting the new definition of the entering variable into all right-hand sides.
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Actually, we only need the entering variable to occur with a non-zero coefficient in the defining equation of the leaving variable.

Of course, such general pivots might lead to infeasible dictionaries or make our solution worse.

As mentioned, there is *often* more than one choice for the entering variable (and sometimes also for the leaving variable). Particular rules that make the choice *unambiguous* are called *pivot rules*.

SOME EXAMPLES FIRST

THE SIMPLEX ALGORITHM

INITIALIZATION/INFEASIBILITY

UNBOUNDEDNESS

GEOMETRY

New Example

Consider the following example:

Example

$$
\begin{aligned}
\max_x \quad & -2x_1 - x_2 \\
\text{s.t.} \quad & -x_1 + x_2 \leq -1 \\
& -x_1 - 2x_2 \leq -2 \\
& x_2 \leq 1 \\
& x_1, x_2 \geq 0
\end{aligned}
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⇝ The initial dictionary is *not* feasible. Why?

⇝ Up to now, we only considered problems for which the right-hand sides were all *non-negative*. This *ensured* that the *initial dictionary was feasible*. Now, we discuss what to do when this is not the case as the above example.

THE PROBLEM IN GENERAL

Given an LP:

$$\begin{aligned} \max_x \quad & \sum_{j=1}^{n} c_j x_j \\ s.t. \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad , \quad i = 1, 2, \cdots, m \\ & x_j \geq 0 \quad , \quad j = 1, 2, \ldots, n. \end{aligned}$$

The initial dictionary looks like

$$\begin{aligned} \zeta = & \qquad \sum_{j=1}^{n} c_j x_j \\ w_i = & \quad b_i - \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, 2, \cdots, m \end{aligned}$$

– The solution associated with this dictionary is obtained by setting: $x_j = 0 \,,\; w_i = b_i$

– This solution is feasible if and only if all b_i's are non-negative.

⇝ *But what if they are not?*

AUXILIARY PROBLEM

We handle this difficulty by introducing an auxiliary problem for which

(1) a feasible dictionary is *easy to find*, and

(2) an optimal dictionary provides *a feasible dictionary for the original problem*,

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Simply set $x_j = 0$ for $j = 1, \cdots, n$, and then pick x_0 sufficiently large.

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$$

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Simply set $x_j = 0$ for $j = 1, \cdots, n$, and then pick x_0 sufficiently large.

Note.
The original problem has a feasible solution iff the auxiliary problem has a feasible solution with $x_0 = 0$.
In other words, the original problem has a feasible solution iff the optimal solution to the auxiliary problem has zero objective value.

FEASIBLE DICTIONARY

Even though the auxiliary problem clearly has feasible solutions, we have not yet shown that it has an easily obtained feasible dictionary. It is best to illustrate how to obtain a feasible dictionary with an example.

Consider again the example

$$\begin{aligned} \max_x \quad & -2x_1 - x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \leq -1 \\ & -x_1 - 2x_2 \leq -2 \\ & x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

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Feasible Dictionary

Next, we introduce slack variables and write down an initial *infeasible* dictionary:

$$
\begin{array}{rrrrr}
\xi = & & & & -1\,x_0 \\
\hline
w_1 = & -1 + & x_1 - & x_2 + & x_0 \\
w_2 = & -2 + & x_1 + & 2x_2 + & x_0 \\
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To turn it feasible, *all we need* is to do a pivot with variable x_0 entering and the *most infeasible* basic variable, w_2, leaving. Why?

$$
\begin{array}{rrrrr}
\xi = & -2 + & 1x_1 + & 2x_2 - & 1w_2 \\
\hline
w_1 = & 1 & - & 3x_2 + & w_2 \\
x_0 = & 2 - & x_1 - & 2x_2 + & w_2 \\
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⇝ Note that we now have a feasible dictionary, so we can apply the simplex method as defined earlier in this chapter.

Reducing Infeasibility

Consider our feasible dictionary:

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\end{array}
$$

We pick x_2 to enter and w_1 to leave the basis. We get

$$
\begin{array}{rcrrrr}
\xi & = & -1.33 + & \boxed{1}\, x_1 - & 0.67w_1 - & 0.33w_2 \\
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x_2 & = & 0.33 & - & 0.33w_1 + & 0.33w_2 \\
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Now, for the second step, we pick x_1 to enter and x_0 to leave the basis.

REDUCING INFEASIBILITY

We get:

$$
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This dictionary is *optimal* for the auxiliary problem. Just note that

If optimal $\xi < 0$, the original LP is *infeasible*!

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We now drop x_0 from the equations and *reintroduce the original objective function*:

$$\zeta = -2x_1 - x_2 = -3 - w_1 - w_2.$$

What did we do to get from the old definition of the objective to the new one?

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What did we do to get from the old definition of the objective to the new one? Substitution!
Hence, the starting feasible dictionary for the *original problem* is

$$\begin{array}{rrrr} \zeta = & -3 - & w_1 - & w_2 \\ \hline x_2 = & 0.33 - & 0.33w_1 + & 0.33w_2 \\ x_1 = & 1.33 + & 0.67w_1 + & 0.33w_2 \\ w_3 = & 0.67 + & 0.33w_1 - & 0.33w_2 \end{array}$$

REDUCING INFEASIBILITY

We get:

$$\begin{array}{rrrrr} \xi = & & - & x_0 & & \\ \hline x_2 = & 0.33 & & - & 0.33w_1 + & 0.33w_2 \\ x_1 = & 1.33 - & x_0 + & 0.67w_1 + & 0.33w_2 \\ w_3 = & 0.67 + & x_0 + & 0.33w_1 - & 0.33w_2 \end{array}$$

This dictionary is *optimal* for the auxiliary problem. Just note that

If optimal $\xi < 0$, the original LP is *infeasible*!

We now drop x_0 from the equations and *reintroduce the original objective function*:

$$\zeta = -2x_1 - x_2 = -3 - w_1 - w_2.$$

What did we do to get from the old definition of the objective to the new one? Substitution! Hence, the starting feasible dictionary for the *original problem* is

$$\begin{array}{rrrr} \zeta = & -3 - & w_1 - & w_2 \\ \hline x_2 = & 0.33 - & 0.33w_1 + & 0.33w_2 \\ x_1 = & 1.33 + & 0.67w_1 + & 0.33w_2 \\ w_3 = & 0.67 + & 0.33w_1 - & 0.33w_2 \end{array}$$

As it turns out, this dictionary is *optimal* for the *original* problem (since the coefficients of all the variables in the equation for ζ are negative), but we *cannot* expect to be this lucky in general.

Two-Phase Simplex

⇝ All we normally can expect is that the dictionary so obtained will be *feasible* for the original problem, at which point we continue to apply the simplex method until an optimal solution is reached.

⇝ The process of solving the auxiliary problem to find an initial feasible solution is often referred to as Phase I, whereas the process of going from a feasible solution to an optimal solution is called Phase II. The overall algorithm is called Two-Phase Simplex Algorithm.

Some examples first

The Simplex Algorithm

Initialization/Infeasibility

Unboundedness

geometry

UNBOUNDED EXAMPLE

Consider the following dictionary:

$$
\begin{array}{rrrrr}
\zeta = & 0 + & 2\,x_1 - & x_2 + & 1\,x_3 \\
\hline
w_1 = & 4 + & 5x_1 - & 3x_2 + & x_3 \\
w_2 = & 10 + & 1x_1 + & 5x_2 - & 2x_3 \\
w_3 = & 7 + & & 4x_2 - & 3x_3 \\
w_4 = & 6 + & 2x_1 + & 2x_2 - & 4x_3 \\
w_5 = & 6 + & 3x_1 + & & 3x_3
\end{array}
$$

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Consider the following dictionary:

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– x_1 could be increased to improve ζ

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- x_1 could be increased to improve ζ
- Which basic variable decreases to 0 first?

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- x_1 could be increased to improve ζ
- Which basic variable decreases to 0 first?
- None of the basic variables will decrease. x_1 can grow without bound, ζ along with it.

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Consider the following dictionary:

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$$

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- Which basic variable decreases to 0 first?
- None of the basic variables will decrease. x_1 can grow without bound, ζ along with it.

Unboundedness occurs!

Unboundedness

Note.
Given a feasible dictionary, unboundedness occurs when there exists a non-basic variable with positive coefficient in the objective function whose increase is not bounded by any of the existing basic variables.

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UNBOUNDEDNESS

Note.
Given a feasible dictionary, unboundedness occurs when there exists a non-basic variable with positive coefficient in the objective function whose increase is not bounded by any of the existing basic variables.

$\rightsquigarrow$ As the non-basic variable goes up, the objective function increases without bound.

$\rightsquigarrow$ Going back to the rule for selecting the leaving variable:

$$\texttt{pick } l \texttt{ from } \{i \in \mathscr{B} : \bar{a}_{ik} > 0 \texttt{ and } \frac{\bar{b}_i}{\bar{a}_{ik}} \texttt{ is minimal}\}$$

unboundedness, will happen when $\forall i \in \mathscr{B} : \; : \bar{a}_{ik} \leq 0$.

ANOTHER EXAMPLE

As another example consider the following dictionary

$$
\begin{array}{rrrr}
\zeta = & 5 + & 1\,x_3 - & 1x_1 \\
\hline
x_2 = & 5 + & 2x_3 - & 3x_1 \\
x_4 = & 7 & - & 4x_1 \\
x_5 = & & & x_1
\end{array}
$$

We have: $k = 3, \mathscr{B} = \{2, 4, 5\}$ and

$$\bar{a}_{23} = -2, \bar{a}_{43} = 0, \bar{a}_{53} = 0$$

all non-positive.

Unboundedness occurs!

SOME EXAMPLES FIRST

THE SIMPLEX ALGORITHM

INITIALIZATION/INFEASIBILITY

UNBOUNDEDNESS

GEOMETRY

When the number of variables in a linear programming problem is *three* or *less*,
$\rightarrow$ we can graph the set of feasible solutions
$\rightarrow$ we can also graph the level sets of the objective function.

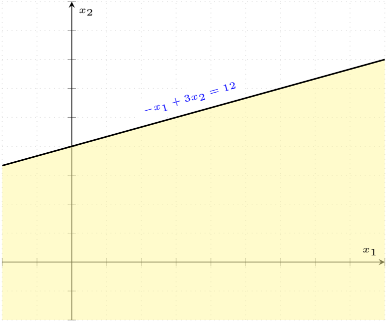
This way, finding the the optimal solution on this picture is usually trivial.
To illustrate, consider the following problem:

$$\begin{array}{llll} \max\limits_{x} & +3x_1 & +2x_2 & \\ \text{s.t.} & -x_1 & +3x_2 & \leq 12 \\ & +x_1 & +x_2 & \leq 8 \\ & +2x_1 & -x_2 & \leq 10 \\ & & x_1, x_2 \geq 0 & \end{array}$$

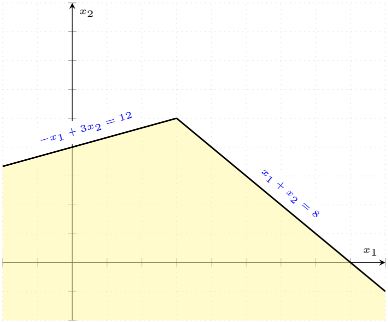
Each constraint (including the non-negativity constraints on the variables) is a *half-plane*.

$\rightsquigarrow$ These half-planes can be determined by first graphing the equation one obtains by replacing the inequality with an *equality* and then check some specific point.

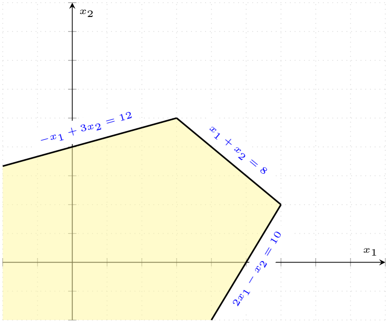
$\rightsquigarrow$ The set of feasible solutions is just the *intersection* of these half-planes.



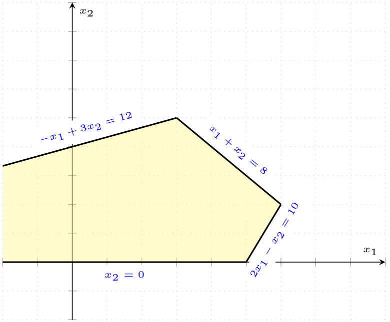
$$(x_1, x_2) \;\;:\;\; -x_1 + 3x_2 \leq 12$$



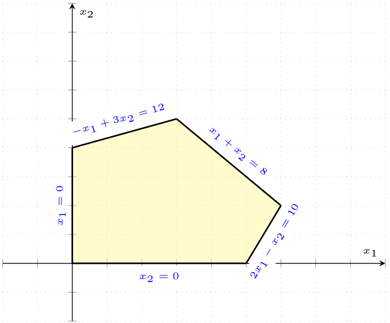
$$(x_1, x_2) \quad : \quad \left\{ \begin{array}{rcrcl} -x_1 & + & 3x_2 & \leq & 12 \\ +x_1 & - & x_2 & \leq & 8 \end{array} \right.$$



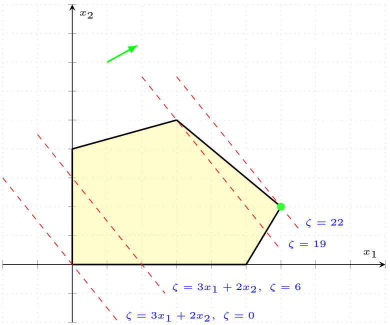
$$(x_1, x_2) \;:\; \left\{ \begin{array}{rcrcr} -x_1 & + & 3x_2 & \leq & 12 \\ +x_1 & - & x_2 & \leq & 8 \\ +2x_1 & - & x_2 & \leq & 10 \end{array} \right.$$

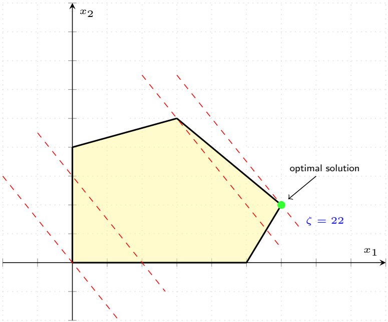


$$(x_1, x_2) \quad : \quad \left\{ \begin{array}{rcrcl} -x_1 & + & 3x_2 & \leq & 12 \\ +x_1 & - & x_2 & \leq & 8 \\ +2x_1 & - & x_2 & \leq & 10 \\ & & x_2 & \geq & 0 \end{array} \right.$$



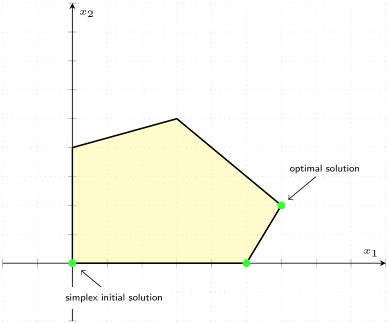
$$
(x_1, x_2) \quad : \quad \begin{cases} -x_1 & + & 3x_2 & \leq & 12 \\ +x_1 & - & x_2 & \leq & 8 \\ +2x_1 & - & x_2 & \leq & 10 \\ & & x_2 & \geq & 0 \\ x_1 & & & \geq & 0 \end{cases}
$$


x_2
x_1
$\zeta = 22$
$\zeta = 19$
$\zeta = 3x_1 + 2x_2,\ \zeta = 6$
$\zeta = 3x_1 + 2x_2,\ \zeta = 0$



Observation.
Algorithms of this type do exist but in *higher dimensions* the algebra required to implement such an algorithm gets quite complicated.

⇝ It turns out that the *simplex method* is algebraically much simpler and, on average performs well.



For the problem at hand:

⇝ the simplex method starts at $(0, 0)$ and jumps to adjacent vertices (green dots) of the feasible set until it finds a vertex that is an optimal solution. Starting at $(0, 0)$, it only takes two simplex pivots to get to the optimal solution.