Collaborative transmission in wireless sensor networks

Introduction to probability theory

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Overview and Structure

- Wireless sensor networks
- Wireless communications
- Basics on probability theory
- Randomised search approaches
- Cooperative transmission schemes
- Distributed adaptive beamforming
  - Feedback based approaches
  - Asymptotic bounds on the synchronisation time
  - Alternative algorithmic approaches
  - Alternative Optimisation environments
- An adaptive communication protocol

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Outline

Basics of probability theory

1. Introduction
2. Notation
3. Calculation with probabilities
4. The Markov inequality
5. The Chernoff bound

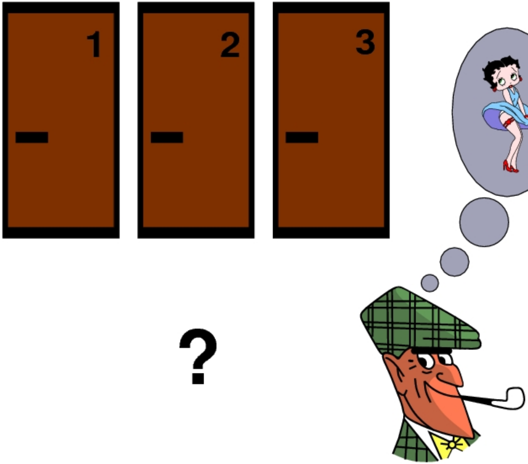
Probability in everyday life

We are confronted with Probability constantly:

- Weather forecasts
- Quiz shows
- ...

Example

The treasure behind the doors



Example

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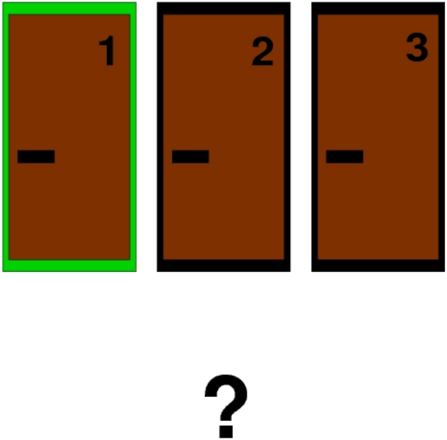

Example

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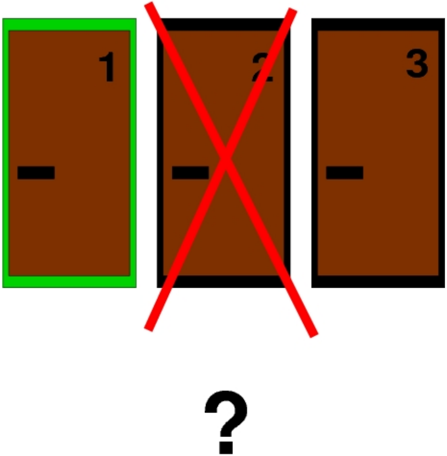


Example

The treasure behind the doors

- What shall the candidate do?
  - Alter his decision?
  - Retain his decision?
  - Does it make a difference?

Example

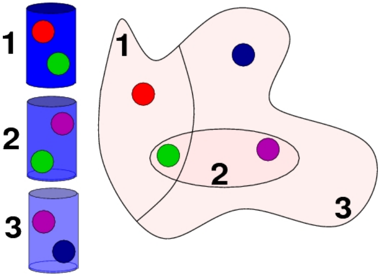
The treasure behind the doors

- What shall the candidate do?
  - Alter his decision?
  - Retain his decision?
  - Does it make a difference?
- We will consider the solution to this Problem in some minutes

Outline

Notation

Experiments, Events and sample points



- The results of experiments or observations are called events.
- Events are sets of sample points.
- The sample space is the set of all possible events.

Notation

Experiments, Events and sample points

- What is the sample space for the experiment of tossing a coin two times?

Example

Sample spaces

- Three distinct balls (a,b,c) are to be placed in three distinct bins.

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	abc			ab	ab	c		c		ac	ac	b	
2		abc		c		ab	ab		c	b		ac	ac
3			abc		c		c	ab	ab		b		b

14	15	16	17	18	19	20	21	22	23	24	25	26	27
b		bc	bc	a		a		a	a	b	b	c	c
	b	a		bc	bc		a	b	c	a	c	a	b
ac	ac		a		a	bc	bc	c	b	c	a	b	a

Example
Sample spaces

- Suppose that the three balls are not distinguishable.

Event / Bin	1	2	3	4	5	6	7	8	9	10
1	***			**	**	*		*		*
2		***		*		**	**		*	*
3			***		*		*	**	**	*

Example
Sample spaces

- Indistinguishable balls and indistinguishable bins

Bin \ Event	1	2	3
1	***	**	*
2		*	*
3			*

Notation

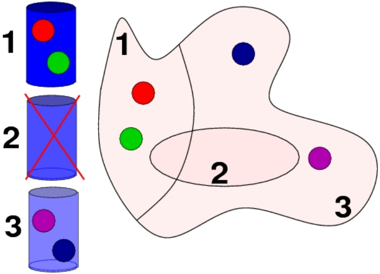
Probability space

Probability space

A probability space (Π, P) consists of a sample space Π and a probability measure $P : \Pi \rightarrow [0,1]$. This function satisfies the following conditions

- For each subset $\Pi' \subseteq \Pi, 0 \leq P(\Pi') \leq 1$
- $P(\Pi) = 1$
- For each $\Pi' \subseteq \Pi, P(\Pi') = \sum_{\chi \in \Pi'} P(\chi)$

Impossible events



Impossible event

With $\chi = \{\}$ we denote the fact that event χ contains no sample points. It is impossible to observe event χ as an outcome of the experiment.

Probability of events

Probability of events

Given a sample space Π and an event $\chi \in \Pi$, the occurrence probability $P(\chi)$ of event χ is the sum probability of all sample points from χ:

$$P(\chi) = \sum_{x \in \chi} P(x). \tag{1}$$

Statistical independence

Independence

A collection of events χ_i that form the sample space Π is independent if for all subsets $\Pi' \subseteq \Pi$

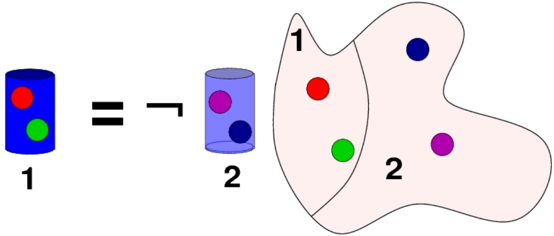
$$P\left(\bigcap_{\chi_i \in \Pi'} \chi_i\right) = \prod_{\chi_i \in S} P(\chi_i). \tag{2}$$

- Statistical independence is required for many useful results in probability theory.
- Be careful to apply such results not in cases where independence between sample points is not provided.

Outline

Calculation with probabilities

Negation of events



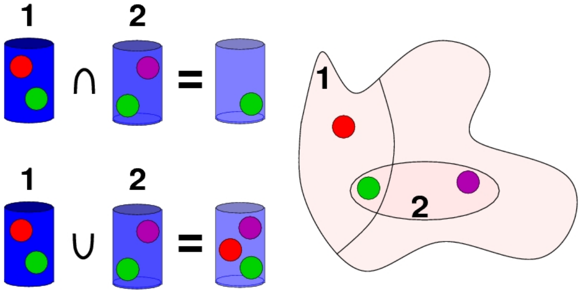
For every event χ there is an event $\neg\chi$ that is defined as 'χ does not occur'.

Negation of events

The event consisting of all sample points x with $x \notin \chi$ is the complementary event (or negation) of χ and is denoted by $\neg\chi$.

Calculation with probabilities

Subsumming events

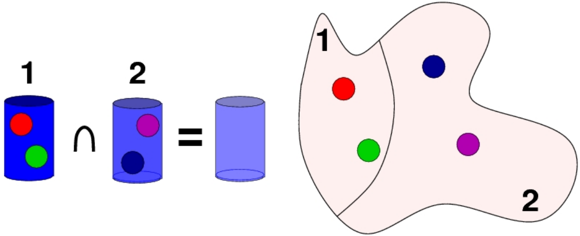


$$\chi_1 \cap \chi_2 = \{x | x \in \chi_1 \wedge x \in \chi_2\} \quad (3)$$

$$\chi_1 \cup \chi_2 = \{x | x \in \chi_1 \vee x \in \chi_2\} \quad (4)$$

Calculation with probabilities

Mutual exclusive events



Mutual exclusive events

When the events χ_1 and χ_2 have no sample point x in common, the event $\chi_1 \cap \chi_2$ is impossible: $\chi_1 \cap \chi_2 = \{\}$.
The events χ_1 and χ_2 are mutually exclusive.

Calculation with probabilities

Combining probabilities

- To compute the probability $P(\chi_1 \cup \chi_2)$ that either χ_1 or χ_2 or both occur we add the occurrence probabilities

$$P(\chi_1 \cup \chi_2) \leq P(\chi_1) + P(\chi_2) \tag{5}$$

Calculation with probabilities

Combining probabilities

- To compute the probability $P(\chi_1 \cup \chi_2)$ that either χ_1 or χ_2 or both occur we add the occurrence probabilities

$$P(\chi_1 \cup \chi_2) \leq P(\chi_1) + P(\chi_2) \tag{5}$$

- The '$\leq$'-relation is correct since sample points might be contained in both events:

$$P(\chi_1 \cup \chi_2) = P(\chi_1) + P(\chi_2) - P(\chi_1 \cap \chi_2). \tag{6}$$

Example
Coin tosses

Question

What is the probability that in two coin tosses either head occurs first or tail occurs second ?

Example

Coin tosses

Events	coin tosses	probability
head - head		$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
head - tail		$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
tail - head		$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
tail - tail		$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

Example
Coin tosses

Events	coin tosses	probability	sum probability
head - head		$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	
head - tail		$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	
tail - tail		$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$

Calculation with probabilities

Conditional probability

Conditional probability

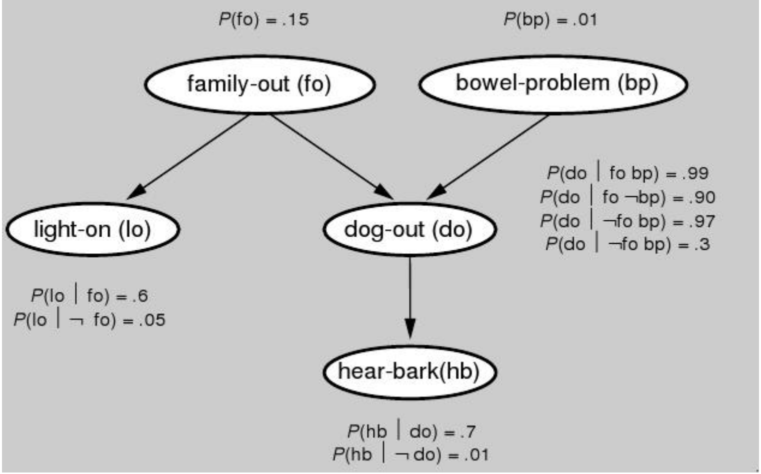
The conditional probability of two events χ_1 and χ_2 with $P(\chi_2) > 0$ is denoted by $P(\chi_1|\chi_2)$ and is calculated by

$$\frac{P(\chi_1 \cap \chi_2)}{P(\chi_2)} \tag{7}$$

$P(\chi_1|\chi_2)$ describes the probability that event χ_1 occurs in the presence of event χ_2.

Example

Conditional probability



Calculation with probabilities

Bayes Rule

With rewriting and some simple algebra we obtain the Bayes rule:

Bayes Rule

$$P(\chi_1|\chi_2) = \frac{P(\chi_2|\chi_1) \cdot P(\chi_1)}{\sum_i P(\chi_2|\chi_i) \cdot P(\chi_i)}. \tag{8}$$

- This equation is useful in many statistical applications.
- With Bayes rule we can calculate $P(\chi_1|\chi_2)$ provided that we know $P(\chi_2|\chi_1)$ and $P(\chi_1)$.

Calculation with probabilities

Expectation

Expectation

The expectation of an event χ is defined as

$$E[\chi] = \sum_{x \in \mathbb{R}} x \cdot P(\chi = x) \tag{9}$$

Example
Expectation

Example

Consider the event χ of throwing a dice. The Sample space is given by $S_\chi = \{1, 2, 3, 4, 5, 6\}$.

What is the expectation of this event?

Example

Expectation

Example

Consider the event χ of throwing a dice. The Sample space is given by $S_\chi = \{1, 2, 3, 4, 5, 6\}$.

What is the expectation of this event?

- The expectation of this event is

$$E[\chi] = \frac{1}{6} \cdot (1 + 2 + 3 + 4 + 5 + 6) = 3.5 \qquad (10)$$

Calculation with probabilities

Calculation with expectations

Linearity of expectation

For any two random variables χ_1 and χ_2,

$$E[\chi_1 + \chi_2] = E[\chi_1] + E[\chi_2]. \tag{11}$$

Multiplying expectations

For an independent random variables χ_1 and χ_2,

$$E[\chi_1 \cdot \chi_2] = E[\chi_1] \cdot E[\chi_2]. \tag{12}$$

Calculation with probabilities

Law of large numbers

Law of large numbers

Let $\{\overline{\chi}\}$ be a sequence of mutually independent random variables with a common distribution. If the expectation $E[\overline{\chi}]$ exists, then for every $\varepsilon > 0$ and $n \rightarrow \infty$

$$P\left\{\left|\frac{\chi_1 + \cdots + \chi_n}{n} - E[\overline{\chi}]\right| > \varepsilon\right\} \rightarrow 0 \tag{13}$$

- Probability that the average value differs from expectation by less than ε approaches one.

Calculation with probabilities

Variance

Variance

The variance of a random variable χ is defined as

$$var[\chi] = E[(\chi - E[\chi])^2]. \quad (14)$$

Calculation with probabilities

Calculation with variance

Add variances

For any independent random variables χ_1 and χ_2

$$var[\chi_1 + \chi_2] = var[\chi_1] + var[\chi_2]. \quad (15)$$

Multiplying variances

For any random variable χ and any $c \in \mathbb{R}$,

$$var[c\chi] = c^2 var[\chi]. \quad (16)$$

The Markov inequality

Estimate the deviation of an event from its expectation

Markov inequality

Let (Π, P) be a probability space and $x : \Pi \longrightarrow \mathbb{R}^+$ a non-negative random variable. For $t \in \mathbb{R}^*$ the following inequality holds:

$$P(x \geq t \cdot E[x]) \leq \frac{1}{t} \quad (17)$$

The Chernoff bound

Estimate the deviation of an event from its expectation

Chernoff bound

Let (Π, P) be a probability space and $x_1, x_2, \ldots, x_n : \Pi \rightarrow \{0, 1\}$ independent random variables with $0 < P(x_i = 1) < 1$ for all $i \in \{1, 2, \ldots, n\}$. For $X := \sum_{1 \leq i \leq n} x_i$ and $\delta > 0$ the following inequality holds:

$$P(X < (1 + \delta)E[X]) < \left(\frac{e^{\delta}}{(1 + \delta)^{1+\delta}}\right)^{E[X]} \quad (18)$$

and for all δ with $0 < \delta < 1$

$$P(X < (1 - \delta)E[X]) < e^{-\frac{E[X]\delta^2}{2}} \quad (19)$$

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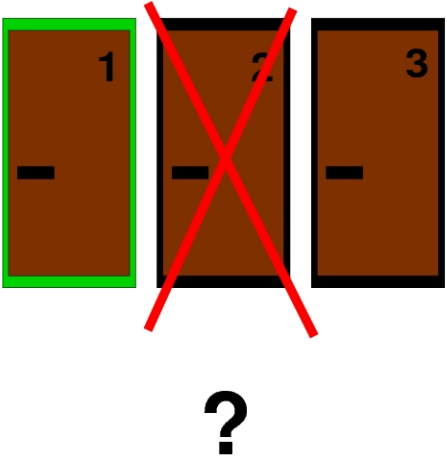

Basics of probability theory

Questions, discussion, remarks

Questions?

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