Algorithms Group **Summer '23**
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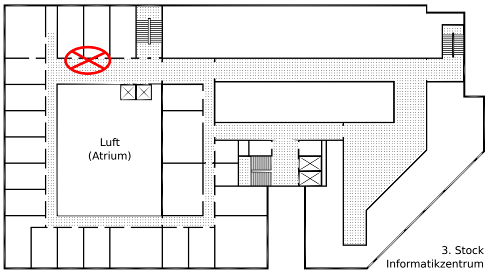
Approximation Algorithms: Homework 5 (6. July)

Solutions are due on July, 19th until 11:00 AM in the cupboard for handing in practice sheets. Please write your name, Matriculation number as well as your Course of study on all pages.



Exercise 1 (Facility Location Problem):
In this problem, we are given:

- a set, C, of clients locations,
- a set, F, of potential facility locations
- distance between any of the above locations
- cost associated with opening each facility

and the goal is open a proper subset of facilities and to assign each client to some opened facility so as to minimize the total facility openning as well as client connection cost. Lets assume that clients and facilities are embedded in a metric space and thus the distance between locations satisfies the triangle inequality. Here is an ILP for the problem:

$$\begin{aligned} \min \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in C} c_{ij} x_{ij} & \\ s.t. \quad \sum_{i \in F} x_{ij} \geq 1, & \quad \forall j \in C \\ y_i \geq x_{ij}, & \quad \forall j \in C,\ \forall i \in F \\ x_{ij} \in \{0,1\}, & \quad \forall j \in C,\ \forall i \in F \\ y_i \in \{0,1\}, & \quad \forall i \in F \end{aligned}$$

where, y_i decides wether to open facility i and $x_{i,j}$ decides whether to serve client j via facility i. Let $n = |C|, m = |F|$, and $B(j,r) = \{i \in F : c_{ij} \leq r\}$ for some $r \geq 0$. Having some fixed radius, r, we say two clients j and j' intersects (at r) if some facility i appeares both in $B(j,r)$ and $B(j',r)$. Now, consider the rounding based idea given in Algorithm 1.

a) Argue that the algorithm correctly assigns each client to an opened facility

b) prove that the total facility opening cost paid by the algorithm's output solution is at most twice the one paid by (x^*, y^*).

c) prove that the total client connection cost paid by the algorithm's output solution is at most 6 times the one paid by (x^*, y^*).

   Thereby leading to a factor 6 approximation algorithm for the facility location problem.

Algorithm 1: LP Rounding For Facility Location Problem

Data: the above ILP above

$(x^*, y^*) \leftarrow$ optimal solution to the LP relaxation of the ILP ;
foreach $j \in C$ **do**
    Compute $\alpha_j = \sum_{i \in F} c_{ij} x^*_{ij}$;
Renumber clients such that: $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$;
for $j = 1$ *to* n **do**
    if j *is already assigned to a facility* **then**
        continue;
    Open cheapest facility i in $B(j, 2\alpha_j)$ and assign j to i;
    for *remaining client* $j' > j$ **do**
        if j' *intersects with* j **then**
            assign j' to i
return list of opened facilities and the client assignment;

(10+20+20 P.)