Classroom Examples of Robustness Problems in Geometric Computations

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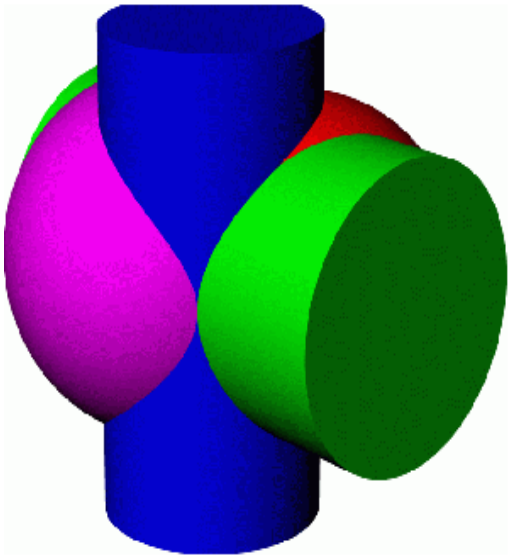
Overview

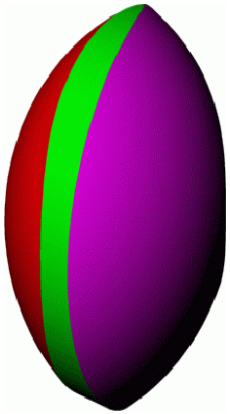
- Overview
- The Orientation Predicate
- A 2D Convex Hull Algorithm

Overview

- The algorithms of computational geometry are designed for a machine model with exact real arithmetic.
- Floating point arithmetic incurs round-off error
- Substituting floating point arithmetic for the assumed real arithmetic may cause implementations to fail.
- I will show you some drastic examples of failures due to roundoff error in the hope
  - That you list more attentively to the alternatives presented later (and which require a fair amount of math)

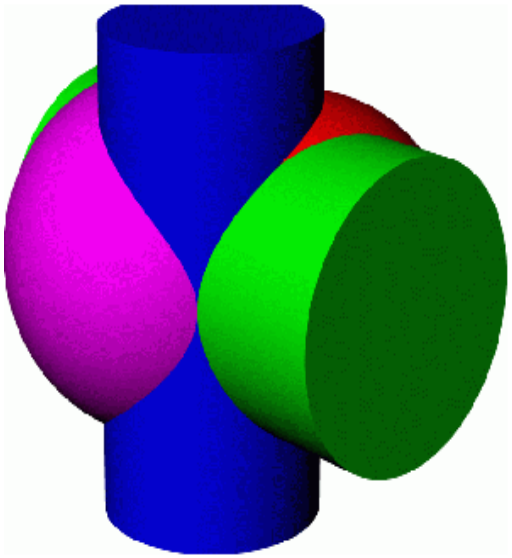
Intersection of Four Simple Solids

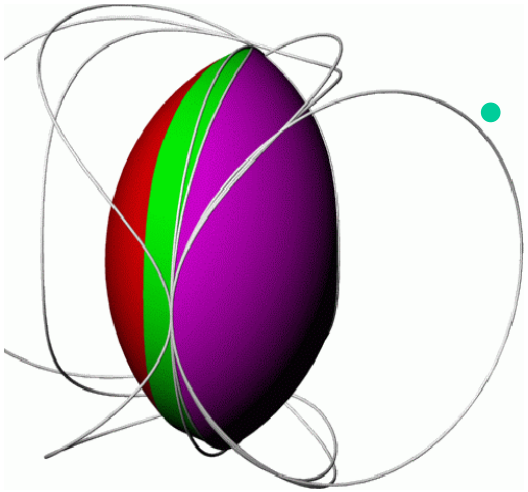




- Rhino3D:
  - $(((s_1 \cap s_2) \cap c_2) \cap c_1) \rightarrow$ successful
  - $(((c_1 \cap c_2) \cap s_1) \cap s_2) \rightarrow$ ``Boolean operation failed''

Intersection of Four Simple Solids





- output is a combinatorial object plus coordinates (not a point set)

- Rhino3D:
  - $(((s_1 \cap s_2) \cap c_2) \cap c_1) \rightarrow$ successful
  - $(((c_1 \cap c_2) \cap s_1) \cap s_2) \rightarrow$ ``Boolean operation failed''

Overview

- …, I believe that the program fails due to floating point rounding errors, but the example is too complicated to be followed through in detail
- We do the following:
  - First study the effect of floating point arithmetic on one of the basic geometric predicates, namely the orientation precidate for three points
  - And then study the global effects on a simple 2d convex hull algorithm.

2D-Orientation of Three Points

$$\text{Orientation}(p, q, r) = \text{sign} \begin{vmatrix} p_x & p_y & 1 \\ q_x & q_y & 1 \\ r_x & r_y & 1 \end{vmatrix}$$

$$\text{Orientation}(p, q, r) = \text{sign}((q_x - p_x)(r_y - p_y) - (q_y - p_y)(r_x - p_x))$$

Implemented with IEEE 784 double precision

→ float_orient (p, q, r)

→ **Float_orient is not a correct implementation of orientation.**

2D-Orientation of Three Points

Orientation(p, q, r) = sign($(q_x\text{-}p_x)(r_y\text{-}p_y) - (q_y\text{-}p_y)(r_x\text{-}p_x)$)

$$p: \begin{pmatrix} 0.5 + x \cdot u \\ 0.5 + y \cdot u \end{pmatrix}$$

$$q: \begin{pmatrix} 12 \\ 12 \end{pmatrix}$$

$$r: \begin{pmatrix} 24 \\ 24 \end{pmatrix}$$

$$0 \le x, y < 256, u = 2^{-53}$$

256x256 pixel image

red=pos., yellow=0, blue=neg.

orientation evaluated with `double`

What do you expect

2D-Orientation of Three Points

Orientation(p, q, r) = sign($(q_x\text{-}p_x)(r_y\text{-}p_y) - (q_y\text{-}p_y)(r_x\text{-}p_x)$)

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$$r: \begin{pmatrix} 24 \\ 24 \end{pmatrix}$$

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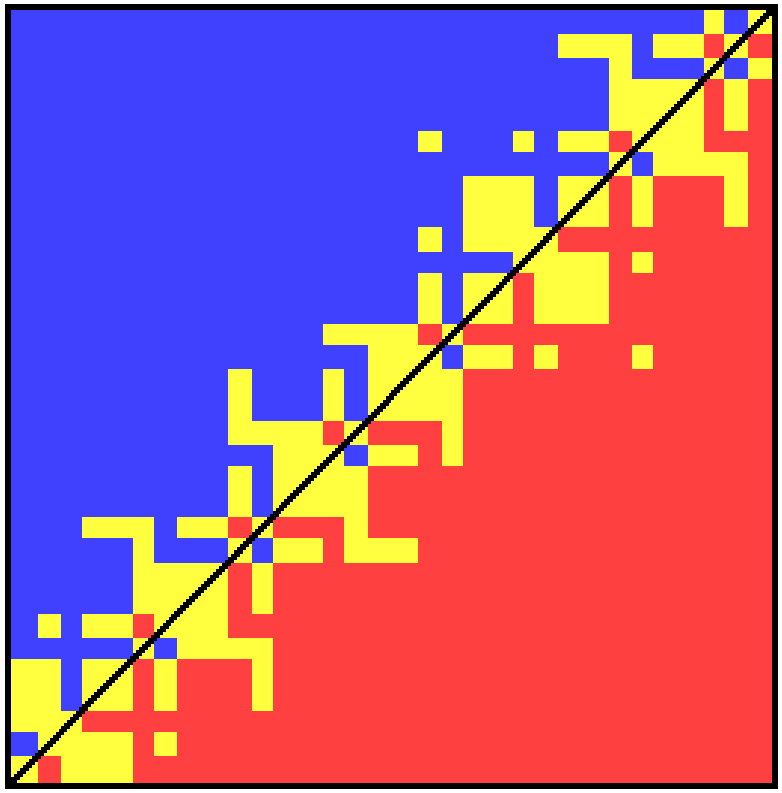

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orientation evaluated with `double`

2D-Orientation of Three Points

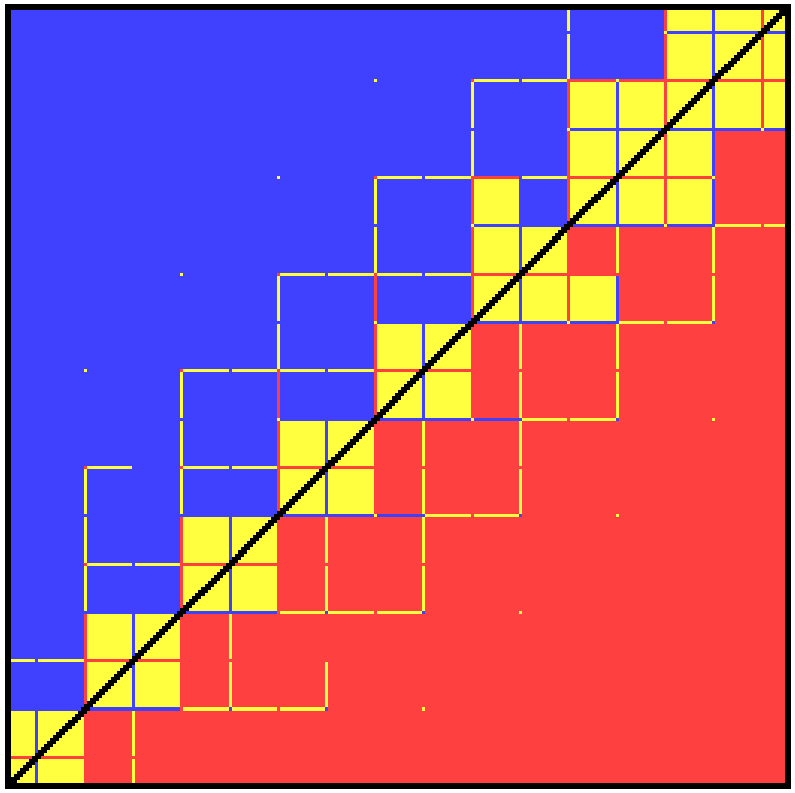
Orientation(p, q, r) = sign($(q_x\text{-}p_x)(r_y\text{-}p_y) - (q_y\text{-}p_y)(r_x\text{-}p_x)$)

$$p: \begin{pmatrix} 0.5 + x \cdot u \\ 0.5 + y \cdot u \end{pmatrix}$$

$$q: \begin{pmatrix} 8.800000000000007 \\ 8.800000000000007 \end{pmatrix}$$

$$r: \begin{pmatrix} 12.1 \\ 12.1 \end{pmatrix}$$

$$0 \le x, y < 256, u = 2^{-53}$$



256x256 pixel image

red=pos., yellow=0, blue=neg.

orientation evaluated with `double`

2D-Orientation of Three Points

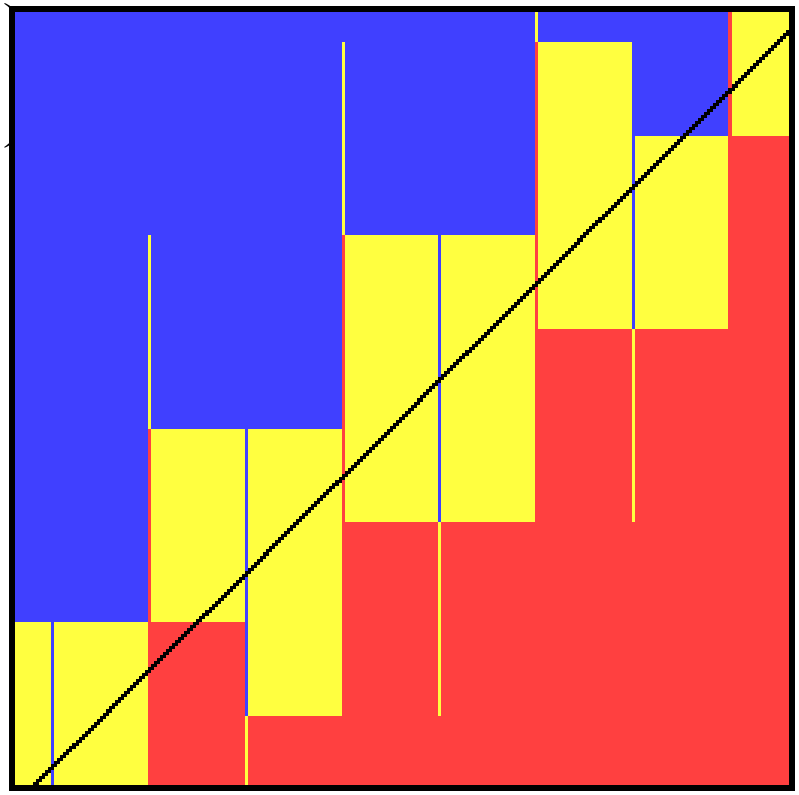
Orientation(p, q, r) = sign($(q_x\text{-}p_x)(r_y\text{-}p_y) - (q_y\text{-}p_y)(r_x\text{-}p_x)$)

$$p: \begin{pmatrix} 0.50000000000002531 + x \cdot u \\ 0.50000000000001710 + y \cdot u \end{pmatrix}$$

$$q: \begin{pmatrix} 17.300000000000001 \\ 17.300000000000001 \end{pmatrix}$$

$$r: \begin{pmatrix} 24.000000000000500000 \\ 24.000000000000517765 \end{pmatrix}$$

$0 \le x, y < 256, u = 2^{-53}$



256x256 pixel image

red=pos., yellow=0, blue=neg.

orientation evaluated with `double`

2D-Orientation of Three Points

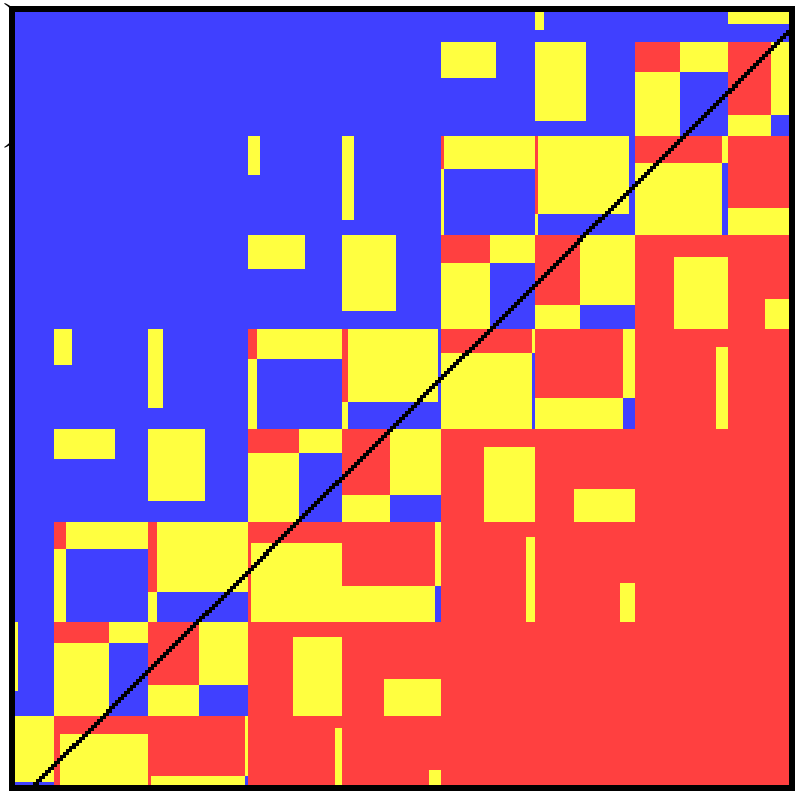
Orientation(p, q, r) = sign($(q_x\text{-}p_x)(r_y\text{-}p_y) - (q_y\text{-}p_y)(r_x\text{-}p_x)$)

$$p: \begin{pmatrix} 0.5000000000\,0002531 + x \cdot u \\ 0.5000000000\,0001710 + y \cdot u \end{pmatrix}$$

$$q: \begin{pmatrix} 17.3000000000\,00001 \\ 17.3000000000\,00001 \end{pmatrix}$$

$$r: \begin{pmatrix} 24.0000000000\,000500000 \\ 24.0000000000\,000517765 \end{pmatrix}$$

$0 \le x, y < 256\,, u = 2^{-53}$

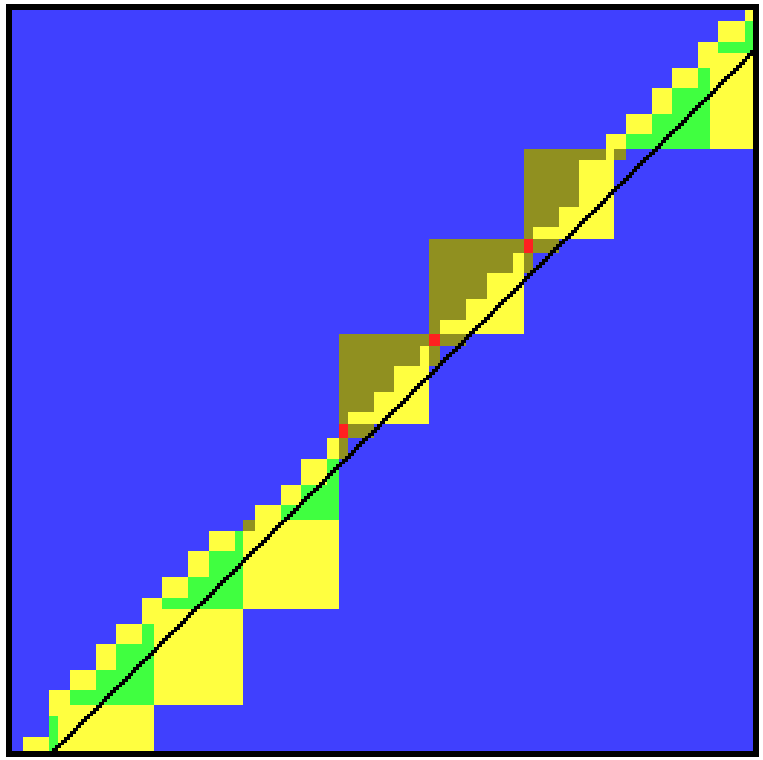


256x256 pixel image

red=pos., yellow=0, blue=neg.

orientation evaluated with **`ext double`**

Classification with respect to two Lines

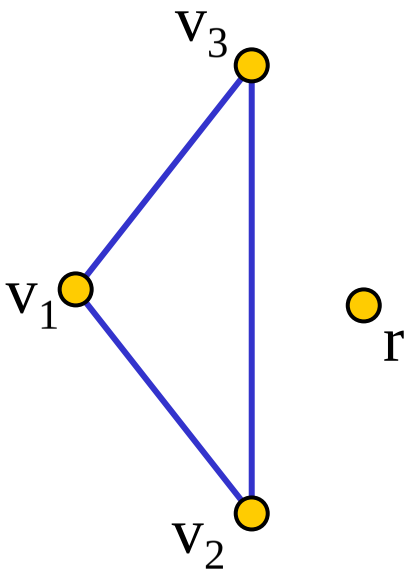


- consider
  - p_1 = (0.5, 0.5)
  - p_2 = (7.3000000000000194, 7.3000000000000167)
  - p_3 = (24.000000000000068, 24.000000000000071)
  - p_4 = (24.00000000000005, 24.000000000000053)
- (p_2, p_3, p_4) form a counter-clockwise triangle

- Classification of ($x(p_1) + x \cdot u$, $y(p_1) + y \cdot u$) with respect to the edges (p_2, p_3) and (p_4,p_2).

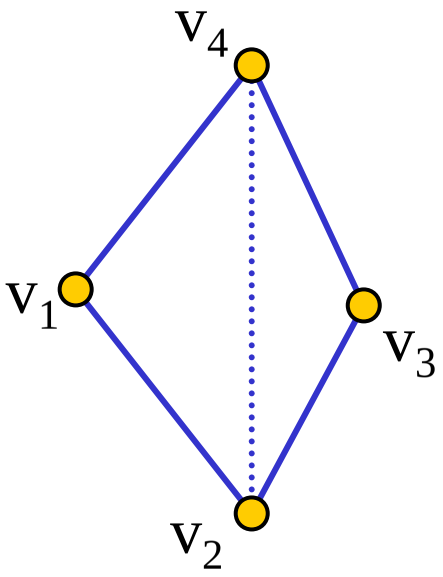
red = sees no edge, ocher = collinear with one, yellow = collinear with both, blue = sees one but not the other, green = sees both

Convex Hulls in the Plane



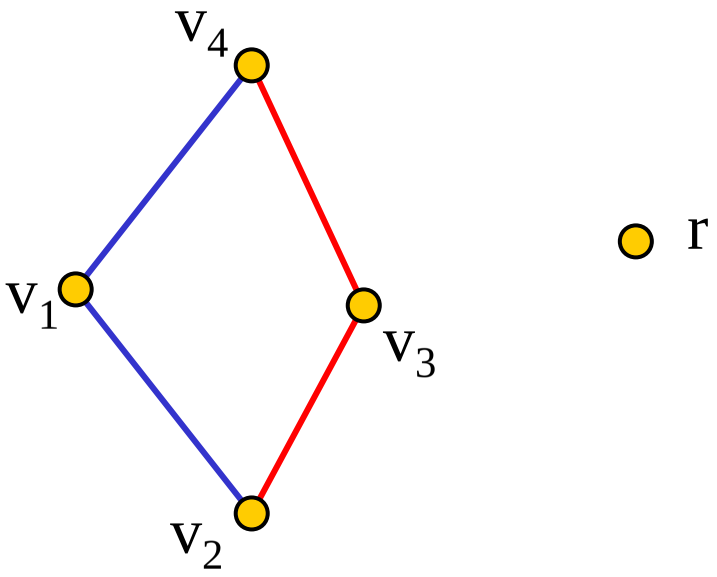
- maintain current hull as a circular list $L=(v_0,v_1,...,v_{k-1})$ of its extreme points in counter-clockwise order
- start with three non-collinear points in S.
- consider the remaining points r one by one.
  - Determine the set of visible edges
  - If none, r is inside and we are done
  - Otherwise, remove all visible edges and add the two tangents from r

Convex Hulls in the Plane



- maintain current hull as a circular list $L=(v_0,v_1,...,v_{k-1})$ of its extreme points in counter-clockwise order
- start with three non-collinear points in S.
- consider the remaining points r one by one.

Correctness Properties



- **[Property A]** A point *r* is outside *CH* iff there is an *i* such that the edge (v_i, v_{i+1}) is visible for *r*. (orientation(v_i,v_{i+1},*r*) > 0)
- **[Property B]** If *r* is outside *CH*, then the set of edges that are weakly visible (= orientation is non-negative) from *r* forms a non-empty consecutive subchain; so does the set of edges that are not weakly visible from *r*.

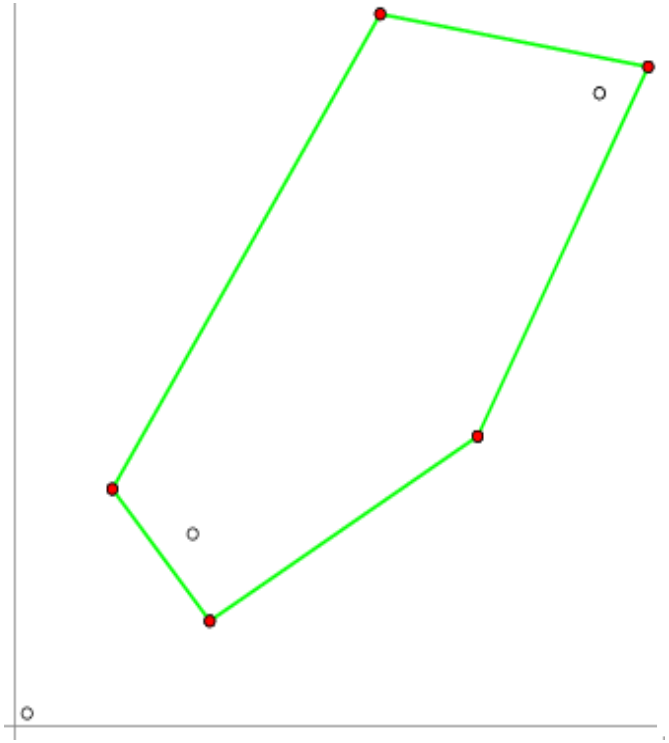
… and next

- Instances leading to violations of properties **(A)** and **(B)** when executed with `double`'s
- and in all possible ways
  - a point outside sees no edge
  - a point inside sees an edge
  - a point outside sees all edges
  - a point outside sees a non-contiguous set of edges
- examples involve nearly collinear points, of course
- examples are realistic as many real-life instances contain collinear points (which become nearly collinear by conversion to `double`'s)

Global Consequences

- A point outside sees no edge of the current hull.

p_1=(7.3000000000000194, 7.3000000000000167)
p_2=(24.000000000000068, 24.000000000000071)
p_3=(24.00000000000005, 24.000000000000053)
p_4=(0.50000000000001621, 0.50000000000001243)
p_5=(8, 4) p_6=(4, 9) p_7=(15, 27)
p_8=(26, 25) p_9=(19, 11)



float_orient(p_1,p_2,p_3) > 0
float_orient(p_1,p_2,p_4) < 0
float_orient(p_2,p_3,p_4) < 0
float_orient(p_3,p_1,p_4) < 0

Global Consequences

- A point outside sees all edges of the current hull.

p_1=(200, 49.200000000000003)
p_2=(100, 49.600000000000001)
p_3=(-233.33333333333334, 50.93333333333333)
p_4=(166.66666666666669, 49.333333333333336)

float_orient(p_1,p_2,p_3) > 0
float_orient(p_1,p_2,p_4) > 0
float_orient(p_2,p_3,p_4) > 0
float_orient(p_3,p_1,p_4) > 0

Global Consequences

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float_orient(p_1,p_2,p_3) > 0
float_orient(p_1,p_2,p_4) > 0
float_orient(p_2,p_3,p_4) > 0
float_orient(p_3,p_1,p_4) > 0

Depending on the implementation:

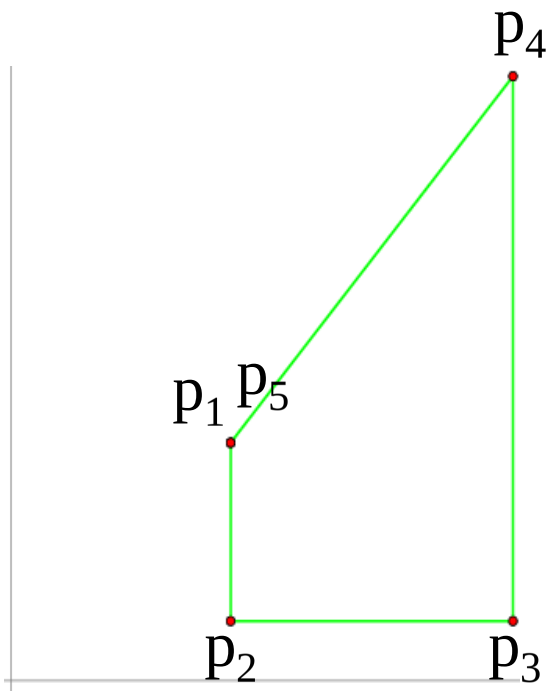
- **Algorithm does not terminate!**
- **Algorithm crashes!**

Global Consequences

- A point inside sees an edge of the current hull.

p_1=(24.00000000000005, 24.000000000000053)
p_2=(24.0, 6.0)
p_3=(54.85, 6.0)
p_4= (54.850000000000357, 61.000000000000121)
p_5=(24.000000000000068, 24.000000000000071)

(p_1,p_2,p_3,p_4) form a convex quadrilateral
p5 is truly inside this quadrilateral, but
float_orient(p_4,p_1,p_5) > 0



Global Consequences

- A point inside sees an edge of the

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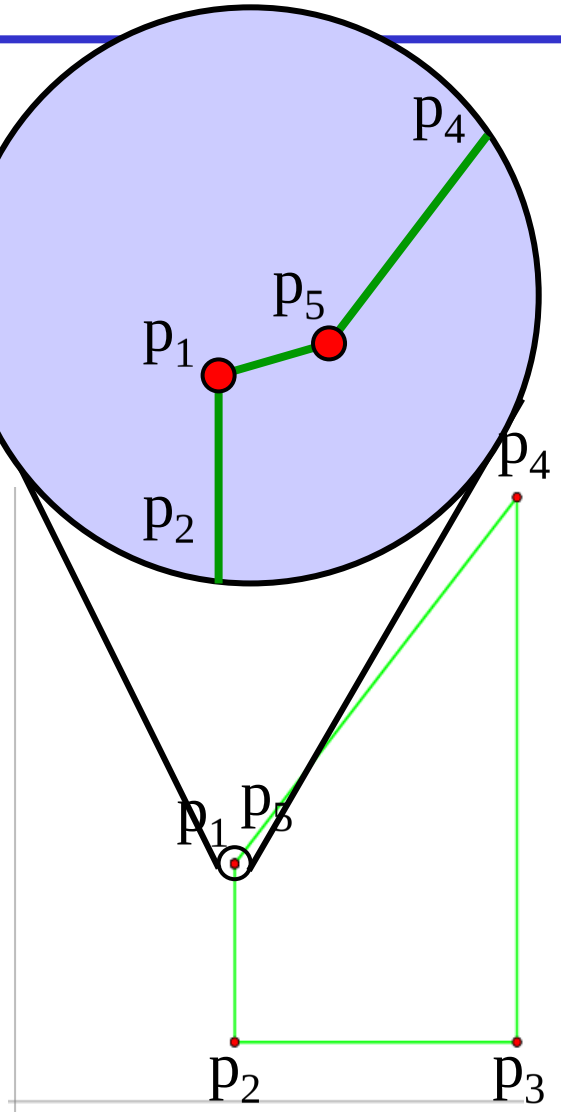
p_2=(24.0, 6.0)
p_3=(54.85, 6.0)
p_4= (54.850000000000357, 61.000000000000121)
p_5=(24.000000000000068, 24.000000000000071)

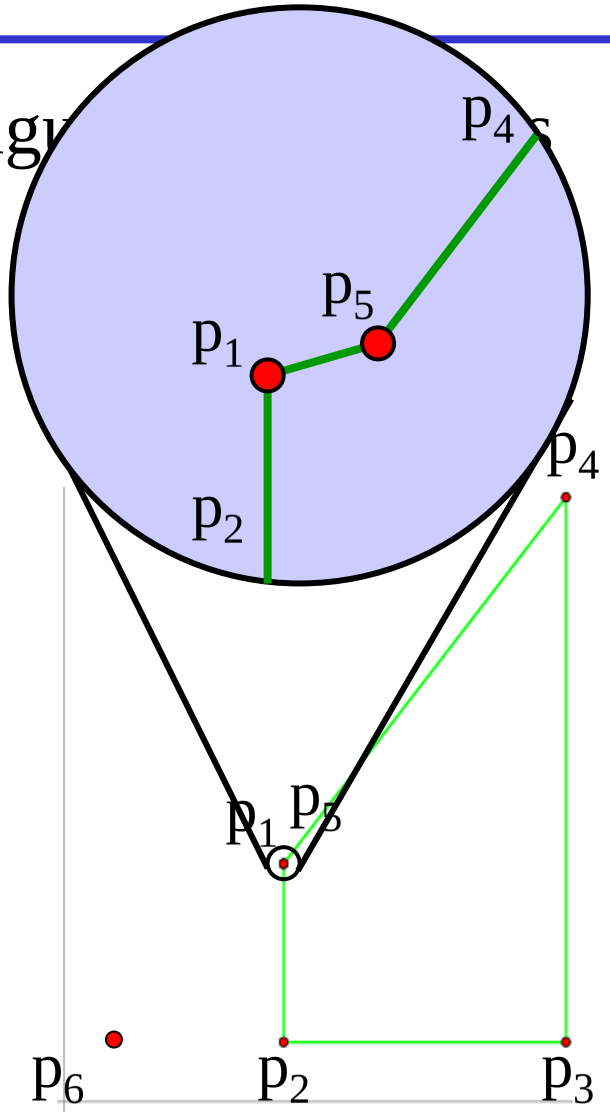
(p_1,p_2,p_3,p_4) form a convex quadrilateral
p5 is truly inside this quadrilateral, but
float_orient(p_4,p_1,p_5) > 0



Global Consequences

- A point outside sees a non-contigu

p_1=(24.00000000000005, 24.000000000000053)
p_2=(24.0, 6.0)
p_3=(54.85, 6.0)
p_4= (54.850000000000357, 61.000000000000121)
p_5=(24.000000000000068, 24.000000000000071)
p_6=(6, 6)



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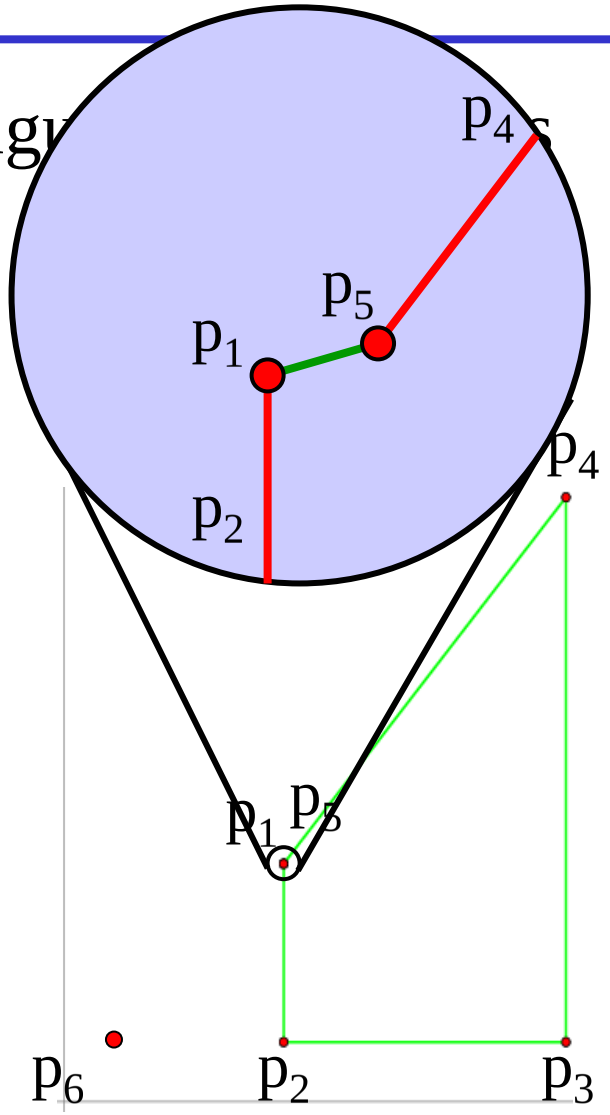
p_3=(54.85, 6.0)
p_4= (54.850000000000357, 61.000000000000121)
p_5=(24.000000000000068, 24.000000000000071)
p_6=(6, 6)

float_orient(p_4,p_5,p_6) > 0
float_orient(p_5,p_1,p_6) < 0
float_orient(p_1,p_2,p_6) > 0



Global Consequences

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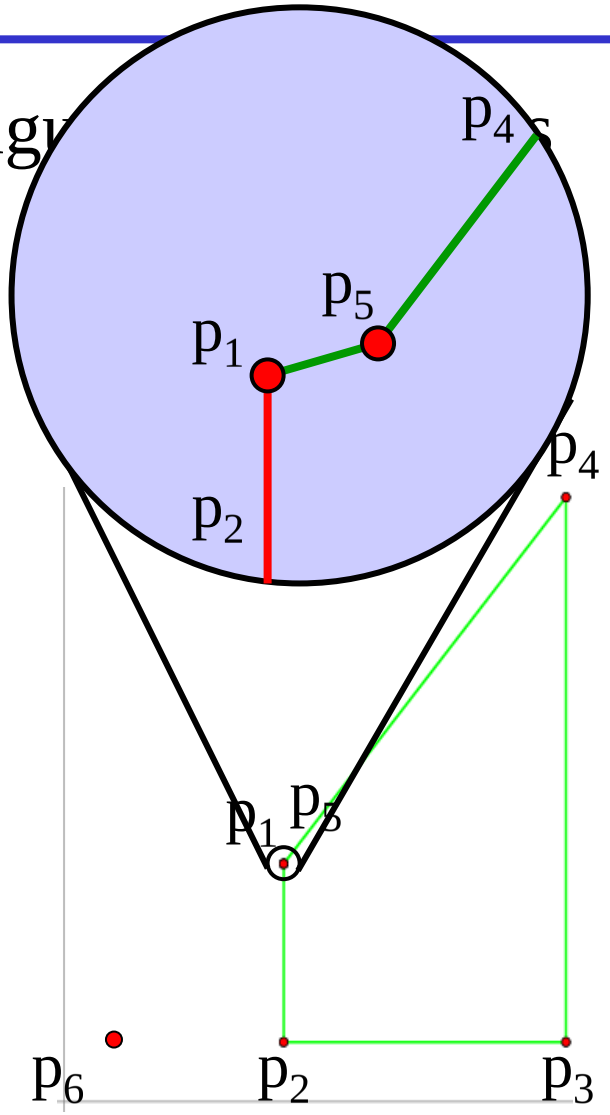
p_1=(24.00000000000005, 24.000000000000053)
p_2=(24.0, **10.0**)
p_3=(54.85, 6.0)
p_4= (54.850000000000357, 61.000000000000121)
p_5=(24.000000000000068, 24.000000000000071)
p_6=(6, 6)



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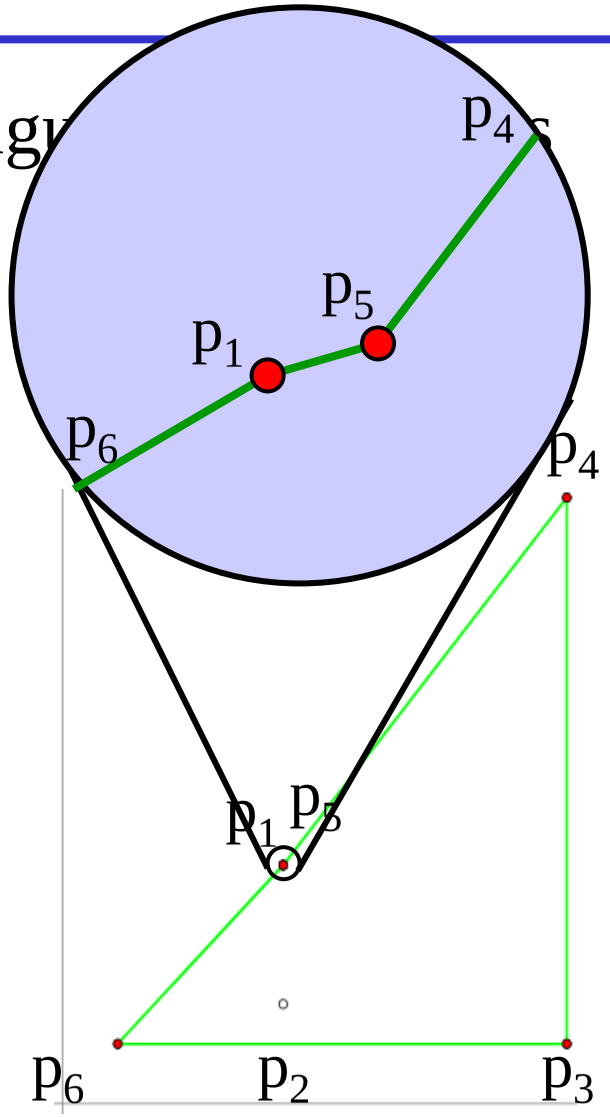

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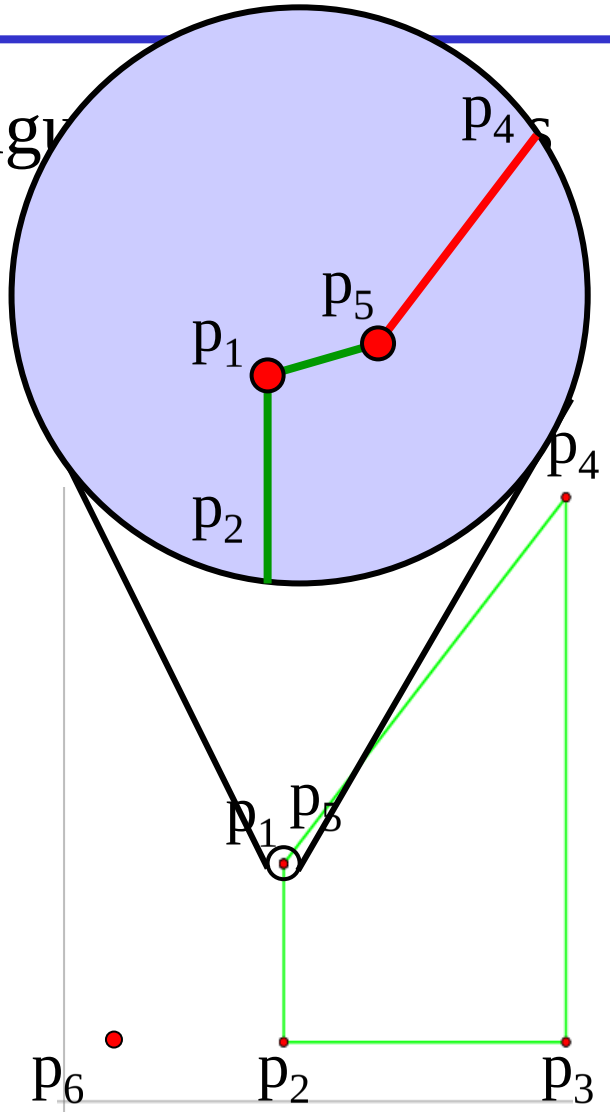
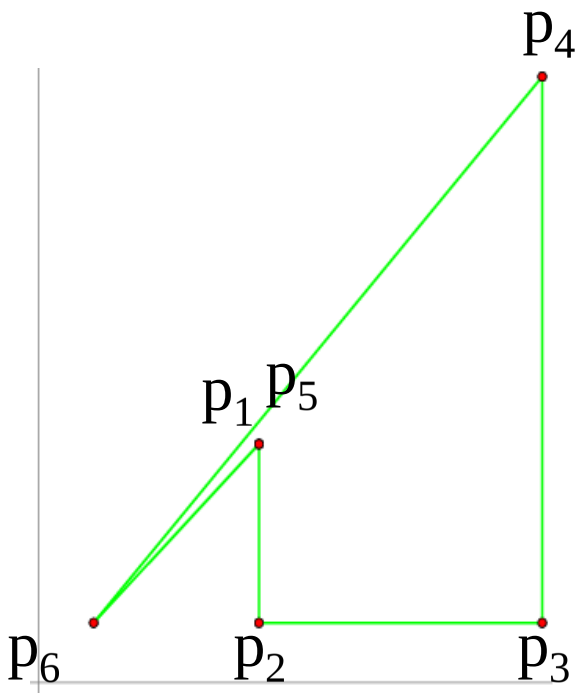
p_6=(6, 6)



Global Consequences

- A point outside sees a non-contiguous set of edges

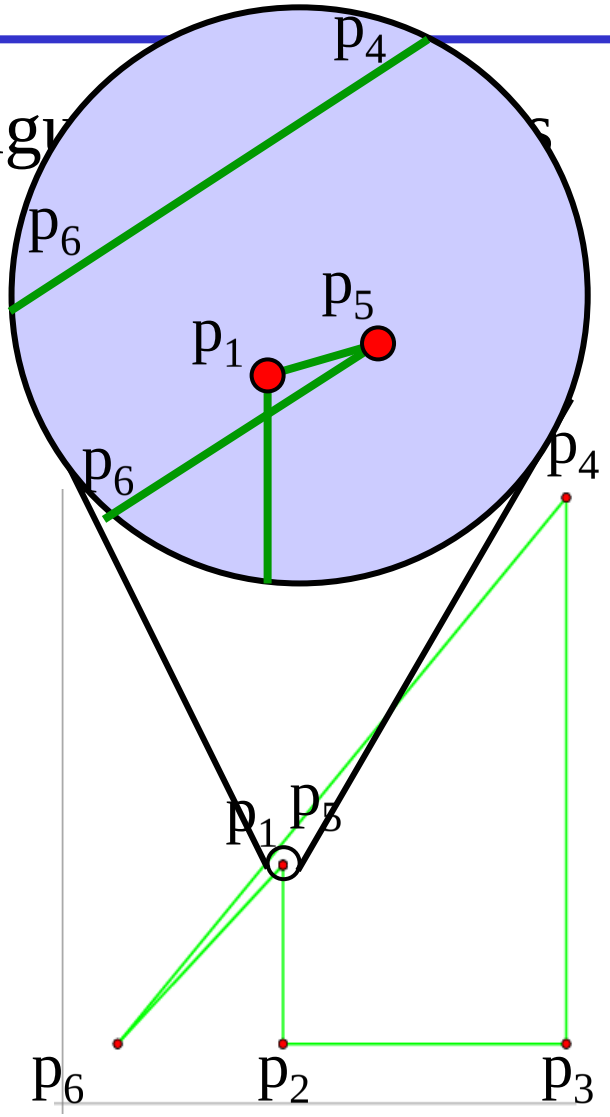
p_1=(24.00000000000005, 24.000000000000053)
p_2=(24.0, **6.0**)
p_3=(54.85, 6.0)
p_4= (54.850000000000357, 61.000000000000121)
p_5=(24.000000000000068, 24.000000000000071)
p_6=(6, 6)



Global Consequences

- A point outside sees a non-contigu[illegible]s

p_1=(24.00000000000005, 24.000000000000053)
p_2=(24.0, **6.0**)
p_3=(54.85, 6.0)
p_4= (54.850000000000357, 61.000000000000121)
p_5=(24.000000000000068, 24.000000000000071)
p_6=(6, 6)



Conclusion

- Classroom examples

  Data sets and C++ programs available online:

  http://www.mpi-sb.mpg.de/~kettner/proj/NonRobust/

- Long version (available soon):
  - More analysis
  - Graham's scan algorithm
  - 3D Delaunay triangulation