

Geometric Algorithms

Smallest Enclosing Disk II

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Sources

- ▶ Emo Welzl

Smallest enclosing disks (balls and ellipsoids)

in New Results and New Trends in Computer Science

Lecture Notes in Computer Science 555 pp. 359-370

Springer-Verlag

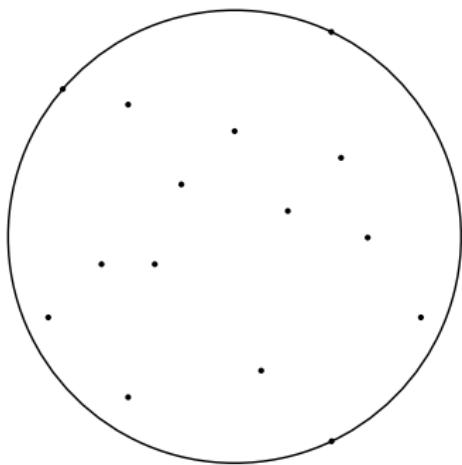
Problem Definition

Given a set P of points:



Problem Definition

Given a set P of points: compute the smallest enclosing disk.



Naming and Special Cases

Naming:

- ▶ P , the set of points
- ▶ $md(P)$, the smallest enclosing disk of P

Special cases:

- ▶ For $P = \emptyset$ set $md(P) = \emptyset$.
- ▶ For $P = \{p\}$ set $md(P) = p$.

Uniqueness

Lemma 1

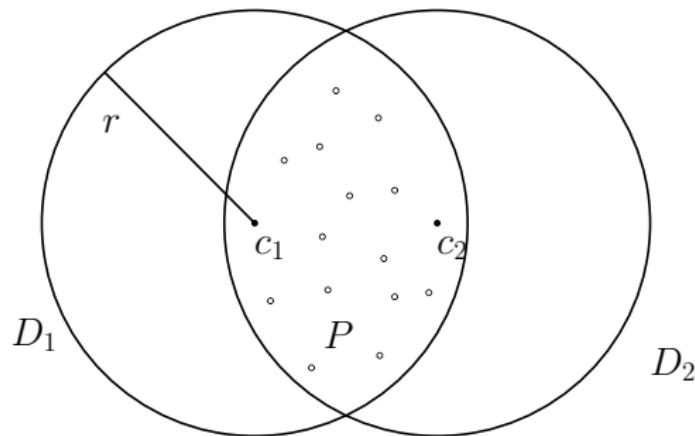
For any point set P , the smallest enclosing disk $md(P)$ is unique.



Uniqueness

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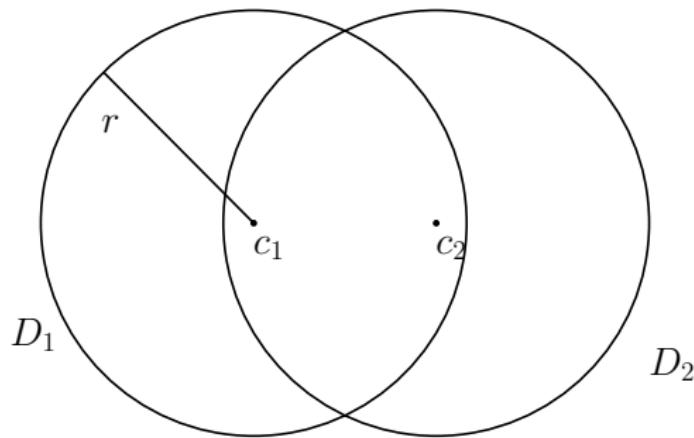
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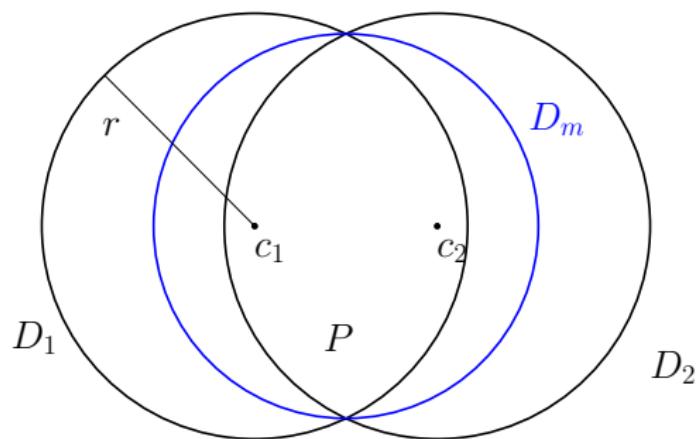
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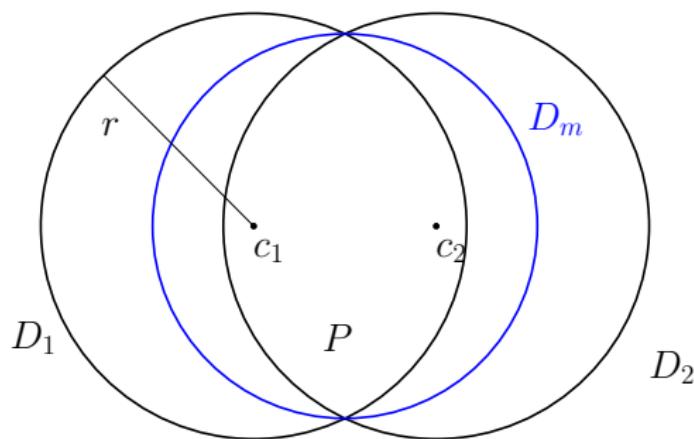
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Lemma 1

For any point set P , the smallest enclosing disk $md(P)$ is unique.



It follows that the problem is well defined for $P \neq \emptyset$.

Minimum Disk with Boundary Constraints

Definition 2

Let P and R be finite point sets in \mathbb{R}^2 , $P \cup R \neq \emptyset$. Then $md_b(P, B)$ is the smallest enclosing disk of $P \cup R$ with $R \subset \partial md_b(P, R)$ if it exists.

Obviously:

- ▶ $md_b(P, \emptyset) = md(P)$
- ▶ $md_b(P \cup R, \emptyset) \subset md_b(P, R)$

Algorithm

Algorithm 1 Function mindisk(P)

1: return mindisk_b(P, \emptyset);

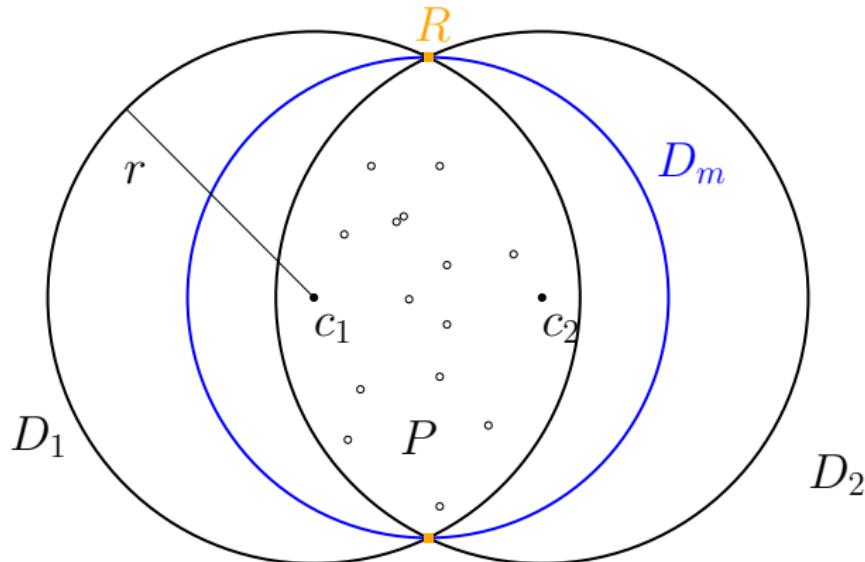
Algorithm 2 Function mindisk_b(P, R)

1: **if** $P = \emptyset$ **then**
2: $D := md_b(\emptyset, R)$;
3: **else**
4: choose random $p \in P$;
5: $D := \text{mindisk_b}(P - \{p\}, R)$;
6: **if** $p \notin D$ **then**
7: $D := \text{mindisk_b}(P - \{p\}, R \cup \{p\})$;
8: **end if**
9: **end if**
10: return D ;

Well defined

Lemma 3

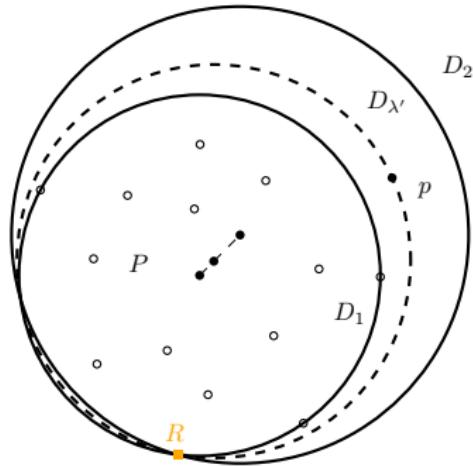
If there exists a disk containing P with R on its boundary, then $md_b(P, R)$ is well defined.



md_b - Point on Boundary

Lemma 4

Provided $md_b(P, R)$ exists and $p \in P$ with
 $p \notin D_1 = md_b(P - \{p\}, R)$, then:
 $md_b(P, R) = md_b(P - \{p\}, R \cup \{p\})$



At most three points required

Lemma 5

*Provided $md_b(P, R)$ exists, there is $S \subset P$ with
 $|S| \leq \max\{0, 3 - |R|\}$ such that $md_b(P, R) = md_b(S, R)$*

Proof.

Obvious since a disk is defined by at most 3 points on the boundary. □

(Exercise)

Improved Algorithm

Algorithm 3 Function mindisk_b(P, R)

```
1: if  $P = \emptyset$  or  $|R| = 3$  then
2:    $D := md_b(\emptyset, R);$ 
3: else
4:   choose random  $p \in P;$ 
5:    $D := \text{mindisk\_b}(P - \{p\}, R);$ 
6:   if  $p \notin D$  then
7:      $D := \text{mindisk\_b}(P - \{p\}, R \cup \{p\});$ 
8:   end if
9: end if
10: return  $D;$ 
```

Complexity

Let $t_j(n)$ the expected number of calls of $p \notin D$ in `mindisk_b(P, R)` for $|P| = n$ and $|R| = 3 - j$, then

We would like to know $t_3(n)$.

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- ▶ $t_0(n) = 0$ since $|R| = 3$
- ▶ $t_j(0) = 0$ since $P = \emptyset$
- ▶ $t_j(n) \leq t_j(n-1) + 1 + \frac{j}{n}t_{j-1}(n-1)$ for $0 < j \leq 3$,
where $\frac{j}{n} = \text{Prob}(p \notin (P - \{p\}, R))$

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We get the following **expected** run times:

- ▶ $t_1(n) \leq t_0(n-1) + 1 = n$
- ▶ $t_2(n) \leq t_2(n-1) + 1 + \frac{2}{n}t_1(n-1) = 3n$
- ▶ $t_3(n) \leq t_3(n-1) + 1 + \frac{3}{n}t_2(n-1) = 10n$

Formulation with one Permutation

Algorithm 4 Function $\text{mindisk}(P)$ – P an ordered sequence

- 1: Compute random permutation π for $1 \dots |P|$
 - 2: return $\text{mindisk_b}(\pi(P), \emptyset)$;
-

Algorithm 5 Function $\text{mindisk_b}(P, R)$ – P an ordered sequence

- 1: **if** $P = \emptyset$ or $|R| = 3$ **then**
 - 2: $D := \text{md}_b(\emptyset, R)$;
 - 3: **else**
 - 4: $p := \text{last}(P)$;
 - 5: $D := \text{mindisk_b}(P - \{p\}, R)$;
 - 6: **if** $p \notin D$ **then**
 - 7: $D := \text{mindisk_b}(P - \{p\}, R \cup \{p\})$;
 - 8: **end if**
 - 9: **end if**
 - 10: return D ;
-

Complexity Analysis on Permutation

- ▶ For sequence P , let $T(P, R)$ be the cost of *mindisk_b*(P, R).
- ▶ Let $t_j(n)$ be the expected value of $T(\pi(P), R)$ over all possible insertion sequences $\pi \in S_n$, where $j = \delta - |R|$.
- ▶ Obviously $t_0(n) = 0$ and $t_j(0) = 0$ remain.
- ▶ We want to know:

$$t_3(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), \emptyset)$$

- ▶ Or in general:

$$t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R), \text{ with } |R| = \delta - j.$$

Complexity Analysis on Permutation – Continued

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$$\begin{aligned} t_j(n) &= \frac{1}{n} \sum_{p \in P} [1 \\ &+ \frac{1}{(n-1)!} \sum_{\substack{\pi \in S_{n-1} \\ p = \pi(P)[n]}} [T(\pi(P) - \{p\}, R) \\ &+ \chi(p \notin mdb(P - \{p\}, R)) \cdot T(\pi(P) - \{p\}, R \cup \{p\})]] \end{aligned}$$

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$$t_j(n) \leq 1 + t_j(n-1) + \frac{j}{n} \cdot t_{j-1}(n-1), \text{ which we know.}$$

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Thus, as before $t_3(n) = 10n$.

Summary

Algorithm:

- ▶ algorithm for computing smallest enclosing disk
- ▶ expected $O(n)$ time
- ▶ $O(n)$ space
- ▶ extendable to higher dimensions

Technique: Randomized Incremental Construction (RIC)

- ▶ Usually easy to implement
- ▶ Complexity analysis may be more tricky
- ▶ Backward Analysis