# Geometric Algorithms

## Smallest Enclosing Disk

Michael Hemmer

April 15, 2014

# Sources

- Emo Welzl
  *Smallest enclosing disks (balls and ellipsoids)*
  in New Results and New Trends in Computer Science
  Lecture Notes in Computer Science 555 pp. 359-370
  Springer-Verlag

# Problem Definition

Given a set $P$ of points:

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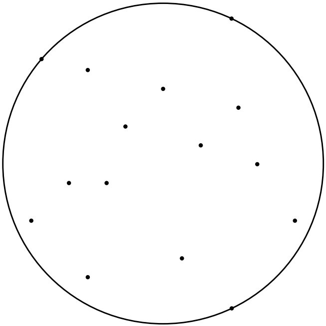
Given a set $P$ of points: compute the smallest enclosing disk.



# Naming and Special Cases

Naming:

- $P$, the set of points
- $md(P)$, the smallest enclosing disk of $P$

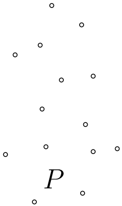
Special cases:

- For $P = \emptyset$ set $md(P) = \emptyset$.
- For $P = \{p\}$ set $md(P) = p$.

# Uniqueness

**Lemma 1**

*For any point set $P$, the smallest enclosing disk $md(P)$ is unique.*



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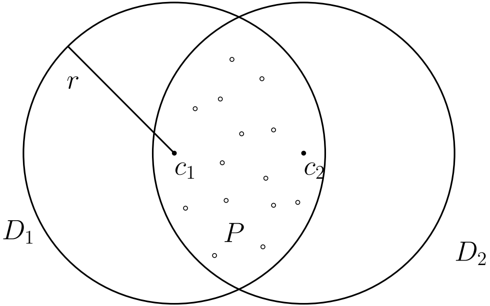

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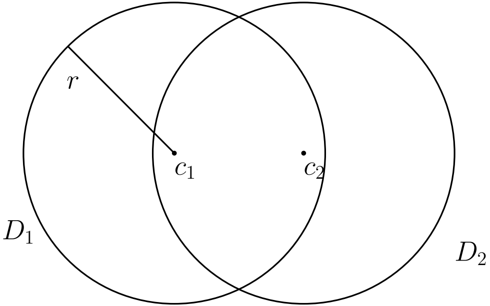

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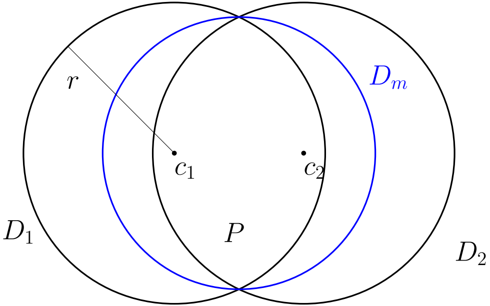

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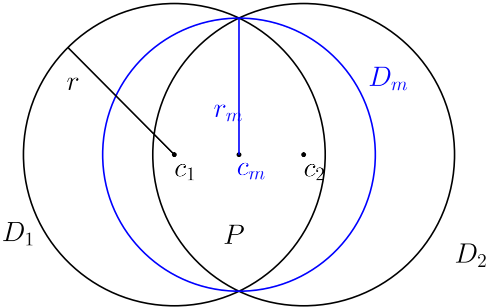

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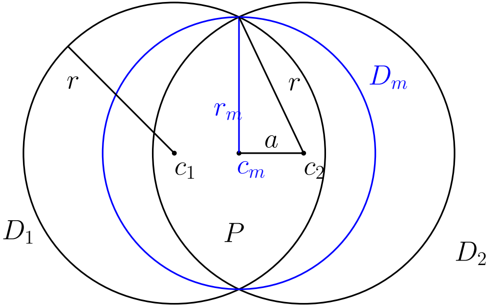

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*For any point set $P$, the smallest enclosing disk $md(P)$ is unique.*

## Proof.

Suppose there are two different smallest enclosing disks $D_1 = (c_1, r)$ and $D_2 = (c_2, r)$, with $P \subset D_1$ and $P \subset D_2$.

The disk $D_m$ with center $(c_1 + c_2)/2$ and radius $sqrt(r^2 - a^2)$, where $a$ is half the distance of $c_1$ and $c_2$, also contains $P$.

Contradiction, since the radius of $D_m$ is smaller. $\square$

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It follows that the problem is well defined for $P \neq \emptyset$.

# Algorithmic Ideas?

Brain Storming :)

# Algorithm

**Algorithm 1** Function mindisk($P$)

```
if P = ∅ then
    D := ∅;
else
    choose random p ∈ P;
    D := mindisk(P − {p});
    if p ∉ D then
        D := mindisk_b(P − {p},p);
    end if
end if
return D;
```

# Sketch Complexity Analysis

- Assume that cost for `mindisk_b`$(A, p)$ costs $c|A|$.
- Cost for `mindisk(`$P$`)` is:

$$t(|P|) \quad = \quad t(|P|-1) + 1 + c(|P|-1)Prob(p \notin md(P - \{p\}))$$

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&\leq t(|P|-1) + 1 + 3c \\
&\leq (1+3c)n
\end{aligned}
$$

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# Minimum Disk with Boundary Constraints

### Definition 2

Let $P$ and $R$ be finite point sets in $\mathbb{R}^2$, $P \cup R \neq \emptyset$. Then $md_b(P, B)$ is the smallest enclosing disk of $P \cup R$ with $R \subset \partial md_b(P, R)$ if it exists.

Obviously:

- $md_b(P, \emptyset) = md(P)$
- $md_b(P \cup R, \emptyset) \subset md_b(P, R)$

# Algorithm

**Algorithm 2** Function mindisk_b($P$,$R$)

**if** $P = \emptyset$ **then**
    $D := md_b(\emptyset, R)$;
**else**
    choose random $p \in P$;
    $D :=$ mindisk_b$(P - \{p\}, R)$;
    **if** $p \notin D$ **then**
        $D :=$ mindisk_b$(P - \{p\}, R \cup \{p\})$;
    **end if**
**end if**
return $D$;

**Algorithm 3** Function mindisk($P$)

return mindisk_b$(P, \emptyset)$;

# Algebraic Formulation

### Definition 3 (Algebraic Formulation)

A disk $D(q, r)$ can be define via function

$$f(p) = 1/r^2 \cdot ||p - q||^2,$$

that is:

$$p \in D(q, r) \Leftrightarrow f(p) \leq 1$$

$$p \in \partial D(q, r) \Leftrightarrow f(p) = 1$$

# Convex Combination of Disks

### Definition 4 (Convex Combination)

For two disks $D_1 = D(q_1, r_1)$ and $D_2 = D(q_2, r_2)$ define disk $D_\lambda$ for $\lambda \in [0, 1]$ via function:

$$f_\lambda(p) = \lambda f_1(p) + (1 - \lambda) f_2(p) \leq 1$$

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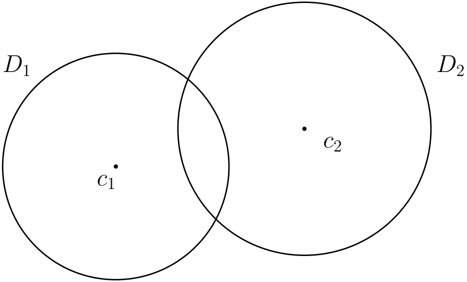
- $D_1 \cap D_2 \subset D_\lambda$
- $\partial D_1 \cap \partial D_2 \subset \partial D_\lambda$
- $D_\lambda$ is a disk
- $r_\lambda$ is smaller than $\max(r_1, r_2)$



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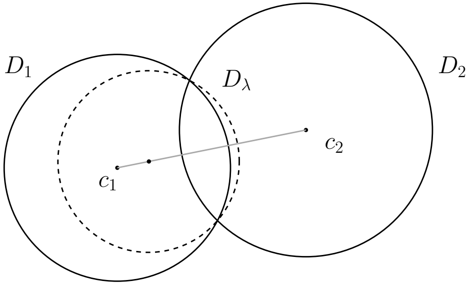

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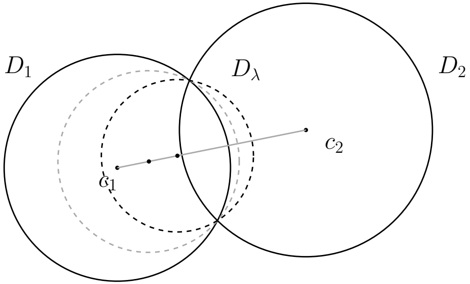

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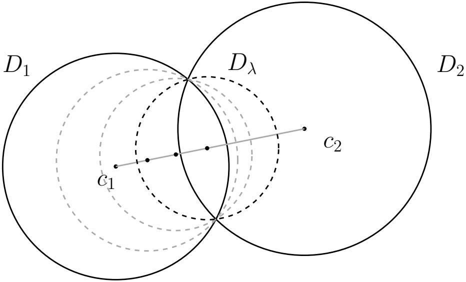

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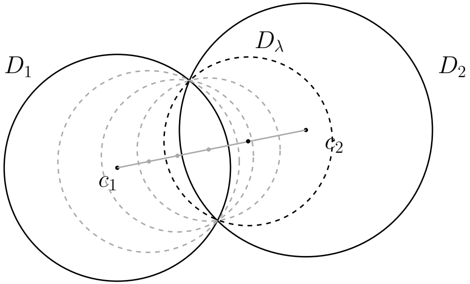

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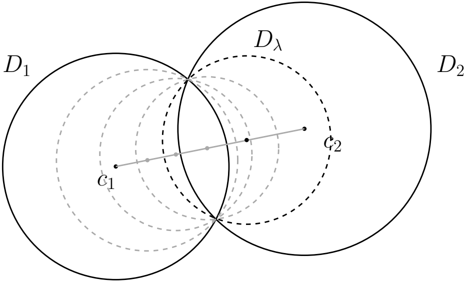



Proof on board or exercise ;)

# $md_b(P, R)$ is well defined

## Lemma 5

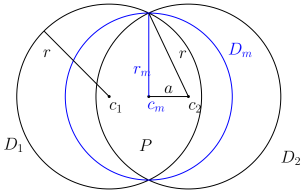
*If there exists a disk containing $P$ with $R$ on its boundary, then $md_b(P, R)$ is well defined.*

## Proof.

Suppose there are two discs $D_1$ and $D_2$ with same radius that contain $P$ and with $R$ on boundary.

Consider $D_\lambda$ for $D_1$ and $D_2$, since $R \subset \partial D_1 \cap \partial D_2$ it follows that $R \subset D_\lambda$.

Same argument as Lemma 1 gives $D_{1/2}$, which has smaller radius; contradiction.

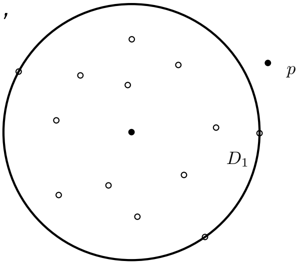



□

# $md_b$ - Point on Boundary

**Lemma 6**

*Provided* $md_b(P, R)$ *exists*
*and*
$p \in P$ *with* $p \notin D_1 = md_b(P - \{p\}, R)$,
*then:*
$md_b(P, R) = md_b(P - \{p\}, R \cup \{p\})$




$\square$

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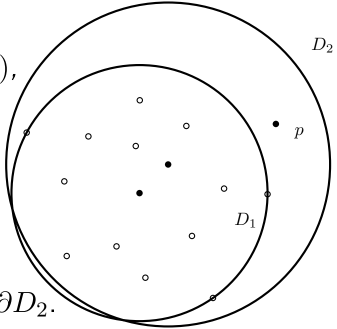


## Proof.

Assume $p \in D_2 = md_b(P, R)$ but $p \notin \partial D_2$.

□

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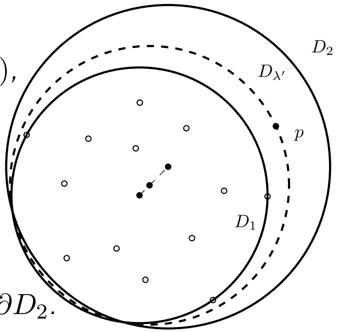


## Proof.

Assume $p \in D_2 = md_b(P, R)$ but $p \notin \partial D_2$.

Consider continues deformation of $D_\lambda$:
There exists a $\lambda' \in (0, 1)$ such that $p \in \partial D_{\lambda'}(D_0, D_1)$.

The radius of $D_\lambda$ is smaller than the one of $D_2$; contradiction. $\square$

# At most three points required

**Lemma 7**

*Provided* $md_b(P, R)$ *exists, there is* $S \subset P$ *with* $|S| \leq \max\{0, 3 - |R|\}$ *such that* $md_b(P, R) = md_b(S, R)$

**Proof.**

Obvious since a disk is defined by at most $3$ points on the boundary. $\square$

(Exercise)

# Improved Algorithm

**Algorithm 4** Function mindisk_b($P$,$R$)

**if** $P = \emptyset$ or $|R| = 3$ **then**
    $D := md_b(\emptyset, R)$;
**else**
    choose random $p \in P$;
    $D :=$ mindisk_b$(P - \{p\}, R)$;
    **if** $p \notin D$ **then**
        $D :=$ mindisk_b$(P - \{p\}, R \cup \{p\})$;
    **end if**
**end if**
return $D$;

# Complexity

Complexity:

- Let $t_j(n)$ the expected number of calls of $p \notin D$ in `mindisk_b(`$P$`,`$R$`)` for $|P| = n$ and $|R| = 3 - j$, then

  We would like to know $t_3(n)$.

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  - $t_0(n) = 0$ since $|R| = 3$
  - $t_j(0) = 0$ since $P = \emptyset$
  - $t_j(n) \leq t_j(n-1) + 1 + \frac{j}{n} t_{j-1}(n-1)$ for $0 < j \leq 3$

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It follows:

- $t_1(n) \leq n$
- $t_2(n) \leq t_2(n-1) + 1 + \frac{2}{n} t_1(n-1)$
- $t_3(n) \leq t_3(n-1) + 1 + \frac{3}{n} t_2(n-1)$

# Complexity

Complexity:

- Let $t_j(n)$ the expected number of calls of $p \notin D$ in `mindisk_b(P,R)` for $|P| = n$ and $|R| = 3 - j$, then

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- $t_3(n) \leq 10n$

# Generalization to smallest enclosing ball in $\mathbb{R}^d$

- Rename function to minball ;)
- Replace constant $3$ by $\delta = d + 1$
- $t_j(n) = nj! \sum_{k=1}^{j} \frac{1}{k!} \leq (e-1)j!n$ (Exercise)

**Theorem 8**

*The smallest enclosing ball of a set of $n$ points in $\mathbb{R}^d$ can be computed in expected time $O(\delta\delta!n)$, where $\delta = d + 1$.*

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**Theorem 8**

*The smallest enclosing ball of a set of $n$ points in $\mathbb{R}^d$ can be computed in expected time $O(\delta\delta!n)$, where $\delta = d + 1$.*

Remark: It is also possible to extend the algorithm to ellipsoids.

# Algorithm for Ball in $\mathbb{R}^d$

**Algorithm 5** Function minball_b($P$,$R$)

**if** $P = \emptyset$ or $|R| = \delta$ **then**
    $D := mb_b(\emptyset, R)$;
**else**
    choose random $p \in P$;
    $D :=$ minball_b$(P - \{p\}, R)$;
    **if** $p \notin D$ **then**
        $D :=$ minball_b$(P - \{p\}, R \cup \{p\})$;
    **end if**
**end if**
return $D$;

# Practical Considerations

- For points in high dimension $d$ the expensive operation is computation of $mb_b(\emptyset, R)$
- Let $s_j(n)$ the expected number of calls of $mb_b(\emptyset, R)$ in `minball_b(`$P$`,`$R$`)` for $|P| = n$ and $|R| = \delta - j$, where $\delta = d + 1$.

# Practical Considerations

- For points in high dimension $d$ the expensive operation is computation of $mb_b(\emptyset, R)$
- Let $s_j(n)$ the expected number of calls of $mb_b(\emptyset, R)$ in `minball_b(`$P$`,`$R$`)` for $|P| = n$ and $|R| = \delta - j$, where $\delta = d + 1$.
  - $s_0(n) = 1$ since $R$ is full
  - $s_j(0) = 1$ since $P = \emptyset$
  - $s_j(n) \leq s_j(n-1) + \frac{j}{n} s_{j-1}(n-1)$ for $0 < j \leq \delta$

# Practical Considerations

- For points in high dimension $d$ the expensive operation is computation of $mb_b(\emptyset, R)$
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- Claim: $s_j(n) \leq (1 + H_n)^j$, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ (Exercise)

# Practical Considerations

- For points in high dimension $d$ the expensive operation is computation of $mb_b(\emptyset, R)$
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- Claim: $s_j(n) \leq (1 + H_n)^j$, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ (Exercise)
- Since $H_n \leq 1 + \ln$, it follows that the number of expected calls to `minball_b` is upper bounded by $(2 + \ln)^\delta$.

# Formulation with one Permutation

**Algorithm 6** Function mindisk($P$) – $P$ an ordered sequence

Compute random permutation $\pi$ for $1 \ldots |P|$

return mindisk_b($\pi(P), \emptyset$);

**Algorithm 7** Function mindisk_b($P$,$R$) – $P$ an ordered sequence

**if** $P = \emptyset$ or $|R| = 3$ **then**

  $D := md_b(\emptyset, R)$;

**else**

  $p := last(P)$;

  $D :=$ mindisk_b($P - \{p\}, R$);

  **if** $p \notin D$ **then**

    $D :=$ mindisk_b($P - \{p\}, R \cup \{p\}$);

  **end if**

**end if**

return $D$;

## Complexity Analysis on Permutation

- For sequence $P$, let $T(P, R)$ be the cost of $mindisk\_b(P, R)$.
- Let $t_j(n)$ be the expected value of $T(P, R)$ over all possible insertion sequences $S_n$, where $j = \delta - |R|$.
- Obviously $t_0(n) = 0$ and $t_j(0) = 0$ remain.
- We want to know:

$$t_3(n) \quad = \quad \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), \emptyset)$$

- Or in general:

$$t_j(n) \quad = \quad \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R), \text{ with } |R| = \delta - j.$$

# Complexity Analysis on Permutation – Continued

$$t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R)$$

# Complexity Analysis on Permutation – Continued

$$
\begin{aligned}
t_j(n) &= \frac{1}{n!}\sum_{\pi\in S_n} T(\pi(P),R) \\
t_j(n) &= \frac{1}{n}\sum_{p\in P}\Big[1 \\
&+ \frac{1}{(n-1)!}\sum_{\substack{\pi\in S_n\\ p=\pi(P)[n]}}\Big[\,T(\pi(P)-\{p\},R) \\
&+ \chi(p\notin md_b(P-\{p\},R))\cdot T(\pi(P)-\{p\},R\cup\{p\})\,\Big]\,\Big]
\end{aligned}
$$

# Complexity Analysis on Permutation – Continued

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t_j(n) &= \frac{1}{n!}\sum_{\pi \in S_n} T(\pi(P), R) \\
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t_j(n) &= \frac{1}{n!}\sum_{\pi \in S_n} T(\pi(P), R) \\
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t_j(n) &= \frac{1}{n}\sum_{p\in P} [\ 1 \\
&+ \frac{1}{(n-1)!}\sum_{\sigma\in S_{n-1}} T(\sigma(P-\{p\}),R) \\
&+ \chi(p\notin md_b(P-\{p\},R))\cdot\frac{1}{(n-1)!}\sum_{\sigma\in S_{n-1}} T(\sigma(P-\{p\}),R\cup\{p\})\ ]
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&+ t_j(n-1) \\
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t_j(n) &= \frac{n}{n} \\
&+ \frac{1}{n} \sum_{p \in P} t_j(n-1) \\
&+ \frac{1}{n} \sum_{p \in P} \chi(p \notin md_b(P - \{p\}, R)) \cdot t_{j-1}(n-1)
\end{aligned}
$$

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t_j(n) &= \frac{n}{n} \\
&+ \frac{1}{n}\sum_{p \in P} t_j(n-1) \\
&+ \frac{1}{n}\sum_{p \in P} \chi(p \notin md_b(P - \{p\}, R)) \cdot t_{j-1}(n-1) \\
t_j(n) &\leq 1 \\
&+ \frac{n}{n} t_j(n-1) \\
&+ \frac{3}{n} \cdot t_{j-1}(n-1)
\end{aligned}
$$

# Complexity Analysis on Permutation – Continued

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t_j(n) &\leq 1 + t_j(n-1) + \frac{j}{n}\cdot t_{j-1}(n-1), \text{ which we know.}
\end{aligned}
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# Complexity Analysis on Permutation – Continued

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t_j(n) &= \frac{1}{n!}\sum_{\pi \in S_n} T(\pi(P), R) \\
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\end{aligned}
$$

Thus, as before $t_3(n) = 10n$.

# Summary

Algorithm:

- algorithm for computing smallest enclosing disk
- expected $O(n)$ time
- $O(n)$ space
- extendable to higher dimensions

Technique: Randomized Incremental Construction (RIC)

- Usually easy to implement
- Complexity analysis may be more tricky
- Backward Analysis