Geometric Algorithms
Smallest Enclosing Disk

Michael Hemmer

April 15, 2014
Sources

Emo Welzl

*Smallest enclosing disks (balls and ellipsoids)*

in New Results and New Trends in Computer Science
Lecture Notes in Computer Science 555 pp. 359-370
Springer-Verlag
Problem Definition

Given a set $P$ of points:
Problem Definition

Given a set $P$ of points: compute the smallest enclosing disk.
Naming and Special Cases

Naming:

- $P$, the set of points
- $md(P)$, the smallest enclosing disk of $P$

Special cases:

- For $P = \emptyset$ set $md(P) = \emptyset$.
- For $P = \{p\}$ set $md(P) = p$. 
Lemma 1

For any point set $P$, the smallest enclosing disk $md(P)$ is unique.
Lemma 1

For any point set $P$, the smallest enclosing disk $md(P)$ is unique.
Uniqueness

Lemma 1

For any point set $P$, the smallest enclosing disk $md(P)$ is unique.
Uniqueness

Lemma 1
For any point set $P$, the smallest enclosing disk $md(P)$ is unique.
Lemma 1

For any point set $P$, the smallest enclosing disk $md(P)$ is unique.
Uniqueness

**Lemma 1**

For any point set $P$, the smallest enclosing disk $md(P)$ is unique.
Uniqueness

Lemma 1
For any point set $P$, the smallest enclosing disk $md(P)$ is unique.

Proof.
Suppose there are two different smallest enclosing disks $D_1 = (c_1, r)$ and $D_2 = (c_2, r)$, with $P \subset D_1$ and $P \subset D_2$.

The disk $D_m$ with center $(c_1 + c_2)/2$ and radius $\sqrt{r^2 - a^2}$, where $a$ is half the distance of $c_1$ and $c_2$, also contains $P$.

Contradiction, since the radius of $D_m$ is smaller. \qed
Lemma 1
For any point set $P$, the smallest enclosing disk $md(P)$ is unique.

Proof.
Suppose there are two different smallest enclosing disks $D_1 = (c_1, r)$ and $D_2 = (c_2, r)$, with $P \subset D_1$ and $P \subset D_2$.

The disk $D_m$ with center $(c_1 + c_2)/2$ and radius $\sqrt{r^2 - a^2}$, where $a$ is half the distance of $c_1$ and $c_2$, also contains $P$.

Contradiction, since the radius of $D_m$ is smaller.

It follows that the problem is well defined for $P \neq \emptyset$. 
Algorithmic Ideas?

Brain Storming :)
Algorithm

Algorithm 1 Function mindisk(P)

1: if $P = \emptyset$ then
2:     $D := \emptyset$;
3: else
4:     choose random $p \in P$;
5:     $D := \text{mindisk}(P - \{p\})$;
6:     if $p \notin D$ then
7:         $D := \text{mindisk}_b(P - \{p\}, p)$;
8:     end if
9: end if
10: return $D$;
Sketch Complexity Analysis

- Assume that cost for \( \text{mindisk}_b(A, p) \) costs \( c|A| \).
- Cost for \( \text{mindisk}(P) \) is:

\[
t(|P|) = t(|P| - 1) + 1 + c(|P| - 1) \text{Prob}(p \not\in \text{md}(P - \{p\}))
\]
Sketch Complexity Analysis

- Assume that cost for $\text{mindisk}_b(A, p)$ costs $c|A|$.
- Cost for $\text{mindisk}(P)$ is:

$$t(|P|) = t(|P| - 1) + 1 + c(|P| - 1) \cdot \text{Prob}(p \notin \text{md}(P - \{p\}))$$

$$\text{Prob}(p \notin P - \{p\}) \leq \frac{3}{|P|} \quad \text{← Backward Analysis!!}$$
Sketch Complexity Analysis

- Assume that cost for $\text{mindisk}_b(A, p)$ costs $c|A|$.
- Cost for $\text{mindisk}(P)$ is:

\[
t(|P|) = t(|P| - 1) + 1 + c(|P| - 1) \text{Prob}(p \not\in \text{md}(P - \{p\})) \\
\leq t(|P| - 1) + 1 + 3c(|P| - 1)/|P|
\]

\[
\text{Prob}(p \not\in P - \{p\}) \leq \frac{3}{|P|} \quad \leftarrow \text{Backward Analysis!!!}
\]
Sketch Complexity Analysis

- Assume that cost for \( \text{mindisk}_b(A, p) \) costs \( c|A| \).
- Cost for \( \text{mindisk}(P) \) is:

\[
\begin{align*}
t(|P|) &= t(|P| - 1) + 1 + c(|P| - 1)Prob(p \notin \text{md}(P - \{p\})) \\
&\leq t(|P| - 1) + 1 + 3c(|P| - 1)/|P| \\
&\leq t(|P| - 1) + 1 + 3c
\end{align*}
\]

\[
Prob(p \notin P - \{p\}) \leq \frac{3}{|P|} \leftarrow \text{Backward Analysis!!!}
\]
Sketch Complexity Analysis

- Assume that cost for $\text{mindisk}_b(A, p)$ costs $c|A|$.
- Cost for $\text{mindisk}(P)$ is:

\[
\begin{align*}
t(|P|) &= t(|P| - 1) + 1 + c(|P| - 1) \Pr(p \notin \text{md}(P - \{p\})) \\
&\leq t(|P| - 1) + 1 + 3c(|P| - 1)/|P| \\
&\leq t(|P| - 1) + 1 + 3c \\
&\leq (1 + 3c)n
\end{align*}
\]

\[
\Pr(p \notin P - \{p\}) \leq \frac{3}{|P|} \quad \leftarrow \text{Backward Analysis!!!}
\]
Definition 2
Let $P$ and $R$ be finite point sets in $\mathbb{R}^2$, $P \cup R \neq \emptyset$. Then $md_b(P, B)$ is the smallest enclosing disk of $P \cup R$ with $R \subset \partial md_b(P, R)$ if it exists.

Obviously:

- $md_b(P, \emptyset) = md(P)$
- $md_b(P \cup R, \emptyset) \subset md_b(P, R)$
Algorithm 2 Function mindisk\_b\((P, R)\)

1: \textbf{if } \(P = \emptyset\) \textbf{then}
2: \hspace{1em} \(D := \text{md}_b(\emptyset, R)\);
3: \textbf{else}
4: \hspace{1em} \text{choose random } p \in P;
5: \hspace{1em} \(D := \text{mindisk}_b(P - \{p\}, R)\);
6: \hspace{1em} \textbf{if } p \notin D \textbf{ then}
7: \hspace{2em} \(D := \text{mindisk}_b(P - \{p\}, R \cup \{p\})\);
8: \hspace{2em} \textbf{end if}
9: \textbf{end if}
10: \textbf{return } D;

Algorithm 3 Function mindisk\((P)\)

1: \textbf{return } \text{mindisk}_b(P, \emptyset);
Definition 3 (Algebraic Formulation)

A disk $D(q, r)$ can be defined via function

$$f(p) = 1/r^2 \cdot ||p - q||^2,$$

that is:

$$p \in D(q, r) \iff f(p) \leq 1$$

$$p \in \partial D(q, r) \iff f(p) = 1$$
Convex Combination of Disks

Definition 4 (Convex Combination)

For two disks $D_1 = D(q_1, r_1)$ and $D_2 = D(q_2, r_2)$ define disk $D_\lambda$ for $\lambda \in [0, 1]$ via function:

$$f_\lambda(p) = \lambda f_1(p) + (1 - \lambda) f_2(p) \leq 1$$
Convex Combination of Disks

Definition 4 (Convex Combination)

For two disks $D_1 = D(q_1, r_1)$ and $D_2 = D(q_2, r_2)$ define disk $D_\lambda$ for $\lambda \in [0, 1]$ via function:

$$f_\lambda(p) = \lambda f_1(p) + (1 - \lambda) f_2(p) \leq 1$$

- $D_1 \cap D_2 \subset D_\lambda$
- $\partial D_1 \cap \partial D_2 \subset \partial D_\lambda$
- $D_\lambda$ is a disk
- $r_\lambda$ is smaller than $\max(r_1, r_2)$

Proof on board or exercise ;)
Convex Combination of Disks

Definition 4 (Convex Combination)

For two disks $D_1 = D(q_1, r_1)$ and $D_2 = D(q_2, r_2)$ define disk $D_\lambda$ for $\lambda \in [0, 1]$ via function:

$$f_\lambda(p) = \lambda f_1(p) + (1 - \lambda)f_2(p) \leq 1$$

- $D_1 \cap D_2 \subset D_\lambda$
- $\partial D_1 \cap \partial D_2 \subset \partial D_\lambda$
- $D_\lambda$ is a disk
- $r_\lambda$ is smaller than $\max(r_1, r_2)$
Convex Combination of Disks

Definition 4 (Convex Combination)

For two disks $D_1 = D(q_1, r_1)$ and $D_2 = D(q_2, r_2)$ define disk $D_\lambda$ for $\lambda \in [0, 1]$ via function:

$$f_\lambda(p) = \lambda f_1(p) + (1 - \lambda) f_2(p) \leq 1$$

- $D_1 \cap D_2 \subset D_\lambda$
- $\partial D_1 \cap \partial D_2 \subset \partial D_\lambda$
- $D_\lambda$ is a disk
- $r_\lambda$ is smaller than $\max(r_1, r_2)$
Convex Combination of Disks

**Definition 4 (Convex Combination)**

For two disks $D_1 = D(q_1, r_1)$ and $D_2 = D(q_2, r_2)$ define disk $D_\lambda$ for $\lambda \in [0, 1]$ via function:

$$f_\lambda(p) = \lambda f_1(p) + (1 - \lambda) f_2(p) \leq 1$$

- $D_1 \cap D_2 \subset D_\lambda$
- $\partial D_1 \cap \partial D_2 \subset \partial D_\lambda$
- $D_\lambda$ is a disk
- $r_\lambda$ is smaller than $\max(r_1, r_2)$
Convex Combination of Disks

Definition 4 (Convex Combination)

For two disks $D_1 = D(q_1, r_1)$ and $D_2 = D(q_2, r_2)$ define disk $D_\lambda$ for $\lambda \in [0, 1]$ via function:

$$f_\lambda(p) = \lambda f_1(p) + (1 - \lambda) f_2(p) \leq 1$$

- $D_1 \cap D_2 \subset D_\lambda$
- $\partial D_1 \cap \partial D_2 \subset \partial D_\lambda$
- $D_\lambda$ is a disk
- $r_\lambda$ is smaller than $\max(r_1, r_2)$
Convex Combination of Disks

**Definition 4 (Convex Combination)**

For two disks $D_1 = D(q_1, r_1)$ and $D_2 = D(q_2, r_2)$ define disk $D_\lambda$ for $\lambda \in [0,1]$ via function:

$$f_\lambda(p) = \lambda f_1(p) + (1 - \lambda) f_2(p) \leq 1$$

- $D_1 \cap D_2 \subset D_\lambda$
- $\partial D_1 \cap \partial D_2 \subset \partial D_\lambda$
- $D_\lambda$ is a disk
- $r_\lambda$ is smaller than $\max(r_1, r_2)$

Proof on board or exercise ;)

\[\begin{array}{c}
\text{Proof on board or exercise ;)}
\end{array}\]
Lemma 5

If there exists a disk containing \( P \) with \( R \) on its boundary, then \( m d_b(P, R) \) is well defined.

Proof.

Suppose there are two discs \( D_1 \) and \( D_2 \) with same radius that contain \( P \) and with \( R \) on boundary.

Consider \( D_\lambda \) for \( D_1 \) and \( D_2 \), since \( R \subset \partial D_1 \cap \partial D_2 \) it follows that \( R \subset D_\lambda \).

Same argument as Lemma 1 gives \( D_{1/2} \), which has smaller radius; contradiction.
Lemma 6

Provided \( md_b(P, R) \) exists and

\( p \in P \) with \( p \notin D_1 = md_b(P - \{p\}, R) \),

then:

\[ md_b(P, R) = md_b(P - \{p\}, R \cup \{p\}) \]
Lemma 6
Provided $md_b(P, R)$ exists and $p \in P$ with $p \notin D_1 = md_b(P - \{p\}, R)$, then:

$$md_b(P, R) = md_b(P - \{p\}, R \cup \{p\})$$

Proof.
Assume $p \in D_2 = md_b(P, R)$ but $p \notin \partial D_2$. 

□
Lemma 6

Provided $md_b(P, R)$ exists and $p \in P$ with $p \notin D_1 = md_b(P - \{p\}, R)$, then:

$$md_b(P, R) = md_b(P - \{p\}, R \cup \{p\})$$

Proof.

Assume $p \in D_2 = md_b(P, R)$ but $p \notin \partial D_2$.

Consider continues deformation of $D_\lambda$:

There exists a $\lambda' \in (0, 1)$ such that $p \in \partial D_\lambda'(D_0, D_1)$.

The radius of $D_\lambda$ is smaller than the one of $D_2$; contradiction.
Lemma 7

Provided $md_b(P, R)$ exists, there is $S \subset P$ with $|S| \leq \max\{0, 3 - |R|\}$ such that $md_b(P, R) = md_b(S, R)$

Proof.
Obvious since a disk is defined by at most 3 points on the boundary.

(Exercise)
Algorithm 4 Function mindisk_b(P,R)

1: if $P = \emptyset$ or $|R| = 3$ then
2:     $D := md_b(\emptyset, R)$;
3: else
4:     choose random $p \in P$;
5:     $D := \text{mindisk}_b(P - \{p\}, R)$;
6: if $p \notin D$ then
7:     $D := \text{mindisk}_b(P - \{p\}, R \cup \{p\})$;
8: end if
9: end if
10: return $D$;
Complexity:

- Let $t_j(n)$ the expected number of calls of $p \notin D$ in $\text{mindisk}_b(P, R)$ for $|P| = n$ and $|R| = 3 - j$, then

We would like to know $t_3(n)$. 
Complexity

Complexity:

- Let $t_j(n)$ the expected number of calls of $p \not\in D$ in mindisk_b($P, R$) for $|P| = n$ and $|R| = 3 - j$, then

We would like to know $t_3(n)$.

- $t_0(n) = 0$ since $|R| = 3$
- $t_j(0) = 0$ since $P = \emptyset$
- $t_j(n) \leq t_j(n - 1) + 1 + \frac{j}{n} t_{j-1}(n - 1)$ for $0 < j \leq 3$
Let $t_j(n)$ the expected number of calls of $p \not\in D$ in $\text{mindisk}_b(P, R)$ for $|P| = n$ and $|R| = 3 - j$, then

We would like to know $t_3(n)$.

- $t_0(n) = 0$ since $|R| = 3$
- $t_j(0) = 0$ since $P = \emptyset$
- $t_j(n) \leq t_j(n - 1) + 1 + \frac{j}{n} t_{j-1}(n - 1)$ for $0 < j \leq 3$

It follows:

- $t_1(n) \leq n$
- $t_2(n) \leq t_2(n - 1) + 1 + \frac{2}{n} t_1(n - 1)$
- $t_3(n) \leq t_3(n - 1) + 1 + \frac{3}{n} t_2(n - 1)$
Complexity:

- Let $t_j(n)$ the expected number of calls of $p \notin D$ in `mindisk_b(P, R)` for $|P| = n$ and $|R| = 3 - j$, then

  We would like to know $t_3(n)$.

  - $t_0(n) = 0$ since $|R| = 3$
  - $t_j(0) = 0$ since $P = \emptyset$
  - $t_j(n) \leq t_j(n - 1) + 1 + \frac{j}{n} t_{j-1}(n - 1)$ for $0 < j \leq 3$

It follows:

  - $t_1(n) \leq n$
  - $t_2(n) \leq t_2(n - 1) + 1 + \frac{2}{n} n$
  - $t_3(n) \leq t_3(n - 1) + 1 + \frac{3}{n} t_2(n - 1)$
Complexity:

- Let \( t_j(n) \) the expected number of calls of \( p \notin D \) in \( \text{mindisk}_b(P,R) \) for \(|P| = n\) and \(|R| = 3 - j\), then

We would like to know \( t_3(n) \).

- \( t_0(n) = 0 \) since \(|R| = 3\)
- \( t_j(0) = 0 \) since \( P = \emptyset \)
- \( t_j(n) \leq t_j(n - 1) + 1 + \frac{j}{n} t_{j-1}(n - 1) \) for \( 0 < j \leq 3 \)

It follows:

- \( t_1(n) \leq n \)
- \( t_2(n) \leq t_2(n - 1) + 3 \)
- \( t_3(n) \leq t_3(n - 1) + 1 + \frac{3}{n} t_2(n - 1) \)
Complexity:

Let $t_j(n)$ the expected number of calls of $p \not\in D$ in $\text{mindisk}_b(P,R)$ for $|P| = n$ and $|R| = 3 - j$, then

We would like to know $t_3(n)$.

- $t_0(n) = 0$ since $|R| = 3$
- $t_j(0) = 0$ since $P = \emptyset$
- $t_j(n) \leq t_j(n - 1) + 1 + \frac{j}{n}t_{j-1}(n - 1)$ for $0 < j \leq 3$

It follows:

- $t_1(n) \leq n$
- $t_2(n) \leq 3n$
- $t_3(n) \leq t_3(n - 1) + 1 + \frac{3}{n}t_2(n - 1)$
Complexity:

- Let $t_j(n)$ the expected number of calls of $p \notin D$ in $\text{mindisk}_b(P, R)$ for $|P| = n$ and $|R| = 3 - j$, then

  We would like to know $t_3(n)$.

  - $t_0(n) = 0$ since $|R| = 3$
  - $t_j(0) = 0$ since $P = \emptyset$
  - $t_j(n) \leq t_j(n - 1) + 1 + \frac{j}{n} t_{j-1}(n - 1)$ for $0 < j \leq 3$

  It follows:

  - $t_1(n) \leq n$
  - $t_2(n) \leq 3n$
  - $t_3(n) \leq t_3(n - 1) + 1 + \frac{3}{n} 3n$
Complexity:

Let $t_j(n)$ the expected number of calls of $p \notin D$ in mindisk$_b(P,R)$ for $|P| = n$ and $|R| = 3 - j$, then

We would like to know $t_3(n)$.

- $t_0(n) = 0$ since $|R| = 3$
- $t_j(0) = 0$ since $P = \emptyset$
- $t_j(n) \leq t_j(n - 1) + 1 + \frac{j}{n} t_{j-1}(n - 1)$ for $0 < j \leq 3$

It follows:

- $t_1(n) \leq n$
- $t_2(n) \leq 3n$
- $t_3(n) \leq t_3(n - 1) + 10$
Complexity:

Let $t_j(n)$ the expected number of calls of $p \notin D$ in $\text{mindisk}_b(P, R)$ for $|P| = n$ and $|R| = 3 - j$, then

We would like to know $t_3(n)$.

- $t_0(n) = 0$ since $|R| = 3$
- $t_j(0) = 0$ since $P = \emptyset$
- $t_j(n) \leq t_j(n - 1) + 1 + \frac{j}{n} t_{j-1}(n - 1)$ for $0 < j \leq 3$

It follows:

- $t_1(n) \leq n$
- $t_2(n) \leq 3n$
- $t_3(n) \leq 10n$
Generalization to smallest enclosing ball in $\mathbb{R}^d$

- Rename function to minball ;)
- Replace constant 3 by $\delta = d + 1$
- $t_j(n) = nj! \sum_{k=1}^{j} \frac{1}{k!} \leq (e - 1)j!n$  \hspace{1cm} (Exercise)

**Theorem 8**

The smallest enclosing ball of a set of $n$ points in $\mathbb{R}^d$ can be computed in expected time $O(\delta \delta!n)$, where $\delta = d + 1$. 
Generalization to smallest enclosing ball in $\mathbb{R}^d$

- Rename function to minball ;)
- Replace constant 3 by $\delta = d + 1$
- $t_j(n) = n j! \sum_{k=1}^{j} \frac{1}{k!} \leq (e - 1) j! n$ (Exercise)

**Theorem 8**

The smallest enclosing ball of a set of $n$ points in $\mathbb{R}^d$ can be computed in expected time $O(\delta \delta! n)$, where $\delta = d + 1$.

Remark: It is also possible to extend the algorithm to ellipsoids.
Algorithm for Ball in $\mathbb{R}^d$

Algorithm 5 Function $\text{minball}_b(P,R)$

1:  if $P = \emptyset$ or $|R| = \delta$ then
2:      $D := \text{mb}_b(\emptyset, R)$;
3:  else
4:      choose random $p \in P$;
5:      $D := \text{minball}_b(P - \{p\}, R)$;
6:      if $p \not\in D$ then
7:         $D := \text{minball}_b(P - \{p\}, R \cup \{p\})$;
8:      end if
9:  end if
10: return $D$;
Practical Considerations

- For points in high dimension $d$ the expensive operation is computation of $mb_b(\emptyset, R)$
- Let $s_j(n)$ the expected number of calls of $mb_b(\emptyset, R)$ in $\text{minball}_b(P, R)$ for $|P| = n$ and $|R| = \delta - j$, where $\delta = d + 1$. 

\begin{align*}
\text{Claim: } s_j(n) &\leq (1 + H_n)^j, \text{ where } H_n = \sum_{k=1}^{n} 1/k \text{ (Exercise)}
\end{align*}

Since $H_n \leq 1 + \ln n$, it follows that the number of expected calls to $\text{minball}_b$ is upper bounded by $(2 + \ln)\delta$. 
Practical Considerations

- For points in high dimension $d$ the expensive operation is computation of $mb_b(\emptyset, R)$
- Let $s_j(n)$ the expected number of calls of $mb_b(\emptyset, R)$ in $\text{minball}_{b}(P, R)$ for $|P| = n$ and $|R| = \delta - j$, where $\delta = d + 1$.
  - $s_0(n) = 1$ since $R$ is full
  - $s_j(0) = 1$ since $P = \emptyset$
  - $s_j(n) \leq s_j(n - 1) + \frac{j}{n} s_{j-1}(n - 1)$ for $0 < j \leq \delta$

- Claim: $s_j(n) \leq (1 + H_n)j$, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ (Exercise)
  - Since $H_n \leq 1 + \ln n$, it follows that the number of expected calls to $\text{minball}_{b}(P, R)$ is upper bounded by $(2 + \ln)\delta$. 


Practical Considerations

▷ For points in high dimension \(d\) the expensive operation is computation of \(mb_b(\emptyset, R)\)

▷ Let \(s_j(n)\) the expected number of calls of \(mb_b(\emptyset, R)\) in \(\text{minball}_b(P, R)\) for \(|P| = n\) and \(|R| = \delta - j\), where \(\delta = d + 1\).
  
  ▷ \(s_0(n) = 1\) since \(R\) is full
  ▷ \(s_j(0) = 1\) since \(P = \emptyset\)
  ▷ \(s_j(n) \leq s_j(n - 1) + \frac{j}{n} s_{j-1}(n - 1)\) for \(0 < j \leq \delta\)

▷ Claim: \(s_j(n) \leq (1 + H_n)^j\), where \(H_n = \sum_{k=1}^{n} \frac{1}{k}\) (Exercise)
Practical Considerations

- For points in high dimension $d$ the expensive operation is computation of $mb_b(\emptyset, R)$
- Let $s_j(n)$ the expected number of calls of $mb_b(\emptyset, R)$ in $\minball_b(P, R)$ for $|P| = n$ and $|R| = \delta - j$, where $\delta = d + 1$.
  - $s_0(n) = 1$ since $R$ is full
  - $s_j(0) = 1$ since $P = \emptyset$
  - $s_j(n) \leq s_j(n - 1) + \frac{j}{n} s_{j-1}(n - 1)$ for $0 < j \leq \delta$
- Claim: $s_j(n) \leq (1 + H_n)^j$, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$ (Exercise)
- Since $H_n \leq 1 + \ln n$, it follows that the number of expected calls to $\minball_b$ is upper bounded by $(2 + \ln n)^\delta$. 
Formulation with one Permutation

**Algorithm 6** Function $\text{mindisk}(P) - P$ an ordered sequence

1: Compute random permutation $\pi$ for $1 \ldots |P|$
2: return $\text{mindisk}_b(\pi(P), \emptyset)$

**Algorithm 7** Function $\text{mindisk}_b(P, R) - P$ an ordered sequence

1: if $P = \emptyset$ or $|R| = 3$ then
2: \quad $D := \text{md}_b(\emptyset, R)$
3: else
4: \quad $p := \text{last}(P)$
5: \quad $D := \text{mindisk}_b(P - \{p\}, R)$
6: \quad if $p \notin D$ then
7: \quad \quad $D := \text{mindisk}_b(P - \{p\}, R \cup \{p\})$
8: \quad end if
9: end if
10: return $D$
Complexity Analysis on Permutation

For sequence $P$, let $T(P, R)$ be the cost of $\textit{mindisk}\_b(P, R)$.

Let $t_j(n)$ be the expected value of $T(P, R)$ over all possible insertion sequences $S_n$, where $j = \delta - |R|$.

Obviously $t_0(n) = 0$ and $t_j(0) = 0$ remain.

We want to know:

$$t_3(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), \emptyset)$$

Or in general:

$$t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R), \text{ with } |R| = \delta - j.$$
Complexity Analysis on Permutation – Continued

\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]
Complexity Analysis on Permutation – Continued

\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) = \frac{1}{n} \sum_{p \in P} [1 \]

\[ + \frac{1}{(n - 1)!} \sum_{\pi \in S_{n-1}} \sum_{p=\pi(P)[n]} [ T(\pi(P) - \{p\}, R) \]

\[ + \chi(p \notin md_b(P - \{p\}, R)) \cdot T(\pi(P) - \{p\}, R \cup \{p\}) ] ] \]
Complexity Analysis on Permutation – Continued

\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) = \frac{1}{n} \sum_{p \in P} \left[ 1 + \frac{1}{(n - 1)!} \sum_{\substack{\pi \in S_n \atop p=\pi(P)[n] \neq p}} T(\pi(P) - \{p\}, R) \right. \]

\[ + \chi(p \notin md_b(P - \{p\}, R)) \cdot T(\pi(P) - \{p\}, R \cup \{p\}) \left. \right] \]
$t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R)$

$= \frac{1}{n} \sum_{p \in P} \left[ 1 + \frac{1}{(n-1)!} \sum_{\pi \in S_n \atop p=\pi(P)[n]} T(\pi(P) - \{p\}, R) \right. $

$+ \chi(p \notin mdb(P - \{p\}, R)) \cdot T(\pi(P) - \{p\}, R \cup \{p\}) \left. \right]$

$= \frac{1}{n} \sum_{p \in P} \left[ 1 + \frac{1}{(n-1)!} \sum_{\pi \in S_n \atop p=\pi(P)[n]} T(\pi(P) - \{p\}, R) \right. $

$+ \chi(p \notin mdb(P - \{p\}, R)) \cdot \frac{1}{(n-1)!} \sum_{\pi \in S_n \atop p=\pi(P)[n]} T(\pi(P) - \{p\}, R \cup \{p\}) \right]
Complexity Analysis on Permutation – Continued

\[
t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R)
\]

\[
t_j(n) = \frac{1}{n} \sum_{p \in P} [1
\]

\[
+ \frac{1}{(n - 1)!} \sum_{\pi \in S_n} T(\pi(P) - \{p\}, R)
\]

\[
+ \chi(p \notin m_{db}(P - \{p\}, R)) \cdot \frac{1}{(n - 1)!} \sum_{\pi \in S_n} T(\pi(P) - \{p\}, R \cup \{p\})
\]
\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) = \frac{1}{n} \sum_{p \in P} [1 + \frac{1}{(n-1)!} \sum_{\pi \in S_n \atop p = \pi(P)[n]} T(\pi(P) - \{p\}, R) \]

\[ + \chi(p \notin md_b(P - \{p\}, R)) \cdot \frac{1}{(n-1)!} \sum_{\pi \in S_n \atop p = \pi(P)[n]} T(\pi(P) - \{p\}, R \cup \{p\}) \]

\[ t_j(n) = \frac{1}{n} \sum_{p \in P} [1 + \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} T(\sigma(P - \{p\}), R) \]

\[ + \chi(p \notin md_b(P - \{p\}, R)) \cdot \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} T(\sigma(P - \{p\}), R \cup \{p\}) \]
Complexity Analysis on Permutation – Continued

\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) = \frac{1}{n} \sum_{p \in P} \left[ 1 + \frac{1}{(n - 1)!} \sum_{\sigma \in S_{n-1}} T(\sigma(P - \{p\}), R) + \chi(p \notin md_b(P - \{p\}, R)) \cdot \frac{1}{(n - 1)!} \sum_{\sigma \in S_{n-1}} T(\sigma(P - \{p\}), R \cup \{p\}) \right] \]
Complexity Analysis on Permutation – Continued

\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) = \frac{1}{n} \sum_{p \in P} [1 + \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} T(\sigma(P - \{p\}), R)] \]

\[ t_j(n) = \frac{1}{n} \sum_{p \in P} [1 + t_j(n-1) + \chi(p \notin md_b(P - \{p\}, R)) \cdot t_{j-1}(n-1)] \]
Complexity Analysis on Permutation – Continued

\[
\begin{align*}
t_j(n) &= \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \\
t_j(n) &= \frac{1}{n} \sum_{p \in P} [1 + t_j(n - 1) + \chi(p \notin md_b(P - \{p\}, R)) \cdot t_{j-1}(n - 1)]
\end{align*}
\]
Complexity Analysis on Permutation – Continued

\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) = \frac{1}{n} \sum_{p \in P} \left[ 1 + t_j(n-1) + \chi(p \not\in md_b(P - \{p\}, R)) \cdot t_{j-1}(n-1) \right] \]

\[ t_j(n) = \frac{n}{n} + \frac{1}{n} \sum_{p \in P} t_j(n-1) + \frac{1}{n} \sum_{p \in P} \chi(p \not\in md_b(P - \{p\}, R)) \cdot t_{j-1}(n-1) \]
Complexity Analysis on Permutation – Continued

\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) = \frac{n}{n} \]
\[ + \frac{1}{n} \sum_{p \in P} t_j(n - 1) \]
\[ + \frac{1}{n} \sum_{p \in P} \chi(p \notin md_b(P - \{p\}, R)) \cdot t_{j-1}(n - 1) \]
Complexity Analysis on Permutation – Continued

\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) = \frac{n}{n} \]

\[ + \frac{1}{n} \sum_{p \in P} t_j(n - 1) \]

\[ + \frac{1}{n} \sum_{p \in P} \chi(p \notin md_b(P - \{p\}, R)) \cdot t_{j-1}(n - 1) \]

\[ t_j(n) \leq 1 \]

\[ + \frac{n}{n} t_j(n - 1) \]

\[ + \frac{3}{n} \cdot t_{j-1}(n - 1) \]
\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]
\[ t_j(n) \leq 1 + \frac{n}{n} t_j(n - 1) + \frac{3}{n} \cdot t_{j-1}(n - 1) \]
Complexity Analysis on Permutation – Continued

\[
t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R)
\]

\[
t_j(n) \leq 1 + \frac{n}{n} t_j(n - 1) + \frac{3}{n} \cdot t_{j-1}(n - 1)
\]
\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) \leq 1 + \frac{n}{n} t_j(n - 1) \]

\[ + \frac{3}{n} t_{j-1}(n - 1) \]

\[ t_j(n) \leq 1 + t_j(n - 1) + \frac{j}{n} \cdot t_{j-1}(n - 1), \text{ which we know.} \]
Complexity Analysis on Permutation – Continued

\[ t_j(n) = \frac{1}{n!} \sum_{\pi \in S_n} T(\pi(P), R) \]

\[ t_j(n) \leq 1 + n t_j(n - 1) + \frac{3}{n} \cdot t_{j-1}(n - 1) \]

\[ t_j(n) \leq 1 + t_j(n - 1) + \frac{j}{n} \cdot t_{j-1}(n - 1), \text{ which we know.} \]

Thus, as before \( t_3(n) = 10n \).
Summary

Algorithm:
- algorithm for computing smallest enclosing disk
- expected $O(n)$ time
- $O(n)$ space
- extendable to higher dimensions

Technique: Randomized Incremental Construction (RIC)
- Usually easy to implement
- Complexity analysis may be more tricky
- Backward Analysis