Online Single Machine Scheduling

we have jobs that we need to schedule on a single machine

∟ formally:

* set J of n independent jobs $J_1, \dots, J_n$
* each J_j: - processing time p_j
  - release time r_j

scheduling problem:

execute jobs in J on a machine such that the total completion time $\sum C_j$ is minimized

completion time of J_j in the schedule

(official notation: $1 | r_j | \sum C_j$)

offline: strongly NP-hard

ONLINE:

FIRST-COME-FIRST-SERVED ALGORITHM (FCFS)

* queue maintained to contained all the jobs that have arrived but have not been executed
* jobs in the queue are listed according to nondecreasing r_j (jobs with same r_j are ordered by nondecreasing p_j)
* when machine becomes idle after completing the execution of a job, the first job in the queue is assigned

to the machine for execution. New job $\rightarrow$ inserted in correct position in queue (II)

Observation: the machine is never left idle when there are jobs in the available queue.

Notation for analysis:

- A any algorithm
- I any problem instance
- $\varphi_A(I)$ — total completion times of I for schedule by A
  $\varphi^*(I)$ — — // ———— for optimal schedule

our „costs" !

Is FCFS competitive?
Which factor?

guesses?

The FCFS schedule for any instance I with n has a block structure:

$$B_1, B_2, \ldots, B_e$$

in each block: no idle time

between two consecutive blocks: idle period

~~Let~~ $s(B_i)$: starting time of B_i

Obviously: $r_j \geq s(B_i)$ for any $J_j \in B_i$

We define another instance I':

it contains $J_1', \dots, J_n'$

processing time of J_j' as J_j

release date of J_j' : $s(B_i)$ if $J_j \in B_i$ in the FCFS schedule for I

optimal schedule for I': same block structure as FCFS schedule for I

AND (w/o proof): jobs in each block in the optimal schedule for I' are executed according to the shortest-job-first rule

We have: $\varphi^*(I) \geq \varphi^*(I')$

because: * all jobs same processing time

* release dates in I' are at least as early as those in I

Assume B_i has jobs $J_{i1}, \dots, J_{ik}$

with $p_{i1} \leq \dots \leq p_{ik_i}$

$\psi^*(I', B_i)$: total completion times of jobs in B_i in the opt. sched. for I' ~~schedule~~

$\psi_{FCFS}(I, B_i)$: —— " —— " —— in the FCFS schedule for I

Then:

$$\psi^*(I', B_i) = \underbrace{k_i\, s(B_i)} + k_i\, p_{i1} + (k_i - 1)\, p_{i2} + \dots + p_{ik_i}$$

each job in B_i starts after $s(B_i)$, so for each of the k_i jobs we add this time

due to the shortest-job-first rule J_{i1} comes first, so all k_i jobs have to „wait" this time

the processing time of J_{i2}, the 2nd job in B_i only accounts $(k_i - 1)$ times as J_{i1} already is processed by then

...

AND:

$$\psi_{FCFS}(I, B_i) \leq \underbrace{k_i\, s(B_i)} + \boxed{p_{i1} + 2p_{i2} + \dots + k\, p_{ik_i}}$$

In the worst case we start with the longest in B_i, so p_{ik_i} counts for all jobs and so on

$$\leq k_i\, \psi^*(I', B_i). \qquad (**)$$

$$\Rightarrow \psi_{FCFS}(I) = \psi_{FCFS}(I, B_1) + \ldots + \psi_{FCFS}(I, B_\ell)$$

$$\overset{(**)}{\leq} k_1 \psi^*(I', B_1) + \ldots + \cancel{\psi}\, k_\ell \psi^*(I', B_\ell)$$

$$\leq n \left(\psi^*(I', B_1) + \ldots + \psi^*(I', B_\ell)\right)$$

$$\leq n\, \psi^*(I)$$

So, we showed:

<u>Theorem 1:</u> $\psi_{FCFS}(I) \leq n \cdot \psi^*(I)$ for any instance I on n jobs.

Is that competitive ratio of n tight?

So: is there an instance that achieves this ratio? nachdenken lassen

<u>I:</u> n jobs

$p_1 = M$, $r_1 = 0$, M arb. large pos. number

$p_j = 1$, $r_j = \varepsilon$ for $j = 2, \ldots, n$, ε arb. small pos. number

<u>optimal schedule:</u>

machine waits intentionally for ε time units until jobs $J_2, \ldots, J_n$ are released,

then executes $J_2, \ldots, J_n$ sequentially,

finally executes the long job J_1

$$\Rightarrow \psi^*(I) = (1+\varepsilon) + (2+\varepsilon) + \ldots + (n-1+\varepsilon) + (M + n - 1 + \varepsilon)$$

$$= M + \frac{1}{2} n (n+1) - 1 + n\varepsilon$$

FCFS schedule:

machine executes J_1, then $J_2, \ldots, J_n$

$$\Rightarrow \psi_{FCFS}(I) = M + (M+1) + \ldots + (M+n-1)$$
$$= nM + \frac{1}{2} n(n-1)$$

$$\Rightarrow \frac{\psi_{FCFS}(I)}{\psi^*(I)} = \frac{nM + \frac{1}{2} n(n-1)}{M + \frac{1}{2} n(n+1) - 1 + n\varepsilon}$$

which is arb. close to n for large M and $\varepsilon \to 0$.