OA, Ü #2, 07.05.13

I

The Bahncardproblem — a short recap

$BP(C, \beta, T)$

C: Bahncard costs

β: price reduction $p \to \beta p$

T: BC valid for time T

adversary $\mapsto$ sequence of travel requests $\sigma = \sigma_1 \sigma_2 \dots$

each σ_i : (t_i, p_i)

t_i: time

p_i: regular price

algorithm: buys Bahncard at times $0 \le \tau_1 < \tau_2 < \dots < \tau_k$

$\hookrightarrow \Gamma_A(\sigma) = (\tau_1, \dots, \tau_k)$ B-schedule of A on σ

$$c_{crit} = \frac{C}{1-\beta}$$

~~We had: Th.3 no det. online alg. better than (2-β)-competitive~~

And: Alg. SUM:

for a travel request (t, p) buy a Bahncard iff

(1) owns no Bahncard

(2) regular T-cost at time t, the sum of all regular requests in $(t-T, t]$, $r_{sum}^{\sigma}(t)$ is at least c_{crit}:

$$r_{sum}^{\sigma}(t) \ge c_{crit}$$

$\rightarrow$ Folie

Deterministic online algorithms

<u>Theorem 3</u>: No deterministic online algorithm for $BP(C,\beta,T)$ can be better than $(2-\beta)$-competitive.

<u>Proof:</u> Let A be an online alg for $BP(C,\beta,T)$.

$\varepsilon > 0$ arb. small

As long A does not have a BC $\rightarrow$ adversary shows requests of cost ε (arb. dense $\rightarrow$ all requests in $[0,T)$)

$\hookrightarrow$ If A wants to be better than $1/\beta$-competitive, it must eventually buy a BC

$\hookrightarrow$ adversary stops showing requests

S: accumulated cost of requests so far, not including current request

$$\Rightarrow \quad C_A(\sigma) = C + S + \beta\varepsilon$$

$$C_{OPT}(\sigma) = \begin{cases} S + \varepsilon & \text{if } S + \varepsilon \leq C_{crit} \\ C + \beta(S+\varepsilon) & \text{if } S + \varepsilon \geq C_{crit} \end{cases}$$

$$\Rightarrow \quad \frac{C_A(s)}{C_{OPT}(s)} = \begin{cases} \dfrac{C+s+\beta\varepsilon}{s+\varepsilon} & \text{if } s+\varepsilon \le c_{crit} \\[2ex] \dfrac{C+s+\beta\varepsilon}{C+\beta(s+\varepsilon)} & \text{if } s+\varepsilon \ge c_{crit} \end{cases}$$

$$\ge 2-\beta-\frac{\varepsilon(1-\beta)^2}{C}$$

as both quotients achieve min a $s = c_{crit}-\varepsilon$.
claim follows with $\varepsilon \to 0$. □

<u>Algorithm SUM</u>

for a travel request (t,p) buy a Bahncard iff

(1) owns no Bahncard

(2) regular T-cost at time t, the sum of all regular requests in $(t-T,t]$, $\mathrm{TT}^{\sigma}_{SUM}(t)$ is at least c_{crit}:

$$\mathrm{TT}^{\sigma}_{SUM}(t) \ge c_{crit}$$

→ Folie

+ 1 Min ans.

+ 1 Min reden

Theorem 1: SUM is $(2-\beta)$-competitive for $BP(C,\beta,T)$. IV

Proof: Let $\sigma = \sigma_1 \sigma_2 \ldots$ be a request sequence

$\Gamma_{OPT}(\sigma) = (\tau_1, \ldots, \tau_k)$ optimal $\mathcal{B}$-schedule for σ

$\Rightarrow$ divides time into epochs $[\tau_j, \tau_{j+1})$ $0 \le j \le k$

(with $\tau_0 = 0$, $\tau_{k+1} = \infty$)

Each epoch starts with expensive phase $[\tau_j, \tau_j + T)$

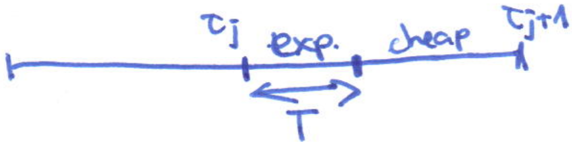
followed by a cheap phase $[\tau_j + T, \tau_{j+1})$

(except for, possibly, first and last)

SUM: buys at most one BC during any epoch

Why?: * Obs. 1(b)

* $(t-T, t]$ must be exp. interval if SUM buys BC at t

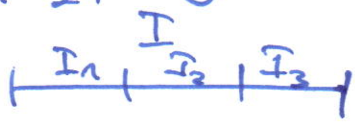


$\Rightarrow$ upper bound for SUM: assume SUM spends C in every expensive phase (+ ticket costs)

- Cheap phase: $c^{I}_{SUM}(\sigma) \le c^{I}_{OPT}(\sigma)$
- expensive phase I: $\begin{pmatrix} c_{SUM} - \text{cost of SUM during } I \\ c_{OPT} - \text{" — OPT — "} \end{pmatrix}$ Incl. cost for BC

We divide I into 3 subphases: I_1, I_2, I_3 (can be empty)

In I_1 and I_3 SUM has BC!



$$s_i = p^{I_i}(\sigma) = \sum_{j : t_j \in I_i} p_j \qquad i = 1,2,3$$

total cost of requests in I_i

$$\Rightarrow \quad C_{SUM} \leq C + s_2 + \beta(s_1 + s_3)$$

$$C_{OPT} = C + \beta(s_1 + s_2 + s_3)$$

$$\Rightarrow \quad \frac{C_{SUM}}{C_{OPT}} \leq \frac{C + s_2 + \beta(s_1+s_3)}{C + \beta(s_1+s_2+s_3)}$$

maximal if
$s_1 = s_3 = 0$
s_2 maximal

$$\leq \frac{C + s_2}{C + \beta s_2}$$

$s_2 \leq c_{crit}$ because def. of SUM
$\rightarrow$ if $s_2 = c_{crit}$ $\rightarrow$ B cuird gelu

$$\leq \frac{C + c_{rit}}{C + \beta c_{rit}}$$

$$c_{crit} = \frac{C}{1-\beta} \Rightarrow \frac{C + \frac{C}{1-\beta}}{C + \beta\frac{C}{1-\beta}} = \frac{\cancel{C}\left(1 + \frac{1}{1-\beta}\right)}{\cancel{C}\left(1 + \frac{\beta}{1-\beta}\right)}$$

$$= \frac{\frac{2-\beta}{1-\beta}}{\frac{1-\cancel{\beta}+\cancel{\beta}}{1-\beta}} = \frac{2-\beta}{\cancel{1-\beta}} \cdot \frac{\cancel{1-\beta}}{1} = 2-\beta$$

$\square$

Randomized Online Algorithms

Algorithm R-SUM: Randomized variant of SUM, which, with probability $q = \frac{1}{1+\beta}$ buys a BC at time $t \iff$ SUM would buy one at t

optimal → see next proof!

Example: Deutsche Bahn: $\beta = \frac{1}{2} \Rightarrow q = \frac{1}{3/2} = 2/3$

Theorem 2: R-SUM is $\frac{2}{1+\beta}$-competitive for $BP(C, \beta, T)$.

Again: epochs (first expensive phase $[\tau_j, \tau_j + T)$, then cheap phase $[\tau_j + T, \tau_{j+1})$)

* cheap phases: analysis as in the deterministic case
* expensive phases: $I = I_1 \cup I_2 \cup I_3$ (R-SUM BC in I_1 and I_3)

BC with prob. q

$$C_{R\text{-}SUM} \leq q \cdot C + s_2 + q \cdot \beta (s_1 + s_3) + (1-q)(s_1 + s_3)$$
$$= q \cdot C + s_2 + (q \cdot \beta + 1 - q)(s_1 + s_3)$$

$$C_{OPT} = C + \beta (s_1 + s_2 + s_3)$$

expensive phase ↳ buys

↳ reduced tickets

$$\Rightarrow \frac{C_{R\text{-}SUM}}{C_{OPT}} = \frac{q \cdot C + s_2 + (q\beta + 1 - q)(s_1 + s_3)}{C + \beta(s_1 + s_2 + s_3)}$$

$$C_{R\text{-}SUM} = q \cdot C + s_2 + (q\beta + 1 - q)(s_1 + s_3)$$

$$\overset{q = \frac{1}{1+\beta}}{=} C \cdot \frac{1}{1+\beta} + s_2 + \left(\frac{\beta}{1+\beta} + 1 - \frac{1}{1-\beta}\right)(s_1 + s_3)$$

V

$$= C\,\frac{1}{1+\beta} + s_2 + \frac{\beta + \cancel{1} + \beta - \cancel{1}}{1+\beta}\,(s_1+s_3)$$

für

$$s_2 \leq c_{crit} = \frac{C}{1-\beta} \;\leq\; C\cdot\frac{1}{1+\beta} + C\,\frac{1}{1-\beta} + \frac{2\beta}{1+\beta}\,(s_1+s_3)$$

$$= C\cdot\frac{1-\cancel{\beta}+1+\cancel{\beta}}{(1+\beta)(1-\beta)} + \frac{2\beta}{1+\beta}\,(s_1+s_3)$$

$$= C\,\frac{2}{(1+\beta)(1-\beta)} + \frac{2\beta}{1+\beta}\,(s_1+s_3)$$

$$C_{OPT} = C + \beta(s_1+s_2+s_3) = C + \beta\cdot s_2 + \beta(s_1+s_3) \geq \underbrace{c_{crit}}_{=\frac{C}{1-\beta}} + \beta(s_1+s_3)$$

$$\Rightarrow \quad \frac{C_{R\text{-}SUM}}{C_{OPT}} \leq \frac{C\,\frac{2}{(1+\beta)(1-\beta)} + \frac{2\beta}{1+\beta}\,(s_1+s_3)}{\frac{C}{1-\beta} = \cancel{c_{crit}} + \beta(s_1+s_3)}$$

$$= \frac{2}{(1+\beta)}\;\frac{\frac{C}{1-\beta} + \beta(s_1+s_3)}{\frac{C}{1-\beta} + \beta(s_1+s_3)} = \frac{2}{1+\beta}$$

$\square$