Exercise 1 (Minimum-Degree Spanning Trees):
Consider the local search algorithm for finding a minimum-degree spanning tree as presented in the tutorial on May 10. Complete the proof of Theorem 3: The algorithm finds a locally optimal tree in polynomial time.

For this consider the potential function $\Phi(T): \Phi(T) = \sum_{v \in V} 3^{d_T(v)}$ for a tree $T$.
We concluded that $\Phi(T) \leq n3^{\Delta(T)}$, hence, the initial potential is at most $n3^n$. In addition, a Hamiltonian path has the lowest possible potential: $2 \cdot 3 + (n - 2)3^2 > n$.

(a) Show that for each move, the potential function of the resulting tree is at most $1 - \frac{2}{27n^4}$ times the potential function previously.

(b) Consider the situation after $\frac{27}{2}n^4 \ln 3$ local moves and conclude that we obtained a locally optimal tree.

(10+10 Punkte)

Exercise 2 (Vertex Cover in Trees):

(a) Give a polynomial-time algorithm for the Vertex Cover Problem when only trees are used as an input. (Hint: What statement is possible on the leaves of the trees and their participation in an optimal Vertex Cover?)

(b) Let $T_n$ be the complete binary tree of depth $n$ (that is, on each path from the root $r$ to an arbitrary leaf we have $n$ nodes). Show: for $n$ odd the root is never included in an optimal Vertex Cover.
Exercise 3 (Independent Set Problem):
A set of vertices $U \subseteq V$ is called independent, if for all $u, v \in U$: $\{u, v\} \notin E$. For the Independent Set Problem we ask for an independent set of maximum cardinality.
Show: $C$ is a Vertex Cover of $G = (V, E)$ iff $U = V \setminus C$ is an independent set.
Moreover, show: $C$ is an optimal solution for the Vertex Cover Problem iff $U = V \setminus C$ is an optimal solution for the Independent Set Problem.

(10 Punkte)

Exercise 4 (NP-Completeness of the Dominating Set Problem):
Dominating Set Problem:
Instance: Graph $G = (V, E)$, positive integer $K \leq |V|$.
Question: Is there a subset $V' \subseteq V$ such that $|V'| \leq K$ and such that every vertex $v \in V \setminus V'$ is joined to at least one member of $V'$ by an edge in $E$?

Vertex Cover Problem:
Instance: Graph $G = (V, E)$, positive integer $C \leq |V|$.
Question: Does $G$ contain a vertex cover of size at most $C$?

Show the Dominating Set Problem to be NP-complete by reducing Vertex Cover to it.

(10 Punkte)